NONEXCLUSIVE COMPETITION FOR A FREELANCER UNDER ADVERSE SELECTION

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Outline

• Highlights

- Introduction
- The Model
- Equilibrium Characterization
- The Main Results
- Concluding Remarks

- In modern labor markets, nonexclusivity is becoming more and more the rule
- Multiple parties compete nonexclusively for a freelancer under adverse selection
- The freelancer is subject to a capacity constraint and has strictly convex cost
- We provide necessary and sufficient conditions for the existence of a pure-strategy equilibrium.
- We prove that Akerlof-like equilibrium trades arise in this setting

Consider a freelancer who...

- has limited working hours (Capacity Constraint)
- has private information regarding the quality of his service. (Adverse Selection)
- can simultaneously work with multiple parties by dividing his working hours accordingly (Nonexclusivity)
- incurs a higher cost for an extra minute of work as the allocated working hours increases (Convex Cost)

Suppose multiple parties are interested in the services of this freelancer. What kind of trades shall we expect to arise in such a setup?

Literature on Adverse Selection

Akerlof, G. (1970). The market for "lemons": Quality uncertainty and the market mechanism., Quarterly Journal of Economics, 84(3), 488–500.

- Sellers are privately informed about the quality of their goods.
 - the goods are non-divisible
 - all trades take place at the same price
- Sellers of high-quality goods end up not trading in equilibrium.

Rothschild, M., and J. E. Stiglitz (1976). *Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information*, Quarterly Journal of Economics, 90, 629–649.

- A population of individuals seeking insurance privately know their risk
 - multiple insurance companies offer contracts
 - each individual chooses at most one contract (exclusive competition)
- Low-quality sellers trade efficiently while high-quality sellers trade a non-zero, but sub-optimal quantity.

Attar, A., Mariotti, T., and Salanie, F. (2011). *Nonexclusive competition in the market for lemons*. Econometrica, 79(6), 1869–1918.

- A seller who faces offers from multiple parties...
 - is subject to a time constraint (capacity)
 - privately knows the quality of his good
 - has linear preferences
 - can work with several buyers (nonexclusivity)
- Multiple buyers (with linear preferences) offer contracts

Equilibrium always exists & aggregate trades are unique:

- If the quality is low enough the seller trades all of his capacity
- Equilibrium price equal to the expected quality of the traded good

Attar, A., Mariotti, T., and Salanie, F. (2014). *Nonexclusive competition under adverse selection*. Theoretical Economics, 9(1), 1–40.

- A seller with a perfectly divisible good... (no capacity constraint)
 - privately knows the quality of his good (High or Low)
 - has strictly quasi-concave preferences
 - can trade with several buyers (nonexclusive)
- Multiple buyers (with linear preferences) offer contracts

Equilibrium exists iff the high-type is not willing to trade at a price equal to the average quality & aggregate trades are unique:

- The high-type does not trade
- Any traded contract yields zero profit (no-cross-subsidization)

	Our paper	Attar et al. (2011)	Attar et al. (2014)
Seller's	Quasilinear with	Linear	Strictly
preferences	strictly convex cost		quasi-concave
Capacity	Yes	Yes	No
Service quality	High or Low	Continuum, discrete or mixed	High or Low
Existence of equilibrium	Exists iff high-type is never willing to trade at a price equal to the average quality OR always willing to trade at the same price	Always exists (for a large class of type distributions)	Exists iff high-type is not willing to trade at a price equal to the average quality

	Our paper	Attar et al. (2011)	Attar et al. (2014)
Cross- subsidization	Depends on the preferences of high-type	Possible in equilibrium	Ruled out in equilibrium
Aggregate Equilibrium Trades	Unique (both types trade at the capacity OR high-type does not trade while low-type either trades efficiently, trades at the capacity, or does not trade)	Unique (if the quality is low enough the freelancer trades at capacity, otherwise does not trade)	Unique (the high-type does not trade, the low-type trades a non-negative quantity)

We characterize the equilibrium trades for this problem under the following setting:

- A freelancer can work with multiple parties simultaneously (nonexclusivity)
 - subject to time constraint (capacity)
 - privately knows the quality of his service (High or Low)
 - has quasilinear preferences (strictly convex cost)
- Multiple parties offer contracts that specify a working hour and a monetary transfer
 - share a common prior regarding the quality of the service
 - have linear preferences

Our results confirm that the Akerlof-like equilibrium outcomes presented in the earlier works extend to our setting: A pure-strategy equilibrium exists in this setting iff one of the following conditions is satisfied:

- (*i*) At the no-trade point, the high-type freelancer is *not* willing to trade for a price equal to the average quality of the service.
- (*ii*) At any feasible trade point, the high-type freelancer is willing to trade for a price equal to the average quality of the service.
 - If (*i*) holds, the high-type does not trade in equilibrium while the aggregate trade of the low-type depends on his preferences. There is no cross-subsidization in equilibrium.
 - If (*ii*) holds, both types trade at the capacity, and there is cross-subsidization in equilibrium.

In all of these equilibria, aggregate equilibrium trades are unique, and the buyers make zero profit.

For the freelancer...

- service quality is H with probability $p_H \in (0,1)$ s.t. $p_H + p_L = 1$
- upper bound on the aggregate working hours is \bar{Q}_C
- his payoff is given by $u_t(Q, M) = M c_t(Q)$ for $t \in \{H, L\}$
- the cost function, c_t is continuously differentiable and strictly convex

• type *t*'s MRS of the good for money is
$$-rac{\partial u_t/\partial Q}{\partial u_t/\partial M}=c_t'(Q)$$

• we assume $c'_{\mathcal{H}}(Q) > c'_{\mathcal{L}}(Q)$ holds for all $Q \in [0, ar{Q}_{\mathcal{C}}]$

The Model

There are $n \ge 2$ identical buyers

- Each buyer i offers a set of contracts $C^i \subset \mathbb{R}^2$
- $(0,0) \in C^i$ for all *i*so that the freelancer is not forced to trade
- Upon trading (q, m) with type t, a buyer earns $u_t q m$ for $t \in \{H, L\}$
- Expected quality of the good is $\nu = p_H \nu_H + p_L \nu_L$ so that $\nu_H > \nu > \nu_L$

As noted by Attar et al. (2011, 2014), we do not need to consider more general mechanisms in our setup.

The freelancer's maximization problem is:

$$\max_{\substack{i:q^i,t^i)\in C^i}} \sum_i m^i - c_t(\sum_i q^i)$$

s.t. $\sum_i q^i \leq \bar{Q}_C$

 C^i is assumed to be compact for all i so that, this problem always has a solution

In a pure strategy equilibria, each t chooses a contract (q_t^i, m_t^i) for all i

Aggregate equilibrium trades are $(Q_t, M_t) = (\sum_i q_t^i, \sum_i m_t^i)$ for each t

Indirect utility function

The maximum payoff that type t freelancer can achieve while trading (q, m) with buyer i is given by:

$$egin{aligned} &z_t^{-i}(m{q},m{m}) = \max_{(q^j,m^j)\in C^j} &m{m} + \sum_{j
eq i} m^j - c_t(m{q} + \sum_{j
eq i} q^j) \ & ext{ s.t. } &\sum_{j
eq i} q^j \leq ar{Q}_C - m{q} \end{aligned}$$

In equilibrium, $U_t = u_t(Q_t, M_t) = z_t^{-i}(q_t^i, m_t^i)$

Note that z_t^{-i} may have discontinuities due to the capacity constraint

Equilibrium Characterization

Considering type t freelancer's optimal choice, (Q_t, M_t) in aggregate, buyer i can fix arbitrary contracts from other buyer's menus which amounts to (Q^{-i}, M^{-i}) and deviate by offering $(q, m) = (Q_t - Q^{-i}, M_t - M^{-i})$.

 $\begin{aligned} \pi_t^i &= \nu_t q_t^i - m_t^i & \text{is the profit of buyer } i \text{ with type } t \in \{H, L\} \\ \pi^i &= p_H \pi_H^i + m_L \pi_L^i & \text{is the expected profit of buyer } i \end{aligned}$

Lemma 1

In equilibrium, for all $q \in [0, \overline{Q}_C]$ and m, if the seller can trade $(Q_t - q, M_t - m)$ with buyers other than i, then

$$u_t q - m > \pi_t^i \quad \text{implies} \quad \nu q - m \le \pi^i.$$

In words, if some buyer *i* can improve his profits with type *t*, then his deviation should be traded by both types of the freelancer and, it should not be profitable in expectation.

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Lemma 1

In equilibrium, for all $q \in [0, \overline{Q}_C]$ and m, if the seller can trade $(Q_t - q, M_t - m)$ with buyers other than i, then

$$u_t q - m > \pi_t^i \quad \text{implies} \quad \nu q - m \le \pi^i.$$

Proof: Buyer *i* deviates to $\{(0,0), (q, m + \epsilon_H), (q_L^i, m_L^i + \epsilon_L)\}$ for $\epsilon_H > \epsilon_L > 0$ (proof for *L* is similar)

- Type *H* can trade $(Q_H q, M_H m)$ with buyers other than *i* and $(q, m + \epsilon_H)$ with *i*
 - prefers trading $(q, m + \epsilon_H)$ since $u_H(Q_H, M_H + \epsilon_H) > U_H + \epsilon_L$
- Type *L* does not prefer (0,0) since $u_L(Q_L, M_L + \epsilon_L) > U_L \ge z_L^{-i}(0,0)$

- if *L* trades $(q_L^i, m_L^i + \epsilon_L)$ then the payoff of buyer *i* strictly increases Hence both types should trade $(q, m + \epsilon_H)$ after the deviation, and the expected payoff of the buyer *i* should not be higher than π^i . Consider the payoff of each type in terms of aggregate equilibrium trades:

$$M_L - c_L(Q_L) \ge M_H - c_L(Q_H)$$

 $M_H - c_H(Q_H) \ge M_L - c_H(Q_L)$

Sum up the above inequalities and employ the fundamental theorem of calculus:

$$c_{H}(Q_{L}) - c_{H}(Q_{H}) \ge c_{L}(Q_{L}) - c_{L}(Q_{H})$$
$$\int_{Q_{H}}^{Q_{L}} c'_{H}(x) dx \ge \int_{Q_{H}}^{Q_{L}} c'_{L}(x) dx$$

Hence, assumption $c'_H(Q) > c'_L(Q)$ implies $Q_L \ge Q_H$ in equilibrium

Equilibrium Characterization

 $S_L = \nu_L (Q_L - Q_H) - (M_L - M_H)$ is the aggregate profit from the additional trade with type L

 $S_H = \nu_H (Q_H - Q_L) - (M_H - M_L)$ is defined similarly for type H

 $\begin{aligned} \Pi_t &= \sum_j \pi_t^j \text{ is the aggregate profit from type } t \in \{H, L\} \\ \Pi &= \sum_j \pi^j = \sum_j (p_L \pi_L^j + p_H \pi_H^j) = p_L \Pi_L + p_H \Pi_H \text{ is the aggregate expected profit} \end{aligned}$

Proposition 1 (Attar et al. (2014))

In any equilibrium, $S_L \leq 0$ and $\Pi = 0$ so that $\pi^j = 0$ for each *j*. Moreover, the following statements hold.

(*i*) In any pooling equilibrium, $M_L = \nu Q_L = M_H = \nu Q_H$.

(ii) In any separating equilibrium, $Q_L > Q_H \ge 0$ holds with $M_H = \nu Q_H$ and $M_L - M_H = \nu_L (Q_L - Q_H)$.

In any equilibrium, $S_{L} \leq 0$ and $\Pi = 0$ so that $\pi^{j} = 0$ for each j. Moreover, the following statements hold.

- (i) In any pooling equilibrium, $M_L = \nu Q_L = M_H = \nu Q_H$.
- (ii) In any separating equilibrium, $Q_L > Q_H \ge 0$ holds with $M_H = \nu Q_H$ and $M_L - M_H = \nu_L (Q_L - Q_H)$.

Proof: Assume $S_L > 0$. Then, $\exists i \text{ s.t. } S_L > s_L^i = \nu_L(q_L^i - q_H^i) - (m_L^i - m_H^i)$

- Buyer i deviates to $(q_H^i + Q_L Q_H, m_H^i + M_L M_H)$
 - Using Lemma 1 yields: $S_L > s_L^i$ implies $p_L(S_L s_L^i) \le p_H S_H$
 - Hence, $S_L > 0$ implies $S_H > 0$

Using $\nu_H > \nu_L$ and $Q_L \ge Q_H$, we reach to the following contradiction:

$$S_H + S_L = (\nu_L - \nu_H)(Q_L - Q_H) \leq 0$$

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In any equilibrium, $S_L \leq 0$ and $\Pi = 0$ so that $\pi^j = 0$ for each *j*. Moreover, the following statements hold.

- (*i*) In any pooling equilibrium, $M_L = \nu Q_L = M_H = \nu Q_H$.
- (ii) In any separating equilibrium, $Q_L > Q_H \ge 0$ holds with $M_H = \nu Q_H$ and $M_L - M_H = \nu_L (Q_L - Q_H)$.

Proof: Assume $\Pi > 0$. Then, $\Pi_H > 0$, and $\exists i \text{ s.t. } \Pi_H > \pi_H^i$

- Buyer *i* deviates to (Q_H, M_H)
 - Using Lemma 1 yields: $\Pi_H > \pi_H^i$ implies $\Pi \pi^i \le p_L S_L$
- In equilibrium, $\pi^j \ge 0$ for all j, and $S_L \le 0$

- Hence, $S_L = 0$, and $\pi^j = 0$ for all $j \neq i$. By $S_H + S_L \leq 0$, $S_H \leq 0$ For all $j \neq i$, $\Pi - \pi^j = \Pi > p_L S_L \geq p_H S_H$ implies $\Pi_H \leq \pi_H^j$ and $\Pi_L \leq \pi_L^j$

$$p_H \Pi_H + p_L \Pi_L = \Pi \le \pi^j = p_H \pi_H + p_L \pi_L = 0$$

a contradiction!

In any equilibrium, $S_L \leq 0$ and $\Pi = 0$ so that $\pi^j = 0$ for each j. Moreover, the following statements hold.

- (i) In any pooling equilibrium, $M_L = \nu Q_L = M_H = \nu Q_H$.
- (ii) In any separating equilibrium, $Q_L > Q_H \ge 0$ holds with $M_H = \nu Q_H$ and $M_L - M_H = \nu_L (Q_L - Q_H)$.

Proof: In any pooling equilibrium, $\Pi = 0$ implies $M_L = \nu Q_L = M_H = \nu Q_H$ In any separating equilibrium, $Q_L > Q_H \ge 0$. Moreover, $\Pi = 0$ implies that (*ii*) holds if $S_L = 0$. Assume that $S_L < 0$

- Recall $\Pi_H > \pi_H^i$ implies $\Pi \pi^i \le p_L S_L$ from the previous slide
 - Using $S_L < 0$ and $\Pi \pi^i = 0$ implies $\Pi_H \le \pi^i_H$ for all *i*
 - Hence, $\Pi_H \leq 0$

This yields the following contradiction:

$$\Pi = \nu Q_H - M_H + p_L S_L \le \Pi_H + p_L S_L < 0 \quad \blacksquare$$

In any equilibrium, $S_L \leq 0$ and $\Pi = 0$ so that $\pi^j = 0$ for each j. Moreover, the following statements hold.

- (i) In any pooling equilibrium, $M_L = \nu Q_L = M_H = \nu Q_H$.
- (ii) In any separating equilibrium, $Q_L > Q_H \ge 0$ holds with $M_H = \nu Q_H$ and $M_L - M_H = \nu_L (Q_L - Q_H)$.

As a consequence, we obtain the following immediate result:

Corollary 1

In any equilibrium, $\Pi_H \ge 0 \ge \Pi_L$ and $S_L = 0$ hold.

In words, the aggregate profit of the buyers gained from additionally trading $(Q_L - Q_H, M_L - M_H)$ with type *L* is always zero in equilibrium. Furthermore, the profit from type *H* subsidizes for the loss from type *L* if an equilibrium exhibits cross-subsidization.

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Conditions on an equilibrium with cross-subsidization:

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) \begin{cases} = \nu & \text{if } Q_{H} < \bar{Q}_{C}, \\ \leq \nu & \text{if } Q_{H} = \bar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

The marginal cost of type H should be less than or equal to ν . Furthermore, the contract offers of any single buyer should not be essential for the aggregate equilibrium trades of type H. That is, type H should be able to trade at the same aggregate level even if a buyer withdraws his offers.

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Equilibrium Characterization

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) egin{cases} =
u & ext{if } Q_{H} < ar{Q}_{C}, \ \leq
u & ext{if } Q_{H} = ar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Proof: Assume that $\Pi_H > 0$ but $c'_H(Q_H) \neq \nu$. Buyer *i* deviates to $\{(0,0), (Q_H + \delta_H, M_H + \epsilon_H)\}$ where $c'_H(Q_H)\delta_H < \epsilon_H < \nu_H\delta_H$.

- Type *H* prefers $(Q_H + \delta_H, M_H + \epsilon_H)$ for small δ_H and ϵ_H
- If Type L also trades this contract, then buyer i increases his payoff
 - Hence, Type L should trade (0,0) yielding at most zero payoff:

$$p_H(\nu_H(Q_H + \delta_H) - M_H - \epsilon_H) = p_H(\Pi_H + \nu_H \delta_H - \epsilon_H) \le 0$$

Letting δ_H and ϵ_H go to zero yields a contradiction. When, $Q_H = \bar{Q}_C$, positive δ values are infeasible, which leaves $c'_H(Q_H) \leq \nu$.

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) iggl\{ =
u \quad ext{if } Q_{H} < ar{Q}_{C}, \ \leq
u \quad ext{if } Q_{H} = ar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Proof: Assume that $U_H > z_H^{-i}(0,0)$. Buyer *i* deviates to $\{(0,0), (Q_H, M_H - \epsilon_H)\}$ for $\epsilon_H > 0$.

- Type *H* prefers $(Q_H, M_H \epsilon_H)$ for small ϵ_H
- If Type L prefers (0,0), then buyer *i* increases his payoff
 - If Type L prefers $(Q_H, M_H \epsilon_H)$, then buyer *i* increases his payoff

This is a contradiction.

Hence, $\exists (Q^{-i}, M^{-i})$ s.t. $u_H(Q^{-i}, M^{-i}) = u_H(Q_H, M_H)$

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) \begin{cases} = \nu & \text{if } Q_{H} < \bar{Q}_{C}, \\ \leq \nu & \text{if } Q_{H} = \bar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Proof: Assume that $Q^{-i} \neq Q_H$. Case 1: $Q^{-i} < Q_H < \bar{Q}_C$ Then, $c'_H(Q_H) = \nu$ and strict convexity of the c_H implies $M^{-i} > \nu Q^{-i}$

- Consider deviations $(q_t, m_t) = (Q_t, M_t) (Q^{-i}, M^{-i})$ for $t \in \{H, L\}$
 - When traded by both types, (q_t, m_t) is profitable
 - Using Lemma 1, $\nu_t q_t m_t \leq \pi_t^i$ should hold
 - This gives $\pi_t^i > (
 u_t
 u)(Q_H Q^{-i})$ for $t \in \{H, L\}$

Averaging yields the contradiction $0 < \pi^i = 0$

Equilibrium Characterization

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) iggl\{ =
u \quad ext{if } Q_{H} < ar{Q}_{C}, \ \leq
u \quad ext{if } Q_{H} = ar{Q}_{C}. \end{cases}$$

Moreover, for each buyer i, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Case 2: $Q_{H} < Q^{-i} < Q_{I}$ Then, $c'_{H}(Q_{H}) = \nu$ and strict convexity of the c_{H} implies $M^{-i} > \nu Q^{-i}$ • Only deviation $(q_L, m_I) = (Q_I, M_I) - (Q^{-i}, M^{-i})$ is feasible - Leading to $\pi_I^i > (\nu_I - \nu)(Q_H - Q^{-i})$, and hence $\pi_I^i > 0$ - Then, $\Pi_H > \Pi = 0$ implies $\exists i$ with $\pi_I^j < \Pi_I < 0$ • Recall $\Pi_L > \pi_I^i$ implies $\Pi - \pi^i \leq p_H S_H$ from the previous proof - Then $p_H S_H > \Pi - \pi^j = 0$ and $S_H + S_I < 0$ implies $S_H = 0$ $S_H + S_I = (\nu_I - \nu_H)(Q_I - Q_H) = 0$, gives $Q_I = Q_H$, a contradiction H. I. Bayrak, N. A. Dalkıran

Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) iggl\{ =
u \quad ext{if } Q_{H} < ar{Q}_{C}, \ \leq
u \quad ext{if } Q_{H} = ar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Case 3:
$$Q^{-i} < Q_H = \bar{Q}_C$$

Using $U_H = z_H^{-i}(0,0)$, $M^{-i} = \nu Q^{-i} + \nu (Q_H - Q^{-i}) + c_H(Q^{-i}) - c_H(Q_H)$
• Strict convexity of c_H gives $c_H(Q_H) < c_H(Q^{-i}) + c'_H(Q_H)(Q_H - Q^{-i})$

comparing with
$$c_H(q_H) \ge v$$
.

$$M^{-i} > \nu Q^{-i} + (\nu - c'_H(Q_H))(Q_H - Q^{-i}) > \nu Q^{-i}$$

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Lemma 2

If in equilibrium $\Pi_H > 0$, then

$$c'_{H}(Q_{H}) egin{cases} =
u & ext{if } Q_{H} < ar{Q}_{C}, \ \leq
u & ext{if } Q_{H} = ar{Q}_{C}. \end{cases}$$

Moreover, for each buyer *i*, the freelancer can trade (Q_H, M_H) with buyers other than *i*.

Case 4: $Q_L \leq Q^{-i}$ In this case, type *L* would strictly prefer (Q^{-i}, M^{-i}) to (Q_L, M_L) , contradicting (Q_L, M_L) being the aggregate equilibrium trade of type *L*.

There is no other case. Hence, one should have $(Q^{-i}, M^{-i}) = (Q_H, M_H)$ in equilibrium.

Equilibrium Characterization

There is no cross-subsidization in equilibrium unless type H trades at the capacity:

Proposition 3

In any equilibrium with $Q_H < \bar{Q}_C$, $\Pi_t = 0$, for any t.

Proof: Assume the contrary. From Corollary 1, $\Pi_H > 0$ and $Q_H < \bar{Q}_C$

• Any buyer *i* with $\pi_H^i > 0$ deviates to {(0,0), $(Q_L - Q_H + \delta_L, M_L - M_H + \epsilon_L), (q_H^i, m_H^i + \epsilon_H)$ } where $c'_L(Q_L + \delta_L)\delta_L < \epsilon_L$ and $\epsilon_H > 0$

• By Lemma 2, type L can trade (Q_H, M_H) with buyers other than i

- Since $S_L = 0$, prefers $(Q_L Q_H + \delta, M_L M_H + \epsilon_L) + (Q_H, M_H)$
- For $\epsilon_H < \epsilon_L \delta_L c'_L (Q_L + \delta_L)$, does not prefer $(q_H^i, m_H^i + \epsilon_H)$
- If type H trades $(q_H^i, m_H^i + \epsilon_H)$, buyer i increases his payoff

Hence, both types should trade $(Q_L - Q_H + \delta, M_{L_{\Box}} - M_{H_{A}} + \epsilon_L)$

Proposition 3

In any equilibrium with $Q_H < \bar{Q}_C$, $\Pi_t = 0$, for any t.

Proof: And buyer *i* should not increase his payoff:

$$u(Q_L - Q_H + \delta_L) - \nu_L(Q_L - Q_H) - \epsilon_L \leq 0$$

• Letting δ_L, ϵ_L go to zero yields $(
u -
u_L)(Q_L - Q_H) \leq 0$

- Then, $\nu > \nu_L$ and $Q_L \ge Q_H$ implies $Q_L = Q_H$ and $\delta_L > 0$
- Then, $\nu \delta_L \leq \epsilon_L$ must hold whenever $c'_L(Q_L) \delta_L < \epsilon_L$
- Then, $c'_L(Q_L) \geq \nu$
- Due to Lemma 2, $c'_H(Q_H) = \nu$ when $Q_H < \bar{Q}_C$

This contradicts the assumption $c'_{H}(Q) > c'_{L}(Q)$

In any equilibrium with no-cross-subsidization, each traded contract yields zero profit, and type H chooses not to trade:

Proposition 4

In any equilibrium with $\Pi_H = \Pi_L = 0$, $\pi_t^i = 0$ and $q_L^i \ge q_H^i = 0$ for all t and i.

Then, In any equilibrium, type H either does not trade or his capacity constraint is binding:

Corollary 2

In any equilibrium, either $Q_H = 0$ or $Q_H = \bar{Q}_C$.

Proposition 4

In any equilibrium with $\Pi_H = \Pi_L = 0$, $\pi_t^i = 0$ and $q_L^i \ge q_H^i = 0$ for all t and i.

Proof: By definition, $\Pi_H = 0$ implies $M_H = \nu_H Q_H$

• Due to Proposition 1, $M_H = \nu Q_H$

This is only possible if $(Q_H, M_H) = (0, 0)$, and $q_H^i = m_H^i = 0$ for all *i*

Equilibrium Characterization

Equilibrium conditions on the marginal cost of both types for any equilibrium with no-cross-subsidization:

Lemma 3

In any equilibrium with $\Pi_H = \Pi_L = 0$, if $Q_L > 0$, then

$$c_{L}'(Q_{L}) \begin{cases} = \nu_{L} & \text{if } Q_{L} < \bar{Q}_{C}, \\ \leq \nu_{L} & \text{if } Q_{L} = \bar{Q}_{C}. \end{cases}$$

Moreover, if $Q_t = 0$, then $c'_t(0) \ge \min \{\nu_t, \nu\}$ for $t \in \{H, L\}$.

Proof: Assume $\Pi_H = \Pi_L = 0$, $Q_L \in (0, \overline{Q}_C)$ and $c'_L(Q_L) \neq \nu_L$

- Buyer *i* deviates to $(Q_L + \delta_L, M_L + \epsilon_L)$ where $c'_L(Q_L)\delta_L < \epsilon_L < \nu_L\delta_L$
- Type L increases his profits by following this deviation
- If type H is not attracted, then buyer i increases his payoff
 - If attracted, then buyer *i* still increases his payoff

This is a contradiction

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(3)

Equilibrium Characterization

Lemma 3

In any equilibrium with $\Pi_H = \Pi_L = 0$, if $Q_L > 0$, then

$$c_L'(Q_L) \begin{cases} = \nu_L & \text{if } Q_L < \bar{Q}_C, \\ \leq \nu_L & \text{if } Q_L = \bar{Q}_C. \end{cases}$$

Moreover, if $Q_t = 0$, then $c'_t(0) \ge \min \{\nu_t, \nu\}$ for $t \in \{H, L\}$.

Proof: When $Q_L = \bar{Q}_C$

- Buyer *i* deviates to $(Q_L + \delta_L, M_L + \epsilon_L)$ where $c'_L(Q_L)\delta_L < \epsilon_L < \nu_L\delta_L$
 - This contract is feasible only for negative δ_L and ϵ_L
- Type L increases his profits by following this deviation
- If type H is not attracted, then buyer i increases his payoff

- If attracted, then buyer *i* still increases his payoff

This contradiction only eliminates the case $c'_L(Q_L) > \nu_L$ Hence, we are left with $c'_L(Q_L) \le \nu_L$

Lemma 3

In any equilibrium with $\Pi_H = \Pi_L = 0$, if $Q_L > 0$, then

$$c_{L}'(Q_{L}) \begin{cases} = \nu_{L} & \text{if } Q_{L} < \bar{Q}_{C}, \\ \leq \nu_{L} & \text{if } Q_{L} = \bar{Q}_{C}. \end{cases}$$

Moreover, if $Q_t = 0$, then $c'_t(0) \ge \min \{\nu_t, \nu\}$ for $t \in \{H, L\}$.

Proof: Assume $Q_t = 0$ and $c'_t(0) < \min \{\nu_t, \nu\}$

- Buyer *i* deviates to (δ_t, ϵ_t) where $c_t'(0)\delta_t < \epsilon_t < \nu_t \delta_t$
 - This contract is feasible only for positive δ_t and ϵ_t
- Type t increases his profits by following this deviation
 - When only traded by t buyer i increases his payoff
- It should be traded by both types, and buyer *i* should not make profits
 - That is $\epsilon_t \geq \nu \delta_t$ for $t \in \{H, L\}$

Equilibrium Characterization

Lemma 3

In any equilibrium with $\Pi_H = \Pi_L = 0$, if $Q_L > 0$, then

$$c_{L}'(Q_{L}) \begin{cases} = \nu_{L} & \text{if } Q_{L} < \bar{Q}_{C}, \\ \leq \nu_{L} & \text{if } Q_{L} = \bar{Q}_{C}. \end{cases}$$

Moreover, if $Q_t = 0$, then $c'_t(0) \ge \min \{\nu_t, \nu\}$ for $t \in \{H, L\}$.

Proof: Then any positive δ_t and ϵ_t with $c'_t(0)\delta_t < \epsilon_t < \nu_t\delta_t$ should satisfy $\epsilon_t \ge \nu \delta_t$ for $t \in \{H, L\}$

- For type H, $c_t'(0) < \min \{\nu_t, \nu\}$ implies $c_H'(0) < \nu$
 - \exists positive δ_H , ϵ_H satisfying $c'_H(0)\delta_H < \epsilon_H < \nu_H\delta_H$ with $\epsilon_H < \nu\delta_H$
 - a contradiction!

• For type L, $c_t'(0) < \min \{\nu_t, \nu\}$ implies $c_L'(0) < \nu_L$

- Then $c'_L(0)\delta_L < \epsilon_L < \nu_L\delta_L$ and $\epsilon_L \ge \nu\delta_L$ imply $\nu\delta_L \le \epsilon_L < \nu_L\delta_L$
- which is impossible!

Characterization of aggregate equilibrium trades as well as necessary conditions for the equilibrium existence:

Theorem 1

If an equilibrium exists, then $\nu \leq c'_H(0)$ or $c'_H(\bar{Q}_C) \leq \nu$. Moreover, the following statements hold.

- (i) If $c'_H(\bar{Q}_C) \leq \nu$, all equilibria are pooling with $Q_L = Q_H = \bar{Q}_C$.
- (ii) If $\nu_L \leq c'_L(0)$ and $\nu \leq c'_H(0)$, all equilibria are pooling with $Q_L = Q_H = 0$.
- (iii) If $c_L'(0) < \nu_L$ and $\nu \leq c_H'(0)$, all equilibria are separating with:

$$Q_L = \begin{cases} Q_L^* & \text{if } c_L'(\bar{Q}_C) > \nu_L, \\ \bar{Q}_C & \text{if } c_L'(\bar{Q}_C) \le \nu_L, \end{cases} \text{ and } Q_H = 0,$$

where Q_L^* satisfies $c'_L(Q_L^*) = \nu_L$.

Theorem 1

(i) If $c'_H(\bar{Q}_C) \leq \nu$, all equilibria are pooling with $Q_L = Q_H = \bar{Q}_C$. (ii) ... (iii) ...

Proof: Assume that an equilibrium exists and $Q_H = \bar{Q}_C$

- In this case, we must have $Q_L = Q_H = ar{Q}_C$ due to $ar{Q}_C \ge Q_L \ge Q_H$
 - Then, $\Pi_H > 0$ follows from Proposition 1
 - Then, $c'_H(\bar{Q}_C) \leq \nu$ follows from Lemma 2

Theorem 1

(*i*) ...

(ii) If $\nu_L \leq c'_L(0)$ and $\nu \leq c'_H(0)$, all equilibria are pooling with $Q_L = Q_H = 0$. (iii) ...

Proof: Assume that an equilibrium exists and $Q_H < \bar{Q}_C$. Then, there is no cross-subsidization due to Proposition 2, and $Q_L \ge Q_H = 0$ by Proposition 3

- If it is a pooling equilibrium, then it must be $Q_L = Q_H = 0$
 - Then, $c_t'(0) \geq \min \{
 u_t,
 u \}$ for $t \in \{H, L\}$ follows from Lemma 3

Theorem 1

(i) ...

(*ii*) ...

(iii) If $c_L'(0) < \nu_L$ and $\nu \leq c_H'(0)$, all equilibria are separating with:

$$Q_L = \begin{cases} Q_L^* & \text{if } c_L'(\bar{Q}_C) > \nu_L, \\ \bar{Q}_C & \text{if } c_L'(\bar{Q}_C) \le \nu_L, \end{cases} \text{ and } Q_H = 0, \end{cases}$$

where Q_L^* satisfies $c'_L(Q_L^*) = \nu_L$..

Proof: If it is a separating equilibrium, then it must be $Q_L > Q_H = 0$ • If $Q_L^* < \bar{Q}_C$, then by strict convexity of c_L , $c'_L(\bar{Q}_C) > \nu_L$ - In this case, $Q_L = Q_L^*$ follows from Lemma 3 • If $Q_L^* \ge \bar{Q}_C$, then by strict convexity of c_L , $c'_L(\bar{Q}_C) \le \nu_L$ - By Lemma 3, $Q_L = \bar{Q}_C$ follows In either case, $c'_L(0) < \nu_L$ follows from strict convexity of c_L And, $c'_H(0) \ge \nu$ follows from Lemma 3 H. Bayrak, N. A. Dalkuran Nonexclusive competition for a freelancer

Theorem 1

If an equilibrium exists, then $\nu \leq c'_{H}(0)$ or $c'_{H}(\bar{Q}_{C}) \leq \nu$. Moreover, the following statements hold.

- (i) If $c'_H(\bar{Q}_C) \leq \nu$, all equilibria are pooling with $Q_L = Q_H = \bar{Q}_C$.
- (ii) If $\nu_L \leq c'_L(0)$ and $\nu \leq c'_H(0)$, all equilibria are pooling with $Q_L = Q_H = 0$.
- (iii) If $c_L'(0) < \nu_L$ and $\nu \leq c_H'(0)$, all equilibria are separating with:

$$Q_L = egin{cases} Q_L^* & ext{if } c_L'(ar Q_C) >
u_L, \ ar Q_C & ext{if } c_L'(ar Q_C) \le
u_L, \end{cases}$$
 and $Q_H = 0,$

where Q_L^* satisfies $c'_L(Q_L^*) = \nu_L$.

Proof: Since both the hypotheses and the conclusions of (*i*), (*ii*), and (*iii*) are mutually exclusive and collectively exhaustive of all aggregate equilibrium trades, the proof is complete.

Any aggregate equilibrium trade can be supported by at least two buyers posting the same linear tariffs (described below). The necessary conditions for equilibrium existence given in Theorem 1 are also sufficient:

Theorem 2

An equilibrium exists if and only if $\nu \leq c'_H(0)$ or $c'_H(\bar{Q}_C) \leq \nu$. Moreover, the following statements hold.

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) = \nu_L q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

(ii) If $c'_H(\bar{Q}_C) \le \nu$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) = \nu q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

Theorem 2

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u_L q, \quad 0 \leq q \leq ar{Q}_C,$$

while the other buyers remain inactive.

(*ii*) ...

Proof: Assume $\nu \leq c'_{H}(0)$ holds

- Let $n \ge k \ge 2$ buyers post the tariff given in (i)
- In aggregate, competitors of any buyer post $M^-(Q^-) =
 u_L Q^-$
 - Where, $Q^- \in [0, ar Q]$ and $ar Q \geq ar Q_C$

• Suppose a buyer deviates and ends up trading (q_L, m_L) and (q_H, m_H) At least one of these contracts should give positive profits if the deviating buyer has a profitable deviation

Theorem 2

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u_L q, \quad 0 \leq q \leq ar{Q}_C,$$

while the other buyers remain inactive.

(*ii*) ...

Proof: If (q_H, m_H) gives profits: $\nu_H q_H > m_H$

- Define $Q_t^- \in [0, ar{Q}_C]$ as the trades with the deviator's competitors
- Define the total quantity traded by $t \in \{H,L\}$ as $\hat{Q}_t = q_t + Q_t^-$

- Similarly
$$\hat{M}_t = m_t + M^-(Q_t^-)$$

• Since type H prefers trading (\hat{Q}_H, \hat{M}_H) : $u_H(\hat{Q}_H, \hat{M}_H) \ge u_H(0, 0)$

Together with $\nu \leq c'_{H}(0)$, the above inequality implies $\hat{M}_{H} > \nu \hat{Q}_{H}$

Theorem 2

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u_L q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

(*ii*) ...

Proof: Maximize $u_L(q_H + Q^-, m_H + M^-(Q^-))$ over feasible Q^-

- This optimization problem is subject to $0 \le Q^- \le ar{Q}_C q_H$
 - A feasible solution is $Q^- = \hat{Q}_L q_H$
- Thus, type L can receive at least $u_L(\hat{Q}_L, m_H + M^-(\hat{Q}_L q_H))$
 - Rewrite the aggregate transfers as $\hat{M}_H +
 u_L (\hat{Q}_L \hat{Q}_H)$
- Since type L prefers trading (\hat{Q}_L, \hat{M}_L) , $\hat{M}_L \geq \hat{M}_H + \nu_L (\hat{Q}_L \hat{Q}_H)$

Theorem 2

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u_L q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

(*ii*) ...

Proof: Recall $\hat{M}_H > \nu \hat{Q}_H$: The aggregate profit can be at most zero:

$$u \hat{Q}_{H} - \hat{M}_{H} + p_{L}[
u_{L}(\hat{Q}_{L} - \hat{Q}_{H}) - (\hat{M}_{L} - \hat{M}_{H})] \leq 0$$

By the definition of the tariff given in (i), the competitors of the deviator cannot make losses. Hence, the deviator does not have a profitable deviation.

Theorem 2

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u_L q, \quad 0 \leq q \leq \overline{Q}_C,$$

while the other buyers remain inactive.

(*ii*) ...

Proof: If (q_L, m_L) gives profits: $\nu_L q_L > m_L$, and $q_L > 0$

- By the definition of the tariff given in (*i*), competitors of the deviator cannot make losses
 - Hence, $u_L q_L > m_L$ implies $u_L \hat{Q}_L > \hat{M}_L$
- Then $u_L(\hat{Q}_L, \nu_L \hat{Q}_L) > u_L(\hat{Q}_L, \hat{M}_L)$

Since type *L* can trade $(\hat{Q}_L, \nu_L \hat{Q}_L)$ with the competitors of the deviator, we arrive at a contradiction

Theorem 2

(*i*) ...

(ii) If $c'_H(\bar{Q}_C) \le \nu$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u q, \quad 0 \leq q \leq \overline{Q}_C,$$

while the other buyers remain inactive.

Proof: Keeping the same notation, assume $c'_{\mathcal{H}}(\bar{Q}_{\mathcal{C}}) \leq \nu$

- Let $n \ge k \ge 2$ buyers post the tariff given in (ii)
- In aggregate, competitors of any buyer post $M^-(Q^-) =
 u Q^-$
 - Where, $Q^- \in [0, ar{Q}]$ and $ar{Q} \geq ar{Q}_C$
- Suppose a buyer deviates and ends up trading (q_L, m_L) and (q_H, m_H)
- Then, type H prefers (\hat{Q}_H, \hat{M}_H) to $(q_H + Q^-, m_H + M^-(Q^-))$

For
$$Q^-=ar{Q}_{\mathcal{C}}-q_{\mathcal{H}}$$
, this gives $\int_{\hat{Q}_{\mathcal{H}}}^{ar{Q}_{\mathcal{C}}}c'_{\mathcal{H}}(x)dx\geq
u(ar{Q}_{\mathcal{C}}-\hat{Q}_{\mathcal{H}})$

Theorem 2

(*i*) ...

(ii) If $c'_H(\bar{Q}_C) \le \nu$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) =
u q, \quad 0 \leq q \leq \overline{Q}_C,$$

while the other buyers remain inactive.

Proof: Recall that c_H is strictly convex and $c'_H(\bar{Q}_C) \leq \nu$

• $\int_{\hat{Q}_H}^{\bar{Q}_C} c'_H(x) dx \ge
u(ar{Q}_C - \hat{Q}_H)$ can be satisfied only if $\hat{Q}_H = ar{Q}_C$

• Then,
$$\hat{Q}_L \geq \hat{Q}_H$$
 implies $\hat{Q}_L \geq \hat{Q}_H = ar{Q}_C$

- Both types are indifferent between $(q_H,m_H)+(Q_H^-,M_H^-)$ and $(q_L,m_L)+(Q_L^-,M_L^-)$

- Hence, expected payoff of the deviator is $u q_t m_t$
- Since the freelancer prefers (q_t, m_t) , it must be that $m_t \ge \nu q_t$ Hence, the deviator does not have a profitable deviation.

Theorem 2

An equilibrium exists if and only if $\nu \leq c'_{H}(0)$ or $c'_{H}(\bar{Q}_{C}) \leq \nu$. Moreover, the following statements hold.

(i) If $\nu \leq c'_{H}(0)$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) = \nu_L q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

(ii) If $c'_H(\bar{Q}_C) \leq \nu$, any equilibrium can be supported by at least two buyers posting the same tariff

$$m(q) = \nu q, \quad 0 \leq q \leq \bar{Q}_C,$$

while the other buyers remain inactive.

Proof: Since no deviator has a profitable deviation, necessary conditions of Theorem 1 are also sufficient

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Concluding Remarks

Our main results fully characterize the aggregate equilibrium trades of a freelancer under nonexclusive competition

- If an equilibrium exists, each buyer makes zero profit in expectation
 - the aggregate equilibrium trades are unique
 - any equilibrium can be supported by linear tariffs

We provide necessary and sufficient conditions for the equilibrium existence

- If the high-type freelancer is not willing to serve at a price equal to the expected quality, then we obtain an Akerlof-like result
 - The high-type freelancer does not trade in equilibrium
 - The low-type freelancer might trade efficiently, not trade at all, or exhaust her capacity
 - There is no cross-subsidization
- if the high-type freelancer is willing to serve for the price equal to the expected quality at every feasible level,
 - Both types exhaust their capacity
 - There is cross-subsidization

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