The Fisher Equation and Output Growth

ABSTRACT

Although the Fisher equation applies for the case of no output growth, I show that it requires an adjustment to account for non-zero output growth. I demonstrate this using a standard model with constant relative risk aversion utility containing money, nominal bonds and output growth in a risk-free setting.

Failing to incorporate a growth term in regressions will lead to biased estimation results.

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I. Introduction

The Fisher equation is a well-accepted theoretical construct in macroeconomics and finance. It can be expressed as:

$$i = E(\pi) + r \tag{1}$$

where i is the nominal interest rate, $E(\pi)$ is the expected inflation rate and r is the real interest rate. In this paper I show that when there is growth in the level of consumption over time the Fisher equation becomes:

$$i = E(\pi) + \rho + \alpha \cdot E(\dot{y}) \tag{2}$$

where ρ represents the consumer's real rate of time preference, $E(\dot{y})$ represents the expected rate of output (or consumption) growth, and α is the degree of relative risk aversion. The logic is simple: output growth reduces the marginal utility of real income over time increasing the real interest rate:

$$r = \rho + \alpha \cdot E(\dot{y}). \tag{3}$$

In equation 2 expected output growth increases the nominal interest rate as the consumer tries to smooth consumption by saving less. Conversely, a decrease in expected output reduces the nominal interest rate. When no output growth is expected equation 2 is identical to equation 1.

This modification of the Fisher equation to account for growth does not alter Fisher's original view that real and monetary factors are separable in their effect upon the economy. Expected inflation still effects output on a one-to-one basis, *ceteris paribus*, and has no effect upon the real rate of time preference of the consumer.

Fisher (1896) introduces the Fisher equation, see Dimand (1999), and Fisher (1930) contained an extensive discussion of the relation. After more than 100, years Fisher's relation is still a work horse. In the two years 2003-2004 alone there were more than a dozen studies on the Fisher equation, mostly featured in empirical journals. Fisher's contribution here was to observe that real and nominal factors can be separated when analyzing their effects upon the economy. This is a precursor to the theory of the neutrality of money. His discussion of the equation demonstrated an understanding of rational expectations decades before the term was coined.

The Fisher equation has various interpretations. At one extreme we can view r as a fixed constant expressing the consumer's rate of time preference so that expected inflation passes through on a one-to-one basis into nominal interest rates. Even if this view is correct, since the researcher cannot observe the consumer's ex-ante inflation expectation there will still be a stochastic element to empirical studies. Alternatively, we can view the real rate as potentially time-varying. Then, while there will be a one-to-one long run relationship between inflation and nominal interest rates it may not be exact in the short run.

Mundell (1963) and Tobin (1965) provide a theoretical explanation for a time-varying real interest rate. For Mundell (1963) higher inflation reduces wealth by reducing the desired level of real money balances. This stimulates saving, which reduces the real interest rate. Tobin (1965) sees inflation as driving saving away from nominal assets to real assets, reducing the real interest rate by lowering the marginal product of capital. Evans and Lewis (1995) considers the case in which inflation follows a Markov-switching process so that ex-post observed the real interest rate features a non-zero expectation and is serially correlated. In empirical studies Mishkin (1984) and Pelaez (1995) reject the hypothesis that the real interest rate is constant for the OECD and U.S., respectively. Fama (1981) concluded that variation in the real interest rate could be explained by real, rather than by nominal, factors. None of these studies necessarily invalidates the Fisher equation as a long run relation. More recently Rapach (2003) does reject the long run Fisher equation, concluding that interest rates adjust less than one-to-one with inflation in the long run, based upon a VAR study.

One empirical observation that appears in the literature is that the inclusion of real variables in an estimation of the Fisher equation yields a significant coefficient; see Levi and Makin (1978), Evans and Makin (1979), VanderHoff (1984), and Dotsey, Lantz and Scholl 2003). This is in seeming contradiction to the Fisher equation's prediction that only inflation explains the nominal interest rate. The argument put forward by Levi and Makin (1978), Evans and Makin (1979), VanderHoff (1984), and others is that the Phillips curve introduces a short run bias. They viewed this bias as disappearing in the long run because the economy reverts to full employment level of output (Levi and Makin 1979, p. 43).

II. Model

I examine a standard cash-in-advance consumption model with constant relative risk aversion utility function and output growth. The model contains no risk. This assumption is made to simplify notation and so that it is clear that my findings do not depend upon Phillips curve explanations, Mundell-Tobin arguments or on a deviation from the quantity theory of money. There are no frictions in the model.

Production is exogenously determined so we can interpret this as a Lucas-type tree economy. The representative household can be thought of as a worker-shopper pair. The worker sells the exogenously arriving output in a competitive market and the shopper collects the couple's income, as cash balances, at the end of the period and proceeds to shop in the next period. I assume a constant output growth rate. Output evolves according to:

$$y_t = \gamma \cdot y_{t-1}. \tag{4}$$

Throughout I will assume a risk-free steady output and monetary growth so these growth rates have no time subscript.

Government in this model introduces money at a constant growth rate, sells one-period bonds on a discounted basis, and provides a lump sum transfer payment. The money supply evolves according to $\overline{M}_{t+1} = \omega \cdot \overline{M}_t$, where a bar over the variable indicates it is the supply of the variable. The government's finance constraint insures that the revenue from seigniorage plus the net revenue from bond sales and redemptions is equal to the lump sum transfer payment:

$$T_t = (\omega - 1) \cdot \overline{M}_t + (1 + i_t)^{-1} \cdot \overline{B}_{t+1} - \overline{B}_t . \tag{5}$$

I assume that the government does not indefinitely let the size of debt grow faster than output.

The representative consumer has a constant relative risk aversion utility function:

$$\left.\begin{aligned} u(c) &= \frac{1}{1-\alpha} \cdot c^{1-\alpha}, \quad && \alpha \neq 1 \\ u(c) &= \ln c, && \alpha = 1 \end{aligned}\right\} \tag{6}$$

The consumer's budget constraint is:

$$P_t \cdot c_t + M_{t+1} + (1 + i_t)^{-1} \cdot B_{t+1} \leq M_t + B_t + P_t \cdot y_t + T_t \tag{7}$$

where the right hand side represents the consumer's wealth. This consists of money carried into this period from the previous period, the maturing one-period nominal bond, output sold in a competitive market, and the lump sum government transfer payment. The left hand side of equation 6 shows how the consumer allocates his wealth. There is a single consumption good. Saving takes the form of nominal money balances and discounted government bonds.

Money demand is motivated by a cash-in-advance constraint placed upon the consumption good. This creates a demand for money, even though money is dominated in return by bonds and equities.

$$P_t \cdot c_t \leq M_t \tag{8}$$

Money is selected a period in advance of its use.

We are only interested in variables in two different time periods, time *t* and *t+1*. Therefore I drop the time subscript and use a prime to denote variables evaluated at time *t*+1. It is necessary to re-pose the problem in a way that is time stationary. I divide equations 6 and 7 by next period's money stock, $\overline{M}'$, to define these new variables:

$$m' = \frac{M'}{\overline{M}'}, \quad p = \frac{P}{\overline{M}'}, \quad b' = \frac{B'}{\overline{M}'}, \quad tr = \frac{T}{\overline{M}'} . \tag{9}$$

Here *m* is the representative consumer's share of money supply. The other variables, *p, b,* and *tr*, are the real goods price, bond value, and transfer payment. Basically, the share of the money stock held by the consumer is selected as a numerairé.

After making the change of variables, we get stationary representations of the consumer's budget constraints.

$$p \cdot c + m' + (1+i)^{-1} \cdot b' = \frac{m+b}{\omega} + p \cdot y + tr \tag{10}$$

$$p \cdot c = \frac{m}{\omega} \tag{11}$$

Each constraint is satisfied as equality since consumption is valued and there is an opportunity cost to holding money.

Let $S = \{y, b, \gamma, \omega'\}$ define the current state of the world. The level of money supply $\overline{M}$ has no effect upon the consumer's decision or on the equilibrium, other than determining the absolute value of *P*.

In equilibrium the representative consumer's share of money stock is one, $m' = m = 1$, and all output is consumed $c = y$.

The consumer's problem can be stated as a value function problem:

$$v(S) = \max_{c, m', b'} \{u(c) + \beta \cdot v(S')\} \tag{12}$$

subject to equations 10 and 11.

Finally, I assume that output does not grow too fast, $\beta \cdot \gamma^{1-\alpha} < 1$. The consumer's first order conditions are as follows:

$$v_c(S) = c^{-\alpha} - p \cdot (\lambda + \mu) = 0 \tag{13}$$

$$v_{m'}(S) = -\lambda + \beta \cdot E[v_{m'}(S')] = 0 \qquad v_m(S) = \frac{1}{\omega} \cdot (\lambda + \mu) \tag{14}$$

$$v_{b'}(S) = -(1+i)^{-1} \cdot \lambda + \beta \cdot E[v_{b'}(S')] = 0 \qquad v_b(S) = \frac{\lambda}{\omega} \tag{15}$$

The second term in equations 14 and 15, v_m and v_b, makes use of the envelope theorem. Combining equations 13-15 together with the constraints, equations 10 and 11, and the two market clearing conditions yields a stationary (in growth rates) equilibrium. In evaluating the envelope condition in equations 14 and 15 it is not necessary to take expectations because there is no uncertainty, however it is important to observe the time superscript of the variables. A priori we know that $\omega' = \omega$ and $\gamma' = \gamma$.

The price level is found from the cash-in-advance constraint, equation 11:

$$p = \frac{1}{\omega \cdot y} \tag{16}$$

Combining equations 13 and 16 I obtain:

$$\lambda + \mu = \omega \cdot y^{1-\alpha} \tag{17}$$

Equation 14 together with equation 17 yields the Lagrange multiplier on wealth:

$$\begin{aligned} \lambda &= \frac{\beta}{\omega'} \cdot (\lambda' + \mu') \\ &= \beta \cdot (\gamma \cdot y)^{1-\alpha} \end{aligned} \tag{18}$$

Finally combining equations 15 and 18 yields the modified Fisher equation:

$$\begin{aligned} i &= \frac{\omega'}{\beta} \cdot \frac{\lambda}{\lambda'} - 1 \\ &= \frac{\omega}{\beta \cdot \gamma^{1-\alpha}} - 1 \end{aligned} \tag{19}$$

Using equation 9 with the price level in equation 16 yields the inflation rate:

$$\begin{aligned}\frac{P'}{P} &= \frac{\omega' \cdot p'}{p} \\ &= \frac{\omega}{\gamma}\end{aligned} \tag{20}$$

The modified Fisher equation presented in equation 19 is perhaps more recognizable if I make the following change of variables:

$$\tilde{i} = \ln(1+i), \quad \pi = \ln\omega - \ln\gamma, \quad \rho = -\ln(\beta), \quad \dot{y} = \ln y_t - \ln y_{t-1} \tag{21}$$

Then the equation becomes:

$$\tilde{i} = \pi + \rho + \alpha \cdot \dot{y}\,. \tag{22}$$

There is a one-to-one relationship between inflation and the nominal interest rate *if we control for output growth*.

Finally, I would like to make a couple of observations concerning the solutions presented in equations 16-20. Firstly, I examine the derivative of the value function with respect to output:

$$v_y(S) = p \cdot \lambda = \beta \cdot \frac{\gamma}{\omega}^{-1} \cdot (\gamma \cdot y)^{-\alpha}\,. \tag{23}$$

This is clearing declining over time if output is growing. This is the reason for an output growth term belonging in the Fisher equation. Next, observe that $p \cdot y$ is constant, confirming that the transformed budget constraint is indeed time stationary.

III. Conclusion

The Fisher equation is only correct when there is no output growth. In practice this condition is not likely to hold. I have derived a modification of the Fisher equation that allows us to directly estimate the consumer's degree of relative risk aversion.

This observation provides theoretical support to the practice of including real variables in the estimation of the Fisher equation. For example Dotsey, Lantz and Scholl's (2003) finding that the ex-ante real interest rate is positively correlated with output, and high rates are correlated with high output in the next quarter is completely consistent with the modified Fisher equation.

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