Existence of the core in a heterogeneous divisible commodity exchange economy

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Abstract

We consider exchange of a heterogeneous divisible commodity modelled as a measurable space. A weak core is shown to exist under rational continuous and convex preferences over characteristic measures of measurable pieces. In addition, if characteristic measures are mutually absolutely continuous a core is shown to exist. Applied to the land trading economy core existence results in Berliant [2] and Dunz [6] are obtained. It is worth noting that in the abstract setting of this paper the existence of a reference measure is not assumed.

Key words: Measurable space exchange economy, core, NTU game.
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1. Introduction

We consider the problem of exchange of a heterogeneous divisible commodity. One notable example of such a commodity is land. This problem is coined in literature as the 'cake division' or 'land trading' problem. A heterogeneous divisible commodity is modelled as a measurable space (X, Σ). In theoretical models of land economics X is assumed to be a Borel measurable subset of Euclidean space R^2 (or more generally R^k), and Σ to be the Borel $\sigma-$algebra $\mathcal{B}(X)$ of subsets of X. It is usual to consider this measurable space with the Lebesgue measure.

Berliant [2] is the first to study the core in the context of land trading economy. He shows the existence of a competitive equilibrium in the case when preferences over land plots are represented by measures absolutely continuous with respect to the Lebesgue measure. He derives this result from the existence of a competitive equilibrium in land trading economy, which is the main result of his paper [2].

Dunz [6] studies the existence of the core also in the context of land trading economy, but for substantially more general preferences. In [6] preferences are given by the utility functions that are compositions of quasi-concave functions with a finite number of characteristics of land parcels. These chracteristics are countably-additive, that is measures, over land parcels. Dunz proves that under these assumptions on preferences and the assumption of absolute continuity of chracteristic measures with respect to the Lebesgue measure the weak core of a land trading economy is nonempty. Assigning a finite number of additive characteristics to land parcels is a common assumption made in empirical literature on land trading. Dunz's proof of the existence of the weak core is direct, and uses Scarf's theorem on the existence of the weak core in nontransferable utility games and a limiting procedure.

Weller [10] consideres a problem of fair division of a measurable space (X, Σ) with a finite number of atomless measures describing agents' preferences over measurable subsets. He shows the existence of an envy-free and efficient partition in this problem.

Here we prove the existence of the core in a general context of a measurable space trading economy. Applied to the land trading economy it gives Dunz's core existence result. It is worth to note that in difference from the above cited results on the core we don't assume the existence of a reference measure with respect to which characteristic measures are absolutely continuous.

We consider a measurable space division problem set in the following way. Let (X, Σ) be a measurable space (the 'cake or 'land plot) and let $\bar{P} = \{A_1, A_2, \dots, A_n\}$ be a measurable ordered partition of X. Let $\mu_1, \mu_2, \dots, \mu_n$ be nonatomic finite vector measures on (X, Σ) of dimension $s_1, s_2, \dots, s_n$, respectively. The interpretation is that there are n persons $N = \{1, 2, \dots, n\}$ each contributing his share A_i $(i \in N)$ and parts of the cake, X, are valued by individuals according to their measures $\mu_1, \mu_2, \ldots, \mu_n$, respectively. The components of vector-measure $\mu_i(B)$ are interpreted as measures of different attributes of a measurable piece B attached to this piece by individual i. We assume that individual i has a preference $\succ_i$ over his subjective attributes profiles $\mu_i(B)$, $B \in \Sigma$ and hence over measurable sets $B \in \Sigma$. We will use the same symbol $\succ_i$ for denoting both of these preferences. No confusion should arise. Every ordered measurable partition $\{B_1, B_2, \ldots, B_n\}$ will be interpreted as a feasible allocation of X.

A *coalition* is an arbitrary nonempty subset of N. The set of all coalitions is denoted as $\mathcal{N}$. All partitions considered further are assumed to be ordered and measurable. Further the terms partition and division will be used interchangably.

Definition 1. We say a coalition $I \subset N$ *improves (weakly improves) upon* a division $P = \{B_1, B_2, \ldots, B_n\}$ if there exists a partition $Q = \{C_i \mid i \in I\}$ of $A(I) = \cup_{i \in I} A_i$ such that $C_i \succ_i B_i$ for all $i \in I$ (not $B_i \succ_i C_i$ for all $i \in I$ and $C_i \succ_i B_i$ for at least one $i \in I$.)

Definition 2. Partition $P = \{B_1, B_2, \ldots, B_n\}$ is said to be *a weak core division (core division)* if there is no coalition that improves (weakly improves) upon division P. The set of all weak core divisions is called the *weak core (core)* of the cake division problem.

We will identify a vector of vectors (perhaps of different dimensions) as a long vector with scalar coordinates arranged in the lexicographic order. Sometimes we will denote coordinates with double indexes, with the first index being the index of the component vector and the second one the index of component in that component vector .

The following theorem is a generalization of a result known as Dubins-Spanier's theorem (see also C. Aliprantis and K. Border [1] page 358) and easily follows from this result. It is to be noted that this theorem was discovered a decade earlier Dubins-Spanier's theorem by Chernoff [4].

Theorem 1. *Let (X, Σ) be a measurable space and let $\mu_1, \mu_2, \ldots, \mu_n$ be nonatomic finite vector measures on (X, Σ) of dimensions $s_1, s_2, \ldots, s_n$, respectively. Then, the following set in R^s, where $s = \sum_{j=1}^{n} s_j$,*

$$\mathcal{R} = \{(\mu_i(B_i))_{i=1}^{n} \in R^s \mid P = (B_1, B_2, \ldots, B_n) \text{ a partition of } X\}$$

is compact and convex.

Proof of Theorem 1 based on Dubins-Spanier's theorem. Let $\mu = (\mu_k)_{k=1}^{s}$ be a vector measure $(\mu_1, \mu_2, \ldots, \mu_n)$ of dimension s. With every partition $P = (B_1, B_2, \ldots, B_n) \in \Pi^n(X)$ we associate the $s \times n$ matrix of reals $M(P) = (\mu_k(B_i))$. Denote by $M^{s \times n}$ the space of all $s \times n$ matrices with real entries. By the Theorem 1 in

Dubins and Spanier [5] the range $\mathcal{R}' \subset M^{s\times n}$ of matrix-valued function M is compact and convex.

Let $L : M^{s\times n} \to R^s$ be a mapping defined in the following way. The first s_1 components of $L(M)$ are the first s_1 entries in the first column of matrix M, the second s_2 components are the entries in the second column of M with the column indexes $s_1 + 1$ through $s_1 + s_2$, and so on. Clearly L is a linear mapping with $L(\mathcal{R}') = \mathcal{R}$. Since $\mathcal{R}'$ is compact and convex it follows that so is $\mathcal{R}$. $\Box$

Corollary 2. ('Generalization of Dubins-Spanier's Theorem). Let (X, Σ) be a measurable space and let $\mu_1, \mu_2, \dots, \mu_n$ be nonatomic finite vector measures on (X, Σ) of the same dimension l. Then the following set

$$M(P) = \{(\mu_1(B_1), \mu_2(B_2), \dots, \mu_n(B_n)) \mid P = (B_1, B_2, \dots, B_n) \in \Pi^n(X)\}$$

in $M^{l\times n}$ is compact and convex.

In fact Theorem 1 and Corollary 2 can be considered as versions of Dubins-Spanier's Theorem [5], that turns out to be convenient in diferrent contexts.

2. The Main Result

We will assume that preferences $\succ_i$ $(i \in N)$ over characteristic measures are rational, continuous and convex, so that they can be represented by continuous quasiconcave utility functions u_i $(i \in N)$ defined on characteristics space $R_+^{s_i}$ $(i \in N)$. We will represent preferences over Σ by introducing utility functions $U_i(B) = u_i(\mu_i(B))$, $(i \in N)$. Thus, in addition to the assumptions of the previous section here we assume rationality (that is, completeness and transitivity) of preferences.

For a (weak) core division $P = \{B_1, B_2, \dots, B_n\}$ the utility profile $(U_1(B_1), U_2(B_2), \dots, U_n(B_n))$ will be called a (weak) core utility profile.

Theorem 3. *If measures μ_i $(i \in N)$ are nonatomic and preferences $\succ_i$ $(i \in N)$ are continuous, rational and convex, then the weak core of the measurable space trading economy is nonempty.*

Proof: The proof will use Scarf's theorem [8] on the existence of the weak core in nontransferable utility games.

For each coalition S define a subset $V(S)$ of R^n in the following way. Consider all partitions $P = \{C_1, C_2, \dots, C_n\}$ of $\cup_{i\in S} A_i$, where $C_j = \emptyset$ for all $j \notin S$. Set $x(P) =$

$(U_1(C_1), U_2(C_2), \ldots, U_n(C_n)) \in R^n$, $A(S) = \cup_{i \in S} A_i$ and

$$V_0(S) = \{x(P) \mid P \text{ is a partition of } A(S)\}.$$

We define $V(S)$ as the smallest comprehensive set in R^n containing $V_0(S)$ and with the property: if $x \in V(S)$ then $x' \in V(S)$, if $x'_j = x_j$ for $j \in S^c$. Otherwise

$$V(S) = V_0(S) + R^{S^c} + R^n_-,$$

where $S^c = N \setminus S$ and R^{S^c} is the corresponding coordinate subspace in R^n. By Theorem 1 the set $\{(\mu_i(C_i))_{i \in S} \mid \{C_1, C_2, \ldots, C_n\}$ is a partition of $A(S)\}$ is compact. Continuity of functions u_i $(i \in N)$ imply that set $V_0(S)$ is compact. Compactness of $V_0(S)$ in turn, implies the closedness of $V(S)$. So sets $V(S)$ $(S \in \mathcal{N}$ are closed.

Thus, we have defined a nontransferable utility (NTU) game (N, V). The weak core, $WC(N, V)$, of this game is defined as the set of all vectors from $V(N) \setminus \cup_{S \in \mathcal{N}} \text{int}\, V(S)$. We will show that nonemptiness of the weak core of this game implies the nonemptiness of the weak core of the measurable space exchange economy. Let $x \in WC(N, V)$. Then by the definition of set $V(N)$ there exists $\bar{x} \in V_0(N)$ with $\bar{x} \geqq x$. Since $\bar{x} \in V_0(N)$, there exists a partition P of X such that $\bar{x} = (U_i(B_i))_{i=1}^n$. We assert P belongs to the weak core of the measurable space exchange economy. Indeed, if a coalition S improves upon partition P, then there exists a partition of $A(S)$ such that $U_i(C_i) > U_i(B_i) = \bar{x}_i$, for $i \in S$. By the definition of $V(S)$ then, $\text{int}\, V(S) \ni \bar{x}$, and hence, since $V(S)$ is comprehensive $\text{int}\, V(S) \ni x$. This contradicts to inclusion $x \in WC(N, V)$. Thus if to show that the weak core of the NTU game (N, V) is nonempty we are done.

According to Scarf's theorem if for an arbitrary balanced collection of coalitions $\mathcal{B}$ the inclusion

$$\cap_{S \in \mathcal{B}} V(S) \subset V(N) \tag{1}$$

holds, then the weak core of a NTU game is nonempty. Thus, if we show that inclusion (1) holds, then the theorem will be proved.

Let $\mathcal{B}$ be a balanced collection of coalitions with the balancing weights $\delta_S \geq 0$, $S \in \mathcal{B}$. Let $x \in \cap_{S \in \mathcal{B}} V_S$. Denote $\mathcal{B}_i$ the subcollection of collection $\mathcal{B}$ consisting of coalitions containing player i. For every coalition S the set $A(S) = \cup_{i \in S} A_i$ has a partition $P_S = \{A^1(S), A^2(S), \ldots, A^n(S)\}$ such that

$$U_j(A^j(S)) \geq x_j, \ \forall j \in S \quad \text{and} \quad A^j(S) = \emptyset \ \text{ for all } \ j \notin S. \tag{2}$$

Denote

$$A_i^j(S) = A_i \cap A^j(S) \quad \text{for} \ \ i, j \in N. \tag{3}$$

Clearly, for each pair $S \in \mathcal{B}$ and $i \in N$ such that $S \ni i$, $\{A_i^1(S), A_i^2(S), \ldots, A_i^n(S)\}$ is a partition of A_i. Fix $i \in N$. Since $\sum_{S \ni i} \delta_S = 1$ by Theorem 1 there exists a partition $\{A_i^1, A_i^2, \ldots, A_i^n\}$ of A_i so that

$$\mu_j(A_i^j) = \sum_{S \ni j} \delta_S \mu_j(A_i^j(S)) \text{ for all } j \in N. \tag{4}$$

Let $B_j = \cup_{i \in N} A_i^j,\ j \in N$. Then $P = \{B_1, B_2, \ldots, B_n\}$ is a partition of X and

$$\mu_j(B_j) = \sum_{i \in N} \mu_j(A_i^j) = \sum_{i \in N} \sum_{S \ni i} \delta_S \mu_j(A_i^j(S)) = \sum_{S \ni j} \delta_S [\sum_{i \in N} \mu_j(A_i^j(S))] = \sum_{S \ni j} \delta_S \mu_j(A^j(S)). \tag{5}$$

Above the second equality follows from (4) and the fourth equality from the fact that $\{A_1^j(S), A_2^j(S), \ldots, A_n^j(S)\}$ is a partition of $A^j(S)$.

Since utility functions u_i are assumed to be quasiconcave and $\sum \delta_S = 1$, using inequalities (2) we obtain from (5)

$$u_j(\mu_j(B_j)) = u_j(\sum_{S \ni j} \delta_S \mu_j(A^j(S))) \geq \min_{S \ni j} u_j(A^j(S)) \geq x_j.$$

So $U_j(B_j) \geq x_j$ for all $j \in N$. Hence $x \in V(N)$. $\square$

3. Examples and existence of the core

As it is noted in the Introduction, Theorem 3 implies the main result of Dunz [6]. Although, this result claims the existence of a core (rather than a weak core) we bring an example demonstrating that under the assumptions made in [6] the core may be empty. Moreover, in [6] continuity of utility functions is not explicitly assumed. We also bring an example to illustrate that without the continuity of utility functions even the weak core may be empty.

Example 1. Let $X = [0,4] \times [0,1],\ A_1 = [0,2] \times [0,1],\ A_2 = (2,3] \times [0,1],\ A_3 = (3,4] \times [0,1]$. Let λ denote the 2-dimensional Lebesgue measure. Let utilities be given by measures $\mu_1,\ \mu_2,\ \mu_3$ defined as $\mu_1 = \lambda$ on $[0,1] \times [0,1]$, and 0 on $(1,4] \times [0,1]$, and $\mu_1 = \mu_2 = \lambda$ on X. Then

$$\begin{array}{lcll} V(1) & = & (1,-,-), & \text{i.e. } V(1) = \{x \in R^3 \mid x_1 \leq 1\}, \\ V(2) & = & (-,1,-), & \\ V(3) & = & (-,-,1), & \\ V(12) & = & (1,2,-), & \text{i.e. } V(12) = \{x \in R^3 \mid x_1 \leq 1,\ x_2 \leq 2\}, \\ V(13) & = & (1,-,2), & \\ V(23) & = & (-,1,1), & \end{array}$$

$$V(123) = F + R^3_-, \text{ where } F = \{x \in R^3 \mid x_1 = 1,\ x_2 \geq 1,\ x_3 \geq 1,\ x_2 + x_3 \leq 3\}.$$

The core of this game is empty. The weak core utility profiles set is equal to F.

Example 2. Consider $X = [0,2] \times [0,2] \subset R^2$ with the Lebesgue measure λ. Let there be two agents with endowments $A_1 = [0,1] \times [0,2]$ and $A_2 = (1,2] \times [0,2]$ and preferensces are defined in the following way. Let both agents have the common characteristics μ_1, μ_2 defined as $\mu_i(B) = \int_B h_i(x) d\lambda(x)$ $(i = 1,2)$ for measurable $B \subset X$, where h_i $(i \in N)$ is the characteristic function of A_i $(i \in N)$. Let $u_1, u_2 : R^2_+ \to R$ be defined as

$$u_1(x_1, x_2) = \begin{cases} x_1 & \text{for } 0 \leq x_1, x_2 \leq 1, \\ 2 - x_1 & \text{for } 1 < x_1 \leq 2, \ 0 \leq x_2 \leq 1, \\ 0 & \text{elswhere.} \end{cases}$$

and $u_2(x_1, x_2) = u_1(x_2, x_1)$.

It is easily verified that functions u_1, u_2 are quasiconcave. It is easy to calculate $V(1) = \{(U_1, U_2) \mid U_1 \leq 0\}$, $V(2) = \{(U_1, U_2) \mid U_2 \leq 0\}$ and $V(12) = \{U_1, U_2) \mid U_1 < 0, \ U_2 < 0\}$. Thus $V(12)$ is open and therefore the weak core and therefore the core is empty.

For coincidence of the weak core and the core some further assumptions are required.

Proposition 4. *If preferences* $\succ_i$ *are the strict parts of rational continuous preferences* $\succcurlyeq_i$*, monotone (for* $x_i, x'_i \in R^{s_i}_+$*,* $x'_i \geq x_i$ *implies* $x'_i \succ_i x_i$*,) and if measures* $\nu_i = \sum_{j=1}^{s_i} \mu_i^j$ $(i \in N)$ *are absoluely continuous with respect to each other, then the weak core and the core coincide.*

Proof: Let a coalition I weakly improve upon a division $P = \{B_1, B_2, \ldots, B_n\}$ via division $Q = \{C_i \mid i \in I\}$ of $A(I) = \cup_{i \in I} A_i$. So we have, not $B_i \succ_i C_i$ for all $i \in I$ and $C_i \succ_i B_i$ for at least one $i \in I$. Since preferences are assumed to be the strict parts of rational preferences [not $B_i \succ_i C_i$] is equivalent to [$C_i \succcurlyeq_i B_i$]. Let $C_{i_0} \succ_{i_0} B_{i_0}$ for $i_0 \in I$. Then by the weak monotony assumption we have $\mu_{i_0}(C_{i_0}) \geq 0$. By the mutual absolutely continuity assumption $\mu_i(C_{i_0}) \geq 0$ for all $i \in N$.

By continuity of $\succ_{i_0}$ there exists $d > 0$ such that for $D \subset C_{i_0}$, with $\mu_{i_0}(D) < d$, we have $C_{i_0} \setminus D \succ_{i_0} B_{i_0}$. By nonatomicity of measure μ_{i_0} such subset D exists. By Theorem 1 there exists a partition D_i $(i \in I)$ for D such that $\mu_i(D_i) = \frac{1}{|I|}\mu_i(D)$ for all $i \in I$. Define $F_{i_0} = C_{i_0} \setminus D$, and $F_i = C_i \cup D_i$ for $i \in I \setminus \{i_0\}$. Then $\{F_i \mid i \in I\}$ is a partition of $A(I)$ such that $F_i \succ_i C_i$ for all $i \in I$. Thus I improves upon division P.
□

Theorem 3 and Proposition 4 imply

Corollary 5. *If in addition to the assumptions of Propsition 4 preferences are convex, then the core of the measurable space trading economy is nonempty.*

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