

STRONG EQUILIBRIUM OUTCOMES OF VOTING GAMES ARE THE GENERALIZED CONDORCET WINNERS

by

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2.2.2001

ABSTRACT¹

We consider voting games induced by anonymous and top-unanimous social choice functions satisfying two additional conditions which we formulate and term “weak neutrality” and “effectivity regularity”. The class of such social choice functions is quite broad, including every “t-refinement” of the Plurality Rule, Plurality with a Runoff, Approval Voting, the Majoritarian Compromise and the Single Transferable Vote, i.e., any selection from either of these social choice rules which is obtained via tie-breaking among candidates according to any total order t on the set of alternatives. We characterize the strong equilibrium outcomes of the voting games determined by such social choice functions. As announced in our title, these turn out to be nothing but generalized Condorcet winners, namely the “(n,q)-Condorcet winners”. In the case of social choice functions (such as those just listed) which are furthermore “top-majoritarian”, they coincide with the classical Condorcet winners.

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¹ **Acknowledgements:** We thank Semih Koray, İpek Sanver, Fuad Aleskerov, Ahmet Alkan, Hakan İnal, Hervé Moulin, Bezalel Peleg, Robert Wilson, Muhamet Yıldız and two anonymous referees for their helpful comments. This paper was first written when Remzi Sanver was a graduate student at Boğaziçi University and an earlier version was completed while he was visiting Universitat Autònoma de Barcelona (UAB), thanks to the kindness of the UAB and the financial support of the Turkish Academy of Sciences. The paper was presented at the 3. International Congress of the Society for the Advancement of Economic Theory, Antalya, June 16-21, 1997, at the annual meeting of ASSET (Southern European Association for Economic Theory), 31 October-1 November 1997, Marseille, and at the Game Theory Seminar, Graduate School of Business, Stanford University, July 1998. We thank the participants. Remzi Sanver gratefully acknowledges partial support from BÜVAK (Boğaziçi University Foundation) and Serem Ltd. This revised version of the paper has benefited from M. R.

1. INTRODUCTION

We know, thanks to Gibbard (1973) and Satterthwaite (1975), that there exists no singleton-valued, strategy-proof and non-dictatorial social choice rule with a range containing at least three alternatives. From Dasgupta, Hammond and Maskin (1979) we also know that if a singleton-valued social choice rule is Nash-implementable on the full domain of preferences, then it must also be truthfully implementable in dominant strategies. Combining these two results leads to the conclusion that no singleton-valued social choice rule whose range contains at least three alternatives can be Nash implemented unless it is dictatorial, a result which was also established by Muller and Satterthwaite (1977).

So, in a real sense, many social choice rules are quite manipulable, and it is impossible to implement them via dominant strategy or Nash equilibria by using direct mechanisms. Regarding implementability via indirect mechanisms, we have many positive results which partially or fully characterize implementable social choice correspondences.²

Alternatively, one could inquire into the “performance” (see, e.g., Hurwicz and Sertel (1999)) of a voting rule derived from a social choice rule. Nash and strong equilibrium outcomes of direct mechanisms associated with many social choice rules have been studied after the pioneering work of Hurwicz (1979). In particular, Thomson (1984) determines Nash equilibria of manipulation games associated with resource allocation mechanisms. Tadenuma and Thomson (1995) consider the problem of fairly allocating an indivisible good and study the direct revelation games associated with subsolutions of the no-envy solution. Otani and Sicilian (1982, 1990) characterize

Sertel’s stay at the IHES (Institut des Hautes Etudes Scientifiques), Bures-sur-Yvettes, France during January, 1999 and January-February, 2000.

² Among these, we have Maskin (1977, 1999), Danilov (1992) for Nash implementation; Maskin (1979), Dutta and Sen (1991) for strong implementation; Moore and Repullo (1988), Abreu and Sen (1990) for subgame perfect implementation; Palfrey and Srivastava (1991), Jackson, Palfrey and Srivastava (1994) for undominated Nash implementation; Jackson (1991), Dutta and Sen (1994) for Bayesian implementation. So we know a lot about what is implementable, and what is not, via various solution concepts. In particular, Maskin (1979) shows that strong implementability of a social choice correspondence is inconsistent with the weak no veto power condition.

equilibrium outcomes of Walras preference games induced by a direct mechanism as well as when a mechanism asking the agents to reveal their demand maps. Sertel and Sanver (1999) characterize equilibrium outcomes of Lindahl endowment pretension games³. Roth (1984) characterizes Nash equilibrium outcomes when stable matching rules are instituted as a direct mechanism in the marriage market.

The “performance” of an outcome function is especially important in the context of voting: If we want to institute a social choice rule as a voting rule, realistically speaking, it has to be through a direct mechanism where people indicate their preferences. Artificial message spaces and indirect mechanisms based on these seem to be out of the question, except in our ivory towers of theory. As a result, we have to reckon with what will happen under “strategic voting” with free coalescing, which we interpret as meaning strong equilibrium. Hence we ask the following question: If we were to declare a singleton-valued social choice rule as the outcome function and naively ask for the agents’ preferences through a “direct” mechanism, what would we actually end up implementing?

A main motivation for studying the strong equilibrium outcomes of the direct mechanism determined by a social choice rule may be found in the following historical procedure, related by Lakeman and Lambert (1959, p.101) :

“Rowland Hill ... records that, when he was teaching in his father’s school, his pupils were asked to elect a committee by standing beside the boy they liked best. This first produced a number of unequal groups, but soon the boys in the largest groups came to the conclusion that not all of them were actually necessary for the election of their favourite and some moved on to help another candidate, while on the other hand a few supporters of an unpopular boy gave him up as hopeless and transferred themselves to the candidate they considered the next best. The final result was that a number of candidates equal to the number required for the committee were each surrounded by the same number of supporters, with only two or three boys left over who were dissatisfied with all those elected.”

³ Sanver (2001) offers a related analysis for the endowment pretension games of a certain axiomatically defined class of taxation rules.

Lakeman and Lambert (1959) consider this as “an admirable example of the use of the Single Transferable Vote.”⁴ As shown in further detail by Sertel and Kalaycıoğlu (1995), it is actually unclear how well the outcome of this procedure would be approximated by the Single Transferable Vote, but in electing a single-member committee by a small number of voters, Hill’s procedure would stop, if it stops, at a strong equilibrium of the voting game induced by instituting the Plurality Rule as the outcome function (a boy receiving the maximal number of votes wins).⁵

Will this procedure stop, and if so at what outcome? Are there social choice rules other than plurality which would lead to the same outcome when instituted as the outcome function? In investigating such questions, we allow a social choice rule to act as an outcome function, and we study its “performance correspondence” i.e., the correspondence which, at each preference profile gives the set of outcomes under the outcome function defined by this social choice rule when each agent is to reveal his preference and the game so determined is resolved at a strong equilibrium of this game.

Identifying outcome functions with their respective mechanisms, it is, of course, quite possible that the social choice rule as a mechanism does not implement itself, as may happen when truth-telling is not an equilibrium of the revelation game determined by the mechanism. Indeed, we characterize the strong equilibrium outcomes under a wide class of voting rules as generalized Condorcet-winners. This coincidence is also a positive result for implementation by use of direct mechanisms, somewhat in contrast with the generally negative results in this area. We also explore the extent to which the notion of a Condorcet winner needs to be generalized in order to ensure its existence at all preference profiles.

In Section 2 we give the preliminaries. Section 3 presents two notions of effectivity and formulates certain properties for social choice rules. Section 4 contains the main characterization results. Section 5 gives the “necessary and sufficient” generalization

⁴ For a comprehensive study of the Single Transferable Vote, see Tideman (1995).

⁵ Thus, there may be ties.

of Condorcet winners so that they always exist. Section 6 is devoted to application of our results to known voting rules. Section 7 records some closing remarks.

2. PRELIMINARIES

Taking any integer $n \geq 2$, we consider a set $\mathbf{N} = \{1, 2, \dots, n\}$ of voters, confronting a non-empty finite set of candidates \mathbf{A} with $|\mathbf{A}| = m \geq 2$. We denote \mathbf{W} for the set of all complete pre-orders⁶ on \mathbf{A} , and imagine that each voter $i \in \mathbf{N}$ has a preference $R_i \in \mathbf{W}$, so that we are always given some “preference profile” $\mathbf{R} = (R_1, \dots, R_n) \in \mathbf{W}^{\mathbf{N}}$. Given any coalition $K \subset \mathbf{N}$ of voters, we denote $\mathbf{R}_K \in \mathbf{W}^K$ for a preference profile restricted to K . For any $R_i \in \mathbf{W}$, P_i will stand for the strict counterpart and I_i for the indifference counterpart of R_i ($i \in \mathbf{N}$).⁷

By a *voting rule* we mean any function $\mathbf{V}: \mathbf{W}^{\mathbf{N}} \rightarrow \mathbf{A}$.⁸ For every preference profile $\mathbf{R} \in \mathbf{W}^{\mathbf{N}}$, a voting rule \mathbf{V} induces a (normal form) “voting game” $\Gamma(\mathbf{V}, \mathbf{R}) = \{(M_i, u_i)\}_{i \in \mathbf{N}}$, where $M_i = \mathbf{W}$ is the message space of the generic agent $i \in \mathbf{N}$, and u_i is a real-valued utility function, defined on the message profile space $\mathbf{M} = \prod_{i \in \mathbf{N}} M_i$, representing the preference of voter i on \mathbf{M} as follows:

$$u_i(\mathbf{m}) \geq u_i(\mathbf{m}') \Leftrightarrow \mathbf{V}(\mathbf{m}) R_i \mathbf{V}(\mathbf{m}') \quad (\mathbf{m}, \mathbf{m}' \in \mathbf{M}; i \in \mathbf{N}).$$

Given any voting rule \mathbf{V} , and denoting σ for the strong equilibrium⁹ solution concept for normal form games, at any preference profile \mathbf{R} , we denote $\sigma[\Gamma(\mathbf{V}, \mathbf{R})] \subset \mathbf{W}^{\mathbf{N}}$ for the set of strong equilibria of the game $\Gamma(\mathbf{V}, \mathbf{R})$ and

⁶ A pre-order is a reflexive and transitive binary relation. Thus, a complete pre-order is a complete (hence also reflexive) and transitive binary relation.

⁷ Thus, $x P_i y$ iff $x R_i y$ holds and $y R_i x$ fails ($x, y \in \mathbf{A}$); $x I_i y$ iff $x R_i y$ and $y R_i x$ ($x, y \in \mathbf{A}$).

⁸ Thus, a voting rule can be considered as a selection from (i.e., singleton-valued refinement of) a social choice rule (“correspondence”) via some tie-breaking rule.

⁹ We say that a joint strategy $\mathbf{m} = \{m_i\}_{i \in \mathbf{N}} \in \mathbf{M}$ is a *strong equilibrium* of the game $\Gamma(\mathbf{V}, \mathbf{R}) = \{(M_i, u_i)\}_{i \in \mathbf{N}}$ if and only if, given any coalition $K \subset \mathbf{N}$, there is no $\mathbf{m}' = \{m'_i\}_{i \in \mathbf{N}} \in \mathbf{M}$ with $m'_j = m_j$ for every $j \in \mathbf{N} \setminus K$ such that $u_i(\mathbf{m}') > u_i(\mathbf{m})$ for each $i \in K$.

$$\mathbf{V}^\sigma(\mathbf{R}) = \{ \mathbf{V}(\mathbf{m}) \in \mathbf{A} \mid \mathbf{m} \in \sigma [\Gamma(\mathbf{V}, \mathbf{R})] \}$$

for the set of strong equilibrium outcomes of the voting rule \mathbf{V} .

We characterize the correspondence $\mathbf{V}^\sigma: \mathbf{W}^N \rightarrow 2^{\mathbf{A}}$ in terms of "(n,q)-Condorcet winners", a generalization of the classical notion of a Condorcet winner. To formulate this notion, we first define a useful "dominance" relation on \mathbf{A} :

Given any $q \in \mathbf{N} \cup \{0, n+1\}$, any $\mathbf{R} \in \mathbf{W}^N$ and any $x, y \in \mathbf{A}$, we say that x (n,q)-dominates y (according to \mathbf{R}), and we write $x D(\mathbf{R};n,q) y$, if and only if $|\{i \in \mathbf{N} \mid x P_i y\}| \geq q$.

The undominated alternatives according to the binary relation $D(\mathbf{R};n,q)$ constitute the (n,q)-Condorcet winners (according to \mathbf{R}). So, given any $\mathbf{R} \in \mathbf{W}^N$, by the (n,q)-Condorcet Set, denoted by $C(\mathbf{R};n,q)$, we mean the set of (n, q)-Condorcet winners, i.e., alternatives which are undominated according to $D(\mathbf{R};n,q)$. Thus, $C(\mathbf{R};n,q) = \{x \in \mathbf{A} \mid [y, z \in \mathbf{A} \text{ and } y D(\mathbf{R};n,q) z] \Rightarrow z \neq x\}$.

Note that, for any $q, r \in \mathbf{N} \cup \{0, n+1\}$, we have $q \leq r \Rightarrow C(\mathbf{R};n,q) \subset C(\mathbf{R};n,r)$. So, given any size n of the electoral population, the proportion of the theoretically possible n -long preference (complete preorder) profiles at which the set of (n,q)-Condorcet winners is non-empty will be a non-decreasing function of $q \in \{0, 1, \dots, n, n+1\}$. In particular, taking any positive integer $n \geq 2$, there is no (n,0)-Condorcet winner, the set of (n,1)-Condorcet winners are the alternatives unanimously considered as best, ..., the (n,n)-Condorcet winners are the alternatives which are weakly Pareto dominated by no alternative (i.e., the set of alternatives compared with which we can find no alternative which everybody finds strictly superior), and of course the set of (n,n+1)-Condorcet winners is the entire set of alternatives. Note also that, defining $n^{**} = \min\{q \in \{0, 1, \dots, n\} \mid q > n - q\}$, $C(\mathbf{R};n,n^{**})$ coincides with the set of Condorcet winners in the classical sense.¹⁰

¹⁰ When n is even, we need to distinguish between strong and weak Condorcet winners. A weak Condorcet winner is an alternative which is found to be at least as good as any other alternative by at

At any preference profile R , the set $V^\sigma(R)$ of strong equilibrium outcomes of a voting rule V can be expressed in terms of $C(R;n,q)$ for a specific value of q derived from the distribution of “effectivity” over the coalitions $K \subset N$, as described in the following section.

3. EFFECTIVITY OF VOTING RULES

Given a voting rule V , we will consider the ability of a coalition to get an alternative chosen under V . Following Moulin and Peleg (1982), Peleg (1984), and Ichiishi (1986), this ability is generally referred to as the “effectivity” of a coalition.¹¹ We give two versions of it.

3.1. Definition: Given a voting rule $V: W^N \rightarrow A$, we say that a coalition $K \subset N$ of voters is *β -effective for $x \in A$* if and only if $\forall R_{N \setminus K} \in W^{N \setminus K}$, there exists some $R_K \in W^K$ such that $V(R_K, R_{N \setminus K}) = x$.

We denote $\beta_V^+(x)$ for the set of β -effective coalitions for the alternative x under the voting rule V . We let $b_V^+(x)$ stand for the cardinality of any minimal cardinality coalition belonging to $\beta_V^+(x)$. By convention, we set $b_V^+(x) = n+1$ whenever $\beta_V^+(x)$ is empty ($x \in A$). We define $b_V^+ = \max_{x \in A} \{ b_V^+(x) \}$.

3.2 Definition: Given a voting rule $V: W^N \rightarrow A$, we say that a coalition $K \subset N$ of voters is *weakly β -effective for $x \in A$* if and only if there exists an alternative $y \in A \setminus \{x\}$ such that for some profile $R_{N \setminus K} \in W^{N \setminus K}$ of the complementary coalition $N \setminus K$ with $\{y\} = \text{argmax}_{i \in N \setminus K} R_i$ at every $i \in N \setminus K$, the coalition K has a profile $R_K \in W^K$ such that $x = V(R_K, R_{N \setminus K})$.

least $n/2$ people. A strong Condorcet winner is an alternative which is found to be at least as good as any other alternative by at strict majority, thus by more than $n/2$ people. Hence, $C(R;n,n^{**})$ is the set of weak Condorcet winners. This distinction vanishes when n is odd.

We denote $\bar{\beta}_V(x)$ for the set of weakly β -effective coalitions for the alternative x under the voting rule V . We let $\bar{b}_V(x)$ stand for the cardinality of any minimal cardinality coalition belonging to $\bar{\beta}_V(x)$. By convention we set $\bar{b}_V(x) = n+1$ whenever $\bar{\beta}_V(x)$ is empty ($x \in \mathbf{A}$). We define $b_V = \min_{x \in \mathbf{A}} \{ \bar{b}_V(x) \}$.

Remark that, for any voting rule V , we always have $b_V^- \leq b_V^+$, as $\beta_V^+(x) \subset \bar{\beta}_V(x)$ for every $x \in \mathbf{A}$. Moreover, for every $x \in \mathbf{A}$ we have $[K \subset K' \subset \mathbf{N} \text{ and } K \in \bar{\beta}_V(x)] \Rightarrow K' \in \bar{\beta}_V(x)$ and $[K \subset K' \subset \mathbf{N} \text{ and } K \in \beta_V^+(x)] \Rightarrow K' \in \beta_V^+(x)$.

4. STRONG EQUILIBRIUM OUTCOMES OF VOTING GAMES AND (n,q)-CONDORCET WINNERS

Now we examine some relations between the (n,q)-Condorcet sets and the set of strong equilibria, V^σ , of voting games. Our first result, for top-unanimous¹² voting rules, connects the (n,q)-Condorcet sets to the strong equilibrium outcomes:

4.1. Theorem: If $V: \mathbf{W}^N \rightarrow \mathbf{A}$ is a top-unanimous voting rule, then we have $C(\mathbf{R}; n, q) \subset V^\sigma(\mathbf{R})$ for $q = b_V^-$ ($\mathbf{R} \in \mathbf{W}^N$).

Proof: Take any top-unanimous voting rule V , and any $\mathbf{R} \in \mathbf{W}^N$. Consider the game $\Gamma(V, \mathbf{R})$. For any $x \in C(\mathbf{R}; n, b_V^-)$ regard any message profile \mathbf{m} where $\{x\} = \text{argmax } m_i$ for each $i \in \mathbf{N}$. As V is top-unanimous, $V(\mathbf{m}) = x$. It suffices to show that $\mathbf{m} \in \sigma[\Gamma(V, \mathbf{R})]$. To this end, take any $y \in \mathbf{A} \setminus \{x\}$. As $x \in C(\mathbf{R}; n, b_V^-)$, we have $|\{i \in \mathbf{N} \mid y P_i x\}| < b_V^- \leq \bar{b}_V(y)$ and so $\{i \in \mathbf{N} \mid y P_i x\} \notin \bar{\beta}_V(y)$. This shows that $\mathbf{m} \in \sigma[\Gamma(V, \mathbf{R})]$ (hence, $x \in V^\sigma(\mathbf{R})$).

¹¹ We thank Bezalel Peleg for bringing to our attention that our Definition 3.1 was the well-known definition of β -effectivity.

¹² A voting rule $V: \mathbf{W}^N \rightarrow \mathbf{A}$ is said to be *top-unanimous* if and only if $V(\mathbf{R}) = x$ whenever $\{x\} = \text{argmax } R_i$ for every $i \in \mathbf{N}$ ($x \in \mathbf{A}$).

4.2. Remark: The reverse of the inclusion established in Theorem 4.1 does not hold. Consider the top-unanimous voting rule $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ where $\mathbf{V}(\mathbf{R}) = x$ whenever $\text{argmax } R_i = \{x\}$ for each $i \in \mathbf{N}$ and $\mathbf{V}(\mathbf{R}) = x^*$ for some fixed $x^* \in \mathbf{A}$ otherwise. Note that $b_{\mathbf{V}}(x) = n$ for every $x \in \mathbf{A} \setminus \{x^*\}$ and $b_{\mathbf{V}}(x^*) = 1$ implying that $b_{\mathbf{V}} = 1$. Consider the case of a doubleton $\mathbf{A} = \{x, x^*\}$ and $\mathbf{N} = \{1, 2\}$. Take a profile $\mathbf{R} = (R_1, R_2)$ where $x P_1 x^*$ and $x^* P_2 x$. Now consider the game $\Gamma(\mathbf{V}, \mathbf{R})$. Take the message profile \mathbf{m} with $\text{argmax } m_i = \{x^*\}$ for each $i \in \{1, 2\}$. Clearly $\mathbf{m} \in \sigma[\Gamma(\mathbf{V}, \mathbf{R})]$. Thus, $x^* \in \mathbf{V}^\sigma(\mathbf{R})$, although $x P_1 x^*$ and so $x^* \notin C(\mathbf{R}; n, b_{\mathbf{V}})$.

Nevertheless, a parallel of the reverse of the inclusion established in Theorem 4.1 holds for anonymous voting rules, as our next result tells us.

4.3. Theorem: If $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ is an anonymous¹³ voting rule, then we have $\mathbf{V}^\sigma(\mathbf{R}) \subset C(\mathbf{R}; n, q)$ for $q = b_{\mathbf{V}}^+(\mathbf{R} \in \mathbf{W}^N)$.

Proof: Take any anonymous voting rule \mathbf{V} . Thus, for any $x \in \mathbf{A}$, the family of β -effective coalitions will be in the form $\beta_{\mathbf{V}}^+(x) = \{K \subset \mathbf{N} \mid |K| \geq k\}$ for some $k \in \mathbf{N}$. Now take any $\mathbf{R} \in \mathbf{W}^N$ and consider any $x \in \mathbf{V}^\sigma(\mathbf{R})$. Thus, there exists $\mathbf{m} \in \mathbf{M}$ with $\mathbf{V}(\mathbf{m}) = x$ and $\mathbf{m} \in \sigma[\Gamma(\mathbf{V}, \mathbf{R})]$. Hence, for any $y \in \mathbf{A} \setminus \{x\}$, $\{i \in \mathbf{N} \mid y P_i x\} \notin \beta_{\mathbf{V}}^+(y)$. Thanks to the structure of the family of β -effective coalitions, we therefore have $|\{i \in \mathbf{N} \mid y P_i x\}| < b_{\mathbf{V}}^+(y) \leq b_{\mathbf{V}}^+$. Thus, there exists no $y \in \mathbf{A}$ with $y D(\mathbf{R}; n, b_{\mathbf{V}}^+) x$, implying that $x \in C(\mathbf{R}; n, b_{\mathbf{V}}^+)$, and thereby showing that $\mathbf{V}^\sigma(\mathbf{R}) \subset C(\mathbf{R}; n, b_{\mathbf{V}}^+)$, as desired.

4.4. Remark : The reverse of the inclusion in Theorem 4.3 does not hold. Consider the constant voting rule $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ (with $|\mathbf{A}| > 1$) where for each $\mathbf{R} \in \mathbf{W}^N$ we have

¹³ We have the usual definition of anonymity, i.e., we say that a voting rule $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ is *anonymous* if and only if, given any permutation $\rho: \mathbf{N} \rightarrow \mathbf{N}$ of voters, we have $\mathbf{V}((R_i)_{i \in \mathbf{N}}) = \mathbf{V}((R_{\rho(i)})_{i \in \mathbf{N}})$ ($(R_i)_{i \in \mathbf{N}} \in \mathbf{W}^N$).

$\mathbf{V}(\mathbf{R}) = \{x^*\}$ for some fixed $x^* \in \mathbf{A}$. Clearly \mathbf{V} is anonymous. Note that $b_{\mathbf{V}}^+(x) = n+1$ for every $x \in \mathbf{A} \setminus \{x^*\}$ and $b_{\mathbf{V}}^+(x^*) = 0$ implying that $b_{\mathbf{V}}^+ = n+1$. Thus, for any profile $\mathbf{R} \in \mathbf{W}^{\mathbf{N}}$ we have $C(\mathbf{R}; n, b_{\mathbf{V}}^+) = \mathbf{A}$, whereas $\mathbf{V}^{\sigma}(\mathbf{R}) = \{x^*\}$.

Thus, we know by Theorems 4.1 and 4.3 that, for any top-unanimous and anonymous voting rule \mathbf{V} , $C(\mathbf{R}; n, b_{\mathbf{V}}^-) \subset \mathbf{V}^{\sigma}(\mathbf{R}) \subset C(\mathbf{R}; n, b_{\mathbf{V}}^+)$ holds at every $\mathbf{R} \in \mathbf{W}^{\mathbf{N}}$. Of course, the tighter the interval $[b_{\mathbf{V}}^-, b_{\mathbf{V}}^+]$, the stronger is our result. The case where $b_{\mathbf{V}}^- = b_{\mathbf{V}}^+$ is of particular interest. To explore this case, for any voting rule \mathbf{V} , we focus on the following subsets of \mathbf{A} :

$$\operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot) = \{x \in \mathbf{A} : b_{\mathbf{V}}^+(x) \leq b_{\mathbf{V}}^+(y) \forall y \in \mathbf{A}\}.$$

We say that a voting rule \mathbf{V} is *weakly neutral* if and only if $b_{\mathbf{V}}^+(x) = b_{\mathbf{V}}^+(y)$ for all $x, y \in \mathbf{A}$ when n is odd and $b_{\mathbf{V}}^+ \leq \min_{x \in \mathbf{A}} \{b_{\mathbf{V}}^+(x)\} + 1$ when n is even.¹⁴ Clearly, weak neutrality is implied by the usual neutrality. It is strictly weaker, in fact, as any singleton-valued refinement of the Plurality Rule (obtained through a tie-breaking rule among alternatives) will exemplify a voting rule which is weakly neutral but not neutral.¹⁵ And it is thanks to its being strictly weaker than neutrality that we are able to combine it with anonymity to reap Theorem 4.5 and the Corollary 4.8 below. (Recall that a social choice function cannot be both neutral and anonymous.)

Next, we say that a voting rule \mathbf{V} is *effectivity regular* if and only if we have

- (i) $b_{\mathbf{V}}^-(x) = b_{\mathbf{V}}^+(x)$ for all $x \in \mathbf{A}$ when n is odd
- (ii) $b_{\mathbf{V}}^+(x) = b_{\mathbf{V}}^-(x)$ for all $x \in \operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot)$ when n is even
- (iii) $b_{\mathbf{V}}^+(x) \leq b_{\mathbf{V}}^-(x) + 1$ for all $x \in \mathbf{A} \setminus \operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot)$ when n is even

¹⁴ Hence for any weakly neutral voting rule \mathbf{V} we have $\operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot) = \mathbf{A}$ when n is odd. If n is even, then either $\operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot) = \mathbf{A}$ or $\{\operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot), \mathbf{A} \setminus \operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot)\}$ forms a partition of \mathbf{A} .

¹⁵ This is also shown in Section 6.

For our purposes, conjoining weak neutrality and effectivity-regularity brings b_V^- and b_V^+ sufficiently close to each other. In fact, when n is odd, for weakly neutral and effectivity-regular voting rules V , we have $b_V^- = b_V^+$. If n is even, then every weakly neutral and effectivity-regular voting rule V gives us $b_V^+ - b_V^- \leq 1$. Combining this with Theorems 4.1 and 4.3 thus directly gives us the following result:

4.5. Theorem (“Quasi-Characterization”): Let $V: \mathbf{W}^N \rightarrow \mathbf{A}$ be a top-unanimous, anonymous, weakly neutral and effectivity-regular voting rule. For any $R \in \mathbf{W}^N$, we have

- (i) $C(R;n,q) = V^\sigma(R)$ for $q = b_V^- = b_V^+$ when n is odd;
- (ii) $C(R;n, b_V^-) \subset V^\sigma(R) \subset C(R;n, b_V^+)$ with $b_V^+ - b_V^- \leq 1$ when n is even.

It is important to check that the properties of top-unanimity, anonymity, weak neutrality and effectivity-regularity which are active in Theorem 4.5 are all independent. We show this in Appendix 1.

Writing $n^* = n^{**} - 1$, in view of Theorem 4.5 it is of interest to see for which voting rules V we have $b_V^-, b_V^+ \in \{n^*, n^{**}\}$. For the strong equilibrium outcomes of the voting game induced by any such rule would coincide with the Condorcet winners in the classical sense.¹⁶ This motivates our formulating the next regularity condition:

4.6. Definition: A voting rule $V: \mathbf{W}^N \rightarrow \mathbf{A}$ is said to be *top-majoritarian* if and only if, for every $K \subset N$ with $|K| \geq n^{**}$, we have $V(R) = x$ whenever $\{x\} = \operatorname{argmax} R_i$ for every $i \in K$ ($x \in \mathbf{A}$).

Of course, top-majoritarianism strictly implies top-unanimity, as well as weak neutrality. Moreover, effectivity regularity is implied by top-majoritarianism if and only if n is odd. Furthermore, apart from these implications, none of the following

five properties is implied by the conjunction of the remaining four: top-unanimity, anonymity, weak neutrality, effectivity-regularity and top-majoritarianism. We show all this in Appendix 2.

The following lemma characterizes the size of the β -effective and weakly β -effective coalitions for top majoritarian and effectivity-regular voting rules.

4.7. Lemma: Let $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ be a top-majoritarian voting rule. We have

- (i) $b_{\mathbf{V}}^- = b_{\mathbf{V}}^+ = n^{**}$ if n is odd;
- (ii) $\{b_{\mathbf{V}}^-, b_{\mathbf{V}}^+\} \subset \{n^*, n^{**}\}$ if n is even

Proof: Consider any top majoritarian voting rule \mathbf{V} .

ad (i): Let n be odd. Taking any $x \in \mathbf{A}$, we check that $b_{\mathbf{V}}^-(x) < n^{**}$ (hence $b_{\mathbf{V}}^+(x) < n^{**}$) and $b_{\mathbf{V}}^+(x) > n^{**}$ (hence $b_{\mathbf{V}}^-(x) > n^{**}$) would contradict that \mathbf{V} is top-majoritarian. So $b_{\mathbf{V}}^-(x) = b_{\mathbf{V}}^+(x) = n^{**}$ holds for every $x \in \mathbf{A}$, showing that $b_{\mathbf{V}}^- = b_{\mathbf{V}}^+ = n^{**}$, proving (i).

ad (ii): Consider now the case where n is even. As \mathbf{V} is top-majoritarian, clearly $\{b_{\mathbf{V}}^-(x), b_{\mathbf{V}}^+(x)\} \subset \{n^*, n^{**}\}$ for all $x \in \mathbf{A}$, which implies $\{b_{\mathbf{V}}^-, b_{\mathbf{V}}^+\} \subset \{n^*, n^{**}\}$ as well, showing (ii), which completes the proof.

We will call a voting rule $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ *regular* whenever it is anonymous, effectivity-regular and top-majoritarian. The following result is a direct corollary to the conjunction of Theorem 4.5, Lemma 4.7 and the fact, shown in Appendix 2, that top-majoritarianism implies top-unanimity and weak-neutrality.¹⁷

¹⁶ When n is even, $C(\mathbf{R};n,n^*)$ is the set of strong Condorcet winners and $C(\mathbf{R};n,n^{**})$ is the set of weak Condorcet winners. We do not have such a distinction when n is odd and $C(\mathbf{R};n,n^{**})$ is the set of Condorcet winners. See footnote 10.

¹⁷ Recall also that $b_{\mathbf{V}}^-(x) \leq b_{\mathbf{V}}^+(x)$ for every $x \in \mathbf{A}$. Moreover, for any $q, r \in \mathbf{N} \cup \{0, n+1\}$, we have $q \leq r \Rightarrow C(\mathbf{R};n,q) \subset C(\mathbf{R};n,r)$.

4.8. Corollary: If $V: \mathbf{W}^N \rightarrow \mathbf{A}$ is a regular voting rule, then for every $R \in \mathbf{W}^N$ we have

- (i) $V^\sigma(R) = C(R; n, n^{**})$ when n is odd;
- (ii) $C(R; n, n^*) \subset V^\sigma(R) \subset C(R; n, n^{**})$ when n is even.

Many interesting voting rules are covered by Theorem 4.5 and Corollary 4.8. We will give examples of these in Section 6. But before this, we recall the infamous common weakness of Condorcet winners on the one hand and strong equilibria on the other: they may fail to exist. So it is natural to search for the lowest value of q for which the existence of an (n, q) -Condorcet winner is guaranteed for every $R \in \mathbf{W}^N$. We turn to this in the next section.

5. ON THE EXISTENCE OF (n, q) – CONDORCET WINNERS

Condorcet (1785) introduced the idea of choosing an alternative which, in comparison with **every** rival alternative, receives the support of a **majority** (possibly different from one rival to another) in its own favor. When a Condorcet winner does not exist, there are two natural ways of mending this rule: lowering the hurdle according to one of the two critical words – “majority” and “every” – in the definition of a Condorcet winner. Copeland (1951) does this for “every” and proposes to choose alternatives receiving majority support against the highest number of rivals. Such an alternative, called a Copeland winner, may lose in pairwise comparisons against a rival by a huge majority. Alternatively, lowering the hurdle of “majority”, one can choose the alternatives receiving the most populous support against all other candidates. This is known as Kramer’s (1977) Rule.¹⁸ The generalized (n, q) -Condorcet winner concept is in the same spirit. In fact, at any given $R \in \mathbf{W}^N$, writing $q(R)$ for the lowest value of

¹⁸ Kramer’s Rule is sometimes called Simpson’s Rule. For more detailed information on these, see Moulin (1983) and Moulin (1988).

q ensuring the existence of (n, q)-Condorcet winner, the set of (n, q(R))-Condorcet winners coincides with the set of Kramer winners.¹⁹

Given n voters, confronting m alternatives, define $q(n, m) = \lfloor n \cdot (m-1) / m \rfloor + 1$, where $\lfloor s \rfloor$ shows the integral part of any real number s. We now state a proposition showing the acyclicity of the binary relation $D(R;n, q)$ when $q \geq q(n, m)$.

5.1. Proposition (Acyclicity): Take any $k \in \{2, \dots, m\}$. For any $R \in \mathbf{W}^N$ we have

$$x_i D(R;n, q(n, m)) x_{i+1} \text{ for all } i \in \{1, \dots, k-1\} \Rightarrow \text{not } x_k D(R;n, q(n, m)) x_1$$

Proof: Take any $k \in \{2, \dots, m\}$, any $R \in \mathbf{W}^N$ and assume that $x_i D(R;n, q(n, m)) x_{i+1}$ for all $i \in \{1, \dots, k-1\}$. Denote $K_i = \{j \in N \mid x_i P_j x_{i+1}\}$ for all $i \in \{1, \dots, k-1\}$. First note that, by the transitivity each $R_i \in \mathbf{W}$, for every $h, h' \in \{1, \dots, k\}$ with $h' > h$, we have $\{i \in N \mid x_h P_i x_{h'}\} \supset \bigcap_{j \in \{h, \dots, h'-1\}} K_j$. Now, since $|K_j| \geq q(n, m)$ for each $j \in \{h, \dots, h'-1\}$, we can compute that $|\bigcap_{j \in \{h, \dots, h'-1\}} K_j| \geq (h'-h) q(n, m) - (h'-h-1) q(n, m)$. Thus, $|\{i \in N \mid x_h P_i x_{h'}\}| \geq (h'-h) q(n, m) - (h'-h-1) q(n, m)$. Setting $h = 1$ and $h' = k$, we have $|\{i \in N \mid x_1 P_i x_k\}| \geq (k-1) q(n, m) - (k-2) q(n, m)$ and $|\{i \in N \mid x_k R_i x_1\}| \leq (k-1) (n - q(n, m))$. The proof is completed by checking that $(k-1) (n - q(n, m)) < q(n, m)$ for any $k \in \{2, \dots, m\}$.

The following theorem is a direct corollary to Proposition 5.1.

5.2. Theorem : For any $R \in \mathbf{W}^N$ we have $C(R;n, q(n, m)) \neq \emptyset$.

It is easy to check that $q(n, m)$ is minimal for Theorem 5.2 to hold, i.e., there exists no $q < q(n, m)$ for which $C(R;n, q) \neq \emptyset$ would hold at every $R \in \mathbf{W}^N$.

Theorem 5.2 tells that $(n, q(n, m))$ Condorcet winners always exist. Thus, for $m=4$ we have 75%, for $m=5$ we have 80%, etc., as “benchmark” majorities, guaranteeing the

¹⁹ Kramer’s Rule is the application of the compromise idea of the Majoritarian Compromise (which was originally called “Approval Voting with a Plurality Floor” by Sertel (1986)) to (n, q) -Condorcet

existence of a generalized $(n, q(n, m))$ -Condorcet winner, i.e., an alternative which is regarded as worse than no alternative by more than this majority of voters.

6. APPLICATIONS

Write \mathbf{T} for the set of total orders²⁰ over \mathbf{A} . We say that a voting rule $\mathbf{V}: \mathbf{W}^{\mathbf{N}} \rightarrow \mathbf{A}$ is a *t-refinement* of a social choice correspondence $\mathbf{F}: \mathbf{W}^{\mathbf{N}} \rightarrow 2^{\mathbf{A}} \setminus \{\emptyset\}$ if and only if there exists some “tie-breaking rule” $\mathbf{t} \in \mathbf{T}$ such that, at any $\mathbf{R} \in \mathbf{W}^{\mathbf{N}}$, $\mathbf{V}(\mathbf{R}) = \operatorname{argmax}_{\mathbf{F}(\mathbf{R})} \mathbf{t}$. Clearly, every *t-refinement* of every anonymous social choice correspondence \mathbf{F} will be anonymous ($\mathbf{t} \in \mathbf{T}$).

6.1 Refinements of anonymous top-majoritarian social choice correspondences

We say that a social choice correspondence $\mathbf{F}: \mathbf{W}^{\mathbf{N}} \rightarrow 2^{\mathbf{A}} \setminus \{\emptyset\}$ is *top-majoritarian* if and only if the following two conditions hold for all $x \in \mathbf{A}$:

- (i) $\mathbf{F}(\mathbf{R}) = \{x\}$ whenever $\{x\} = \operatorname{argmax} R_i \forall i \in \mathbf{K}$ for some $\mathbf{K} \subset \mathbf{N}$ with $|\mathbf{K}| \geq n^{**}$
- (ii) $x \in \mathbf{F}(\mathbf{R})$ whenever $\{x\} = \operatorname{argmax} R_i \forall i \in \mathbf{K}$ for some $\mathbf{K} \subset \mathbf{N}$ with $|\mathbf{K}| = n^*$, if n is even.

6.1 Proposition: Any *t-refinement* of an anonymous top-majoritarian social choice correspondence is regular.

Proof: Let \mathbf{F} be an anonymous top-majoritarian social choice correspondence and \mathbf{V} be a *t-refinement* of \mathbf{F} . It is clear that \mathbf{V} is top-majoritarian and anonymous, it remaining to show that \mathbf{V} is effectivity-regular. From Appendix 2, we know that \mathbf{V} satisfies the requirements of effectivity-regularity when n is odd. To show that it also does so generally, now assume that n is even and focus on (ii) and (iii) in the definition of effectivity-regularity above.

winners: Choose the alternatives which are $(n-q)$ -Condorcet winners for the lowest possible value of q .
²⁰ We recall that a *total order* is a complete, transitive and antisymmetric binary relation.

To focus on (iii) first, we know from Lemma 4.7 that $\{b_V^-(x), b_V^+(x)\} \subset \{n^*, n^{**}\}$ for all $x \in \mathbf{A}$, so that $b_V^+(x) \leq b_V^-(x) + 1$ for all $x \in \mathbf{A}$, implying (iii).

To focus on (ii), we take any $x \in \operatorname{argmin}_{\mathbf{A}} b_V^+(\cdot)$. Since $b_V^-(\cdot) \leq b_V^+(\cdot)$, the only way in which $b_V^-(x) \neq b_V^+(x)$ would be $b_V^-(x) < b_V^+(x)$. But since \mathbf{V} is top-majoritarian, again by Lemma 4.7, we have $\{b_V^-(x), b_V^+(x)\} \subset \{n^*, n^{**}\}$, requiring that $b_V^+(x) = n^{**}$. Having picked $x \in \operatorname{argmin}_{\mathbf{A}} b_V^+(\cdot)$, we must therefore have $b_V^+(y) = n^{**}$ for every $y \in \mathbf{A}$. But this is impossible, for, we can pick $\mathbf{t} \in \mathbf{T}$, $\mathbf{a} = \operatorname{argmax}_{\mathbf{A}} \mathbf{t}$ and choose and regard $\mathbf{R} \in \mathbf{W}^{\mathbf{N}}$ with $\{\mathbf{a}\} = \operatorname{argmax}_{\mathbf{A}} R_j$ for $j = 1, \dots, n/2$, so that $\mathbf{V}(\mathbf{R}) = \mathbf{a}$, showing $b_V^+(\mathbf{a}) = n^*$, contradicting that $b_V^+(y) = n^{**}$ for every $y \in \mathbf{A}$. This shows that $b_V^-(x) = b_V^+(x)$ for every $x \in \operatorname{argmin}_{\mathbf{A}} b_V^+(\cdot)$, establishing (ii) and thus completing the proof.

It is easy to check that the Plurality Rule, Plurality with a Runoff, the Single Transferable Vote, the Majoritarian Compromise (Sertel (1986), Sertel and Yılmaz (1999)) and both forms of Approval Voting (of Fishburn (1978) and Brams and Fishburn (1983), as well as of Sertel (1988)), are all anonymous, top-majoritarian social choice correspondences. So, we know, thanks to Proposition 6.1, that any t-refinement of these will be covered by Corollary 4.8, whereby the strong equilibrium outcomes of its voting game will coincide, at any preference profile, with the Condorcet winners in the classical sense.

6.2 Refinements of Borda's Rule

It is easy to construct examples where $\mathbf{B}: \mathbf{W}^{\mathbf{N}} \rightarrow 2^{\mathbf{A}} \setminus \{\emptyset\}$ (Borda's Rule) selects some singleton $x \in \mathbf{A}$, although n is odd and n^{**} of the voters $i \in \mathbf{N}$ have $\{y\} = \operatorname{argmax}_{\mathbf{A}} R_i$ for some $y \in \mathbf{A} \setminus \{x\}$. Thus, \mathbf{B} has no top-majoritarian t-refinement and escapes Corollary 4.8. Of course, for any voting rule \mathbf{V} equated to a t-refinement of \mathbf{B} , by Theorems 4.1 and 4.3, we have $C(\mathbf{R}; n, b_V^-) \subset \mathbf{V}^\sigma(\mathbf{R}) \subset C(\mathbf{R}; n, b_V^+)$, since Borda's Rule (hence \mathbf{V}) is anonymous and top-unanimous. But b_V^- and b_V^+ may be quite disparate in this case, as we see from the next proposition.

6.2 Proposition: Let $V: W^N \rightarrow A$ be a t-refinement of B . We have

- (i) $b_V^+ = n^{**}$ when n is odd;
- (ii) $b_V^+ \in \{n^*, n^{**}\}$ and when n is even;
- (iii) $b_V^- = \langle (1/m)n \rangle^{21}$ if n is not divisible by m ;
- (iv) $b_V^- \in \{\langle (1/m)n \rangle, \langle (1/m)n \rangle + 1\}$ if n is divisible by m .

Proof: Let V be any t-refinement of B . It is easy to see that for every $x \in A$ we have $b_V^+(x) = n^{**}$ when n is odd and $b_V^+(x) \in \{n^*, n^{**}\}$ when n is even, which is sufficient to show that $b_V^+ = n^{**}$ when n is odd and $b_V^+ \in \{n^*, n^{**}\}$ when n is even. Now, for any $n \geq 2$, consider some $K \subseteq N$ with $|K| = k$ whose members rank some $y \in A$ as the best and some $x \in A \setminus \{y\}$ as the second best alternative, i.e., for every $i \in K$, we have $\text{argmax}_A R_i = \{y\}$ and $\text{argmax}_{A \setminus \{y\}} R_i = \{x\}$. If the remaining agents $N \setminus K$ wish to ensure that $x \in A$ is chosen, then the best thing they can do is to rank x as the best and y as the worst, i.e., for every $i \in N \setminus K$, let $\text{argmax}_A R_i = \{x\}$ and $\text{argmin}_A R_i = \{y\}$. In this case x will get a Borda score $B(x) = k(m-1) + (n-k)m$ and y a Borda score $B(y) = km + (n-k)$. Thus, $B(x) \geq B(y) \Rightarrow k \leq n[(m-1)/m] \Rightarrow n - k \geq n(1/m)$ and similarly $B(x) > B(y) \Rightarrow n - k > n(1/m)$. Thus if n is not divisible by m , for any t-refinement V , a coalition of size $\langle n(1/m) \rangle$ will be weakly β -effective over x . If n is divisible by m , then, depending on the tie-breaking rule used in the refinements, a coalition of size $\langle n(1/m) \rangle$ or $\langle n(1/m) \rangle + 1$ will be weakly β -effective over x , completing the proof.

6.3 k-majority rules

²¹ For any real number s , $\langle s \rangle$ means the lowest integer which is not less than s .

Given any integer $k > n - k$, what we mean by a k -majority rule is a voting rule $\mathbf{V}_k: \mathbf{W}^N \rightarrow \mathbf{A}$ which is defined as follows: For any “status quo” $s \in \mathbf{A}$, every $x \in \mathbf{A}$ and every $R \in \mathbf{W}^N$,

$$\mathbf{V}_k(R) = \begin{cases} x & \text{if there exists } K \subset N \text{ with } |K| \geq k \text{ and } \{x\} = \operatorname{argmax}_{R_i} \text{ for all } i \in K; \\ s & \text{otherwise.} \end{cases}$$

6.3 Proposition: Let $\mathbf{V}_k: \mathbf{W}^N \rightarrow \mathbf{A}$ (for some $k > n - k$) be a k -majority rule. Regardless of the status quo $s \in \mathbf{A}$, \mathbf{V}_k is top-unanimous and anonymous (so that Theorems 4.1 and 4.3 apply). Furthermore, we have $b_{\mathbf{V}_k}^- = n - k + 1$ and $b_{\mathbf{V}_k}^+ = k$.

Proof: Take any integer $k > n - k$ and any status quo $s \in \mathbf{A}$. It is clear that \mathbf{V}_k is top-unanimous and anonymous. One can directly check that we have $b_{\mathbf{V}_k}^+(x) = b_{\mathbf{V}_k}^-(x) = k$ for every $x \in \mathbf{A} \setminus \{s\}$ and $b_{\mathbf{V}_k}^-(s) = b_{\mathbf{V}_k}^+(s) = n - k + 1$, thus implying $b_{\mathbf{V}_k}^- = n - k + 1$ and $b_{\mathbf{V}_k}^+ = k$, completing the proof.

Thus, given any integer $k > n - k$, we have $C(R; n, n-k+1) \subset \mathbf{V}^\sigma(R) \subset C(R; n, k)$ at every $R \in \mathbf{W}^N$ when the k -majority rule \mathbf{V}_k is instituted as the outcome function.

6.4 (n, q)-Condorcet winners as a social choice correspondence

We know by Theorem 5.2 that for any $q \geq q(n, m)$, we have $C(R; n, q) \neq \emptyset$ at every $R \in \mathbf{W}^N$. So, taking any integer $q \geq q(n, m)$, we can use $C(\cdot; n, q) = \mathbf{F}$ as a social choice correspondence $\mathbf{F}: \mathbf{W}^N \rightarrow 2^{\mathbf{A}} \setminus \{\emptyset\}$. Let $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ be any t -refinement of \mathbf{F} . One can check that \mathbf{V} is unanimous and anonymous, so that Theorems 4.1 and 4.3 apply. In fact, we have $b_{\mathbf{V}}^+(x) = b_{\mathbf{V}}^-(x) = n - q + 1$ for every $x \in \mathbf{A}$, thus implying $b_{\mathbf{V}}^- = b_{\mathbf{V}}^+ = n - q + 1$. Thus, given any integer $q \geq q(n, m)$, instituting any t -refinement of $C(R; n, q)$ as outcome function, we have $\mathbf{V}^\sigma(R) = C(R; n, n-q+1)$ at every $R \in \mathbf{W}^N$.

Given any positive integer $q < q(n, m)$, from $C(R; n, q)$ we can still derive a social choice correspondence $F: W^N \rightarrow 2^A \setminus \{\emptyset\}$ by setting $F(R) = s$ for some previously fixed $s \in A$ whenever $C(R; n, q)$ is empty. Let $V: W^N \rightarrow A$ be any t -refinement of F . One can check that V is unanimous and anonymous, whereby Theorems 4.1 and 4.3 apply. One can directly check that $b_V^+(x) = b_V^-(x) = n - q + 1$ for every $x \in A \setminus \{s\}$ and $b_V^-(s) = b_V^+(s) = q$, thus implying $b_V^- = q$ and $b_V^+ = n - q + 1$. Thus, given any positive integer $q < q(n, m)$, we have $C(R; n, q) \subset V^\sigma(R) \subset C(R; n, n - q + 1)$ at every $R \in W^N$ when we institute any t -refinement V of F as outcome function.

7. CLOSING REMARKS

In this paper we characterized the strong equilibrium outcomes of certain voting games. It turns out, for many well-known social choice rules, that if we refine them to singleton-valued functions by use of tie-breaking rules and employ them in the role of the outcome function of a direct mechanism, at the strong equilibria of the voting games so determined, we obtain the Condorcet winners when they exist. It may be regarded as a somewhat striking irony that many social choice rules which themselves fail to be Condorcet-consistent, in fact, turn out to induce a Condorcet-consistent outcome at the strong equilibria of the voting games they determine.²²

Our results also afford an interpretation in the spirit of implementing the classical Condorcet social choice rule on a restricted domain of preference profiles where it is non-empty-valued, e.g. where preferences are single-peaked. On such a domain, we implement the Condorcet social choice rule via strong equilibrium by use of direct mechanisms employing outcome functions which are refinements obtained through tie-breaking rules of certain well-known social choice rules. This should constitute a certain rehabilitation of Condorcet's social choice rule: In particular, when $q = n^{**}$

²² Neither of the social choice rules, which we found in Section 6 to give Condorcet winners at the strong equilibrium of the voting game they determine (the Plurality Rule, Plurality with a Runoff, Approval Voting, the Majoritarian Compromise and the Single Transferable Vote) is Condorcet-consistent in the sense of always choosing a Condorcet winner when it exists. Furthermore, Plurality and the Majoritarian Compromise are both capable of choosing a Condorcet loser (i.e., Condorcet winner with all preferences reversed) even in the presence of a Condorcet winner. See Sertel and Yilmaz (1997).

the Condorcet Rule implements itself under strong equilibrium on the domain of preference profiles where a Condorcet winner exists.

This positive implementation result loses some of its bite when we allow an unrestricted domain of preferences, as it is based on our equivalence result between the set of generalized Condorcet-winners and the set of strong equilibrium outcomes of the voting game determined by instituting a social choice function as the outcome function of a mechanism: This equivalence result may turn out to be the trivial equivalence of the empty set to the empty set when there are no generalized Condorcet-winners at a given preference profile.²³ So one may ask how far we should generalize (and weaken) the notion of a “Condorcet winner” in order to guarantee its existence independent of the preference profile. We know that an (n, q) -Condorcet winner always exists whenever $q \geq \lfloor n \cdot (m-1) / m \rfloor + 1$. Since $n(m-1)/m$ approaches n as the cardinality m of our set of alternatives increases, however, this generalization may have gone too far when we have a large number of alternatives. Indeed, the set of generalized Condorcet winners converges to the set of weakly Pareto optimal outcomes as the cardinality of the set of alternatives increases.

To sum up, we show, for a large class of social choice rules (including the Plurality Rule, Plurality with a Runoff, Approval Voting, the Majoritarian Compromise and the Single Transferable Vote), that when strong equilibrium is used as the solution concept, the social choice rule employed in the role of outcome function of a direct mechanism implements on the domain of profiles with Condorcet winners, not necessarily itself²⁴, but a Condorcet social choice rule.²⁵ These results, although simple and intuitive²⁶, can be considered as positive from an implementation viewpoint, in some contrast with the prevalence of impossibility results in this field.

²³ In fact, one can check the existence of strong equilibria of the voting game at some preference profile, just by checking the existence of (n, q) -Condorcet winners, for the appropriate q , at that preference profile.

²⁴ In fact, Condorcet-consistent social choice correspondences are self-implementable on this domain of preference profiles with Condorcet winners.

²⁵ It is not surprising that Borda’s Rule is not in this list.

²⁶ Regarding the intuitiveness of our results, some caution may be advisable. For it is possible to think, as we admit having done at the outset, that the Condorcet consistency of the outcomes of a voting game under strong equilibrium is more or less a direct consequence of definitions. This paper is witness to our having educated ourselves into realizing that this was far from being so. In fact, Borda’s Rule and

Research Questions: It is natural to question for which shrinkings and for which swellings (expansions) of the strong equilibrium solution concept our characterization result, Theorem 4.5, and its Corollary 4.8 remain valid. The strong equilibrium being hard to shrink, we address the swelling question. At the extreme, it is easy to see that under every social choice function which satisfies the weak no veto power condition, the entire set \mathbf{A} of alternatives is attainable as outcomes of our voting games under Nash equilibrium. As to the case of coalition-proof Nash equilibrium, a generally smaller swelling of the strong equilibrium, the question becomes more interesting.²⁷

Here we focus on the central question of whether, in the case of voting games, all coalition-proof Nash equilibrium outcomes can actually be matched in welfare by strong equilibrium outcomes. The answer is negative, for reason of the following simple example:

In the context of $\mathbf{A} = \{x, y, z\}$ and $\mathbf{N} = \{1, 2, 3\}$, given a preference profile

R_1	R_2	R_3
x	y	z
y	x	x
z	z	y

under Plurality refined by any tie-breaking rule with z beating x and y, regarding any declaration where all three agents find y better than both x and z, such a declaration will be a coalition-proof Nash equilibrium, resulting in the outcome y, while y cannot be attained as a strong equilibrium outcome here. The strong equilibrium outcome here is x, of course, and the welfare of x and y differ for our three agent society.

k-majority rules defy the easy credo which we had to abandon, and we had to formulate the notions of weak neutrality (rather than neutrality) and effectivity regularity in order to be able to state the theorems which render strong equilibrium outcomes of voting games as Condorcet winners, as announced in our title.

²⁷ We thank an anonymous referee for leading us to this question.

It is to be noted, furthermore, that while strong equilibria of a voting game exist precisely when a Condorcet winner exists, coalition-proof Nash equilibria may exist even in the absence of Condorcet winners. The example of

R_1	R_2	R_3
x	y	z
y	z	x
z	x	y

where we use Plurality refined by any tie-breaking rule with y beating x and z , suffices to show this, as here we have no Condorcet winner but any declaration where all three agents find x better than both y and z is a coalition-proof Nash equilibrium of the associated voting game, resulting in the outcome x .

Summarily, the strong equilibrium concept gives a rather perfect fit with Condorcet's notion while the coalition-proof Nash equilibrium concept is too "fat".²⁸ Are there, then, interesting shrinkings or swellings of the strong equilibrium for which our characterization result, Theorem 4.5, and its Corollary 4.8 are preserved?

²⁸ This is even so to the point of denying coalition-proof Nash equilibrium outcomes to be contained in $C(\cdot, n, q)$ for anonymous social choice functions, e.g., for selections of Plurality, so that even a counterpart of Theorem 4.3 fails to hold when we replace the strong equilibrium concept by the coalition-proof Nash equilibrium

APPENDIX 1

The Properties of top-unanimity (TU), anonymity (A), weak-neutrality (WN), and effectivity-regularity (ER) are all independent

Proof:

ad (i) TU is not implied by the conjunction of A, WN and ER.

Let $A = \{x, y\}$ and take any $n \geq 2$. We pick $V: W^N \rightarrow A$ so that $V(R) = x$ if and only if $|\{i \in N \mid x P_i y\}|$ is even ($R \in W^N$). So V chooses y whenever the number of agents who strictly prefer x to y is odd, and chooses x otherwise. One can check that V satisfies A, WN, ER but not TU.

ad (ii) A is not implied by the conjunction of TU, WN and ER.

Let V be a voting rule which chooses the alternative unanimously considered as the best when such an alternative exists, and otherwise picks one of the alternatives from the highest indifference class of a previously fixed agent (“contingent dictator”). V satisfies TU, WN, ER, but not A.

ad (iii) WN is not implied by the conjunction of TU, A, and ER.

Let V be a voting rule which chooses the alternative unanimously considered as the best when such an alternative exists, and otherwise picks some previously fixed $s \in A$. V satisfies TU, A, ER, but not WN.

ad (iv) ER is not implied by the conjunction of TU, A, and WN.

Let V be any t -refinement of Borda’s rule. V satisfies TU, A and WN, but not ER.

Appendix 2

Top-majoritarianism (TM) implies TU as well as WN. Moreover, ER is implied by TM if and only if n is odd. Furthermore, apart from these implications, none of the following five properties is implied by the conjunction of the remaining four: TU, A, WN, ER, TM.

Proof:

- (i) TM implies TU by definition.
- (ii) Now let us see that TM implies WN as well. We know, thanks to Lemma 4.7, that for any top-majoritarian voting rule V , we have $b_V^+(x) = n^{**}$ for every $x \in A$ when n is odd and $b_V^+(x) \in \{n^*, n^{**}\}$ for all $x \in A$ when n is even, whereby V is weakly neutral.
- (iii) To see that TM implies ER when n is odd, we again refer to Lemma 4.7, which, for n odd, gives $b_V^-(x) = b_V^+(x) = n^{**}$ for every $x \in A$, establishing effectivity-regularity.
- (iv) Now we show that ER is not implied by TM when n is even. Indeed, even the conjunction of A and TM fails to imply ER. To see this, let $N = \{1, 2\}$ and $A = \{x, y, z\}$. Consider the following voting rule V : use the Plurality Rule²⁹ whenever it ties at most two alternatives. Ties between x and y are broken in favor of x ; ties between y and z are broken in favor of y ; ties between x and z are broken in favor of z . At a preference profile where the Plurality Rule ties all three alternatives, pick the one which is in the lowest indifference class of the least number of agents. If there are two such alternatives, then break ties

²⁹ in the sense of picking the alternative which is in the highest indifference class of the greatest number of voters

according to the above rule. In case all three alternatives are tied, choose x .
 One can check that \mathbf{V} satisfies A and TM, but not ER.³⁰

(v) A is not implied by the conjunction of ER and TM.

To see that the conjunction of ER and TM fails to imply A, let \mathbf{V} be a voting rule which chooses the alternative considered as the best by a strict majority when such an alternative exists, and otherwise picks one of the alternatives from the highest indifference class of a previously fixed agent. \mathbf{V} satisfies ER and TM, but not A.

(vi) TM is not implied by the conjunction of TU, A, WN and ER.

To see this, let $\mathbf{A} = \{x, y\}$ and take any $n \geq 2$. We define $\mathbf{V}: \mathbf{W}^N \rightarrow \mathbf{A}$ as follows: if there exists an alternative considered unanimously the best then pick it; otherwise, apply the rule described in (i) of Appendix 1.

³⁰ As we have $b_{\mathbf{V}}^+(t) = 2$ and $b_{\mathbf{V}}^-(t) = 1$ for all $t \in \mathbf{A} = \operatorname{argmin}_{\mathbf{A}} b_{\mathbf{V}}^+(\cdot)$ while n is even.

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