

Existence of competitive equilibrium under financial constraints and increasing returns

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Abstract

This paper studies a cash-in-advance model with increasing returns in production. In the model both partial and general price taking equilibrium may exist since working capital of firms limit their input demands. We provide a sufficient condition for the existence of partial equilibrium of a firm owner-manager operating on a non-convex choice set. Furthermore we establish the existence and uniqueness of competitive equilibrium in the special case of logarithmic utility.

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1 Introduction

Convexity of production sets is one of the basic assumptions of neoclassical general equilibrium theory. It is widely observed, however, that for many industries the decreasing returns assumption in production implied by convex technologies is far from reality. Due to this observation, the well known non-existence problem of competitive equilibrium under non-convex technologies gave rise to two branches of literature, one motivated from a normative and the other from a positive view-point. The normative approach deviated from the profit maximization assumption in order to secure or approximate the fundamental theorems of welfare economics. Marginal cost pricing (e.g., Guesnerie, 1975, Beato, 1982, Khan and Vohra, 1987, Vohra, 1988, 1992) and average cost pricing (e.g., Brown and Heal., 1983) are two important lines of research in this spirit. The positive approach, on the other hand, gave up the price taking assumption for the firms, resulting in models of imperfect competition (e.g., Mankiw, 1985, Blanchard and Kiyotaki, 1987) in spirit of Chamberlin (1933). In this paper we also take a positive standpoint, but explore the possibility of keeping both the price taking assumption and the maximizing behavior of the firms in a model with financial constraints.

One of the main obstacles against the existence of an Arrow-Debreu equilibrium under increasing-returns-to-scale (IRS) technologies is the unbounded factor demands in face of a limited endowment of total factors of production. We eliminate this obstacle by reverting to a version of the limited participation models recently used in the business cycles literature (e.g., Fuerst, 1992, Christiano et al., 1997, 1998). In these models firm need cash at the beginning of a production cycle and they meet this need from a competitive loan market. In Basci and Saglam (2001) the case of internal financing under borrowing constraints is studied. Credit rationing is a natural result of increasing returns since under IRS technologies the loan demands are also unbounded for any positive interest rate.

A second important obstacle is the loss of convenience from using the tools of convex analysis

and convex programming, once increasing returns is allowed for. In this paper, our contribution is two fold. First we limit the labor demands by assuming internal financing via owners equity and retained earnings and second we study a non-linear programming problem on a non-convex feasible set.

The paper is organized as follows. In Section 2 general model is presented as well as the sufficient condition for the existence of partial equilibrium of the firm owners. Section 3 gives the existence result for the competitive equilibrium. Section 4 concludes.

2 The Model

In our hypothetical finite-horizon economy, at each time t , we have two agents with two different types; “worker” and “producer”. They differ in their access to production technology. There are two types of commodities: a factor of production, labor L_t and a nonstorable consumption good, apple q_t . Agent 1 (worker) has only labor endowment $\bar{L} > 0$ and has no access to production technology. Agent 2 (producer) has no labor endowment and has an increasing returns to scale (IRS) technology $f(L)$, with $f(0) = 0$, $f' \geq 0$ and $f'' > 0$, to convert labor into apples. One can have apples only through these production possibilities i.e initially there are no endowment of apples.

Agents are indexed by $i = 1, 2$. Preferences of the agents over the consumption good, apple, are represented by the same instantaneous utility function U , defined on nonnegative consumption levels. We assume that neither one of the agents values leisure. The preferences over the lifetime consumption for both types of agents are given by an additively separable form $\sum_{t=0}^T \beta^t U(C_{i,t})$ where $\beta \in (0, 1)$ is the common discount factor, and $C_{i,t}$ is the consumption of agent i at time t . We assume that U is twice continuously differentiable $U' > 0$ and $U'' < 0$. The economy operates with money under cash-in-advance constraints in both labor and apple

markets. Money is perfectly storable and $M_{i,t}$ denotes the money holding of agent i at time t . We assume that initially all the currency in the economy, M_0 , is owned by Agent 2, that is, $M_{1,0} = 0$ and $M_{2,0} = M_0$. Total money stock does not change over time. The paper money is backed by the government with a promised price of $(\frac{1}{p_2})$ in the last period. This assumption is due to the finiteness of time horizon as explained below.

2.1 Markets

We will consider a market organization with three periods ($t=0,1,2$). Each period, due to cash in advance constraints imposed on factor purchases, goods market opens after the labor market closes. In Period 0, Agent 2, since he has initially all the currency in the economy, purchases labor. Then he produces apples with the IRS technology. After the production of apples is complete Agent 1 has money, Agent 2 has apples and goods market opens. Agent 2 sells part of his apples to Agent 1 in return for money and now both Agent 1 and Agent 2 has apples to consume. Agent 2 also has money to be used as working capital in the next period. In the last period, money held by agents is backed by the government by selling apples to them.

With the endowment structure described above and given the strictly positive nominal wages w_t and prices p_t for each period t , finite horizon utility maximization problem of the two agents can be written as

Agent1 (Worker)

$$\begin{aligned}
 \text{(P1)} \quad & \max \sum_{t=0}^{t=2} \beta^t U(C_{1,t}) \\
 & \text{subject to for all } t = 0, 1, 2 \\
 & C_{1,t} = q_t^d \\
 & L_t^s \leq \bar{L}
 \end{aligned}$$

$$M_{1,t+1} = M_{1,t} + w_t L_t^s - p_t q_t^d$$

where $M_{1,t+1}, C_{1,t}, q_t^d, L_t^s \geq 0$ for all t and $M_{1,0} = 0$ is given.

Here, the planned good and labor transactions are denoted by q and L respectively. The superscripts s and d stand for demand and supply in the above formulation and the remaining part of the paper.

Agent 2 (Producer)

$$(P2) \quad \max \sum_{t=0}^{t=2} \beta^t U(C_{2,t})$$

subject to for all $t = 0, 1, 2$

$$C_{2,t} = f(L_t^d) - q_t^s$$

$$w_t L_t^d \leq M_{2,t}$$

$$M_{2,t+1} = M_{2,t} - w_t L_t^d + p_t q_t^s$$

where $M_{2,t+1}, C_{2,t}, q_t^s, L_t^d \geq 0$ for all t and $q_2^s = 0, M_{2,0} = M_0$ is given.

In Period 2 the producer will choose to set $q_t^s = 0$ regardless of $p_2 \geq 0$ determined by the government and hence $M_{2,3} = 0$ since in Period 3 there is no market where money can be used.

As an auxiliary assumption, suppose that $L_t^d = \frac{M_{2,t}}{w_t}$, that is, Agent 2 uses all of his money to purchase labor,¹ then problem (P2) becomes

$$(P2)' \quad \max \sum_{t=0}^{t=2} \beta^t U\left(f\left(\frac{M_{2,t}}{w_t}\right) - \frac{M_{2,t+1}}{p_t}\right)$$

subject to for all $t=0,1,2$

$$f\left(\frac{M_{2,t}}{w_t}\right) - \frac{M_{2,t+1}}{p_t} \geq 0$$

and $M_{2,t} \geq 0$ for all t , $M_{2,3} = 0, M_{2,0} = M_0 > 0$ and $\beta \in (0, 1)$.

¹This assumption will be justified by imposing a “profitability condition” later in Section 3.

An *equilibrium* in this economy consists of a finite sequence of apple prices, money wages, labor demands, labor supplies, apple demands, apple supplies and money holdings by the two agents such that at each date, demands, supplies and money holdings are optimal under the given wage and price sequences, demand equals supply in both labor and apple markets and money holdings sum up to the total money supply at each time.

Formally we say that $\langle p_t, w_t, L_t^d, L_t^s \rangle_{t=0}^2$ and $\langle q_t^d, q_t^s, M_{1,t+1}, M_{2,t+1} \rangle_{t=0}^1$ is an *equilibrium* if

- (i) $\langle L_t^s \rangle_{t=0}^2, \langle q_t^d, M_{1,t+1} \rangle_{t=0}^1$ solves (P1) and
 $\langle L_t^d \rangle_{t=0}^2, \langle q_t^s, M_{2,t+1} \rangle_{t=0}^1$ solves (P2) under $\langle w_t, p_t \rangle_{t=0}^2$
- (ii) $L_t^d = L_t^s \quad \forall t$
- (iii) $q_t^d = q_t^s$ for $t = 0, 1$
- (iv) $M_{1,t+1} + M_{2,t+1} = \bar{M} \equiv M_0$ for $t = 0, 1$

Since at $t = 2$ the goods market will clear with the intervention of government where the goods demand of workers is met by the government at price p_2 , the variables q_2^d, q_2^s and the corresponding $M_{1,3}$ and $M_{2,3}$ are excluded from the above definition.

Most of the following analysis will be an attempt to find a solution to Agent 2's optimization problem. Therefore we will drop the index i in variables of interest and the term "optimization problem" will refer to Agent 2's optimization problem until Section 3.

2.2 Producer's Optimization Problem

Let $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ and $h_i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ for $i = 1, \dots, 5$ be defined as

$$V(M_1, M_2) = U\left(f\left(\frac{M_0}{w_0}\right) - \frac{M_1}{p_0}\right) + \beta U\left(f\left(\frac{M_1}{w_1}\right) - \frac{M_2}{p_1}\right) + \beta^2 U\left(f\left(\frac{M_2}{w_2}\right)\right)$$

$$h_1 = M_1 \geq 0$$

$$\begin{aligned}
h_2 &= M_2 \geq 0 \\
h_3 &= C_0 = f\left(\frac{M_0}{w_0}\right) - \frac{M_1}{p_0} \geq 0 \\
h_4 &= C_1 = f\left(\frac{M_1}{w_1}\right) - \frac{M_2}{p_1} \geq 0 \\
h_5 &= C_2 = f\left(\frac{M_2}{w_2}\right) \geq 0
\end{aligned}$$

where U is the twice continuously differentiable, instantaneous utility function satisfying $U'(\cdot) > 0, U''(\cdot) < 0$ and $\lim_{c \rightarrow 0} U'(C) = \infty$. f denotes the IRS production function and satisfies $f'(L) \geq 0, f''(L) > 0$. All the parameters $\{w_t, p_t, M_0, \beta\}_{t=0}^{T=2}$ are assumed to be strictly positive, β is the discount factor, $\beta \in (0, 1)$. Note that the objective function V is bounded from above.

Three period utility maximization problem of the Agent 2 can then be reformulated as

$$\begin{aligned}
& \max V(M_1, M_2) \quad \text{over the constraint set} \\
& \Gamma = \{(M_1, M_2) \in \mathbb{R}_+^2 \mid h_i(M_1, M_2) \geq 0 \ i = 1, \dots, 5\}
\end{aligned}$$

2.2.1 Properties of the Constraint Set

Figure 1 shows the constraint set Γ , for the case $f(L) = L^2$, on the (M_1, M_2) plane where $M_1, M_2 \in \mathbb{R}_+^2$. It is easy to see that the set $\Gamma \subset \mathbb{R}^2$ is compact (closed and bounded) and non-convex. On the nonlinear section of the boundary $M_2 = p_1 f\left(\frac{M_1}{w_1}\right)$ we have $C_1 = 0$, on the vertical line $M_1 = p_0 f\left(\frac{M_0}{w_0}\right)$ we have $C_0 = 0$ and the horizontal line $M_2 = 0$ is the set of points (M_1, M_2) where $C_2 = 0$. At the corners which are numbered by 1,2,3 we have $C_1 = C_2 = 0, C_1 = C_0 = 0, C_2 = C_0 = 0$ respectively.

At point $(M_1, M_2) \in \text{int}\Gamma$, the distance $d_1 = p_1 f\left(\frac{M_1}{w_1}\right) - M_2 = p_1 C_1$ measures the first period consumption and $d_2 = p_0 f\left(\frac{M_0}{w_0}\right) - M_1 = p_0 C_0$ measures the second period consumption. Distance from the point $(M_1, M_2) = \mathbf{x}$ to the horizontal line (M_1 axis) is a monotone transformation $g(C_2)$ of the third period consumption C_2 . That is $d_3 = M_2 = w_2 f^{-1}(C_2) = g(C_2)$. Note that the production function f is a continuous, strictly increasing function (of L) hence it has an

inverse and $f^{-1} = g$ is also monotone.

2.2.2 Existence of a Solution to Producer's Optimization Problem

If the constraint set Γ were convex and the objective function V were concave and continuous on Γ then the Kuhn-Tucker sufficient conditions would be applicable to our problem. Clearly Γ is not convex and V is not concave, moreover V is not continuous on the boundary, $\partial\Gamma$, for certain types of utility functions like logarithmic ones. Proposition 2 states that even if this is the case, imposing the following condition on V guarantees the existence of a global maximum \mathbf{x}^* of V which is in the interior of the constraint set Γ . The following definition is needed before stating the related condition:

Definition 1 *Let $\Psi : [0, 1] \rightarrow \Gamma$ be a curve and let \mathbf{A} be its tangent vector field. Define the derivative ²*

$$D_{\mathbf{A}}V(\Psi(0)) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \{V(\Psi(\epsilon)) - V(\Psi(0))\}$$

We will impose the following condition on V :

Condition(*): For all $\mathbf{x} \in \partial\Gamma$ there exists $\Psi_{\mathbf{x}} : [0, 1] \rightarrow \Gamma$ with $\Psi_{\mathbf{x}}(0) = \mathbf{x}$ such that $D_{\mathbf{A}}V(\Psi_{\mathbf{x}}(0)) > 0$.

Figure 2 illustrates the Condition (*) on the producer's feasible set.

Proposition 2 *Assume that the function V is continuous on $\text{int}\Gamma$ and that for any given parameter set, V satisfies Condition (*). Then a global maximum $\mathbf{x}^* = (M_1^*, M_2^*) \in \text{int}\Gamma$ exists to the inequality constraint problem $\max V(M_1, M_2)$ over the constraint set Γ . Therefore the point \mathbf{x}^* satisfies the Kuhn-Tucker first order (necessary) conditions for a maximum, that is*

²Indeed this derivative is known as the Lie derivative but we used here a special form of it adapted to scalar functions. (Abraham et al., 1983)

there exists λ^* such that the following conditions are met.

$$[KT1] \quad \lambda_i^* \geq 0 \text{ and } \lambda_i^* h_i(\mathbf{x}^*) = 0 \text{ for } i = 1, \dots, 5$$

$$[KT2] \quad DV(\mathbf{x}^*) + \sum_{i=1}^5 \lambda_i^* Dh_i(\mathbf{x}^*) = 0$$

Proof: Take the point $\bar{\mathbf{x}} \in \partial\Gamma$ such that $V(\bar{\mathbf{x}}) \geq V(\mathbf{x}) \forall \mathbf{x} \in \partial\Gamma$. Such an $\bar{\mathbf{x}}$ exists because we know that the function V is bounded above. Define the set $K_{\mathbf{x},\epsilon} = \{\mathbf{x}' \mid \|\mathbf{x} - \mathbf{x}'\| < \epsilon\} \cap \Gamma$. Since V satisfies Condition (*) \exists at least one $\mathbf{x}_0 \in \text{int}K_{\bar{\mathbf{x}},\epsilon}$ such that $V(\mathbf{x}_0) > V(\bar{\mathbf{x}})$. Note that $\Psi(0) = \bar{\mathbf{x}}$ implies $\bar{\mathbf{x}} \notin \partial\Gamma$ since $\bar{\mathbf{x}}$ is maximal on $\partial\Gamma$. Fix \mathbf{x}_0 and $\epsilon' > 0$ such that $\mathbf{x}_0 \in \partial\Gamma_{\epsilon'}$ where $\Gamma_{\epsilon'} = \Gamma \setminus \bigcup_{\mathbf{x} \in \partial\Gamma} K_{\mathbf{x},\epsilon'}$. (see Figure 3). Note that $\Gamma_{\epsilon'} \subset \Gamma$ is a compact set and V is continuous on $\Gamma_{\epsilon'}$. Then by Weierstrass Theorem $\exists \mathbf{x}^* \in \text{argmax}\{V(\mathbf{x}) \mid \mathbf{x} \in \Gamma_{\epsilon'}\}$ which implies $V(\mathbf{x}^*) \geq V(\mathbf{x}_0) > V(\bar{\mathbf{x}})$. Since V is continuous on the interior of Γ this argument is valid for all $\epsilon > 0$. Hence \mathbf{x}^* maximizes of V on Γ as well. Moreover $\mathbf{x}^* \in \text{int}\Gamma$. \square

Remark 3 Note that this proof is valid regardless of the dimension and geometry of Γ , provided that Γ is compact. This means that we have an existence result for the n -period economy. However there are serious technical difficulties in solving (M_1, \dots, M_n) explicitly. Therefore we preferred to give the existence result for $n = 2$.

For $\mathbf{x}^* \in \text{int}\Gamma$, [KT1] and [KT2] reduce to $DV(\mathbf{x}^*) = 0$. Therefore we have the following set of equations as our F.O.C;

$$-\frac{1}{p_0}U'(f(\frac{M_0}{w_0}) - \frac{M_1}{p_0}) + \frac{\beta}{w_1}U'(f(\frac{M_1}{w_1}) - \frac{M_2}{p_1})f'(\frac{M_1}{w_1}) = 0 \quad (1)$$

$$\frac{\beta}{p_1}U'(f(\frac{M_1}{w_1}) - \frac{M_2}{p_1}) + \frac{\beta^2}{w_2}U'(f(\frac{M_2}{w_2}))f'(\frac{M_2}{w_2}) = 0 \quad (2)$$

If we have a unique point $(\mathbf{x}^*, \lambda^*)$ which satisfies the above F.O.C, it follows that this point also identifies the problem's global maximum \mathbf{x}^* .

Imposing the *Inada Condition*, $\lim_{c \rightarrow 0} U'(C) = \infty$, on the utility function $U(C)$ (instead of Condition (*)) is not enough to ensure that the optimum will not occur at the boundary of (our constraint set) Γ . Two typical examples are $U(C) = \ln C$ and $U(C) = \sqrt{C}$, both of which satisfy the so called “Inada condition”. Somewhat surprisingly $U(C) = \sqrt{C}$ supports this conjecture. If one tries to solve the equations (1) and (2) with $U(C) = \sqrt{C}$ and $f(L) = L^2$, one will see that the existence of the interior solution depends on the parameter values. This violates the statement, in the conclusion of Proposition 2, that there exists an interior global maximum for any given parameter set. For example with all the parameters of interest $\{w_t, p_t, M_0\}_{t=0}^T$, except β , set equal to one, F.O.C leads to the following set of equations to be solved :

$$M_2 = M_1^2 - \frac{1}{4\beta^2} \quad (3)$$

$$M_2 = 4\beta^2 M_1^3 - (4\beta^2 - 1)M_1^2 \quad (4)$$

Equating (3) and (4) we get the cubic equation

$$4\beta^2 M_1^3 - 4\beta^2 M_1^2 + \frac{1}{4\beta^2} = 0 \quad (5)$$

which has double roots when $M_1 = 0$ or $M_1 = \frac{2}{3}$ with $\beta_{critical} = (\frac{27}{64})^{\frac{1}{4}}$. This means that for $\beta < \beta_{critical}$ equation (5) has no positive real solution at all.

The simple reason for $U(C) = \sqrt{C}$ be appearing as a counter example is that the behaviour of the value function $V(M_1, M_2)$ depends on the β values. For $\beta < \beta_{critical}$, V does not satisfy Condition (*), which in fact is a sufficient condition. Following observations can be made :

Case(1): $0 < \beta < \frac{1}{2}$

With $\psi(0) = (0, 0) = \tilde{\mathbf{x}}$, we have $D_{\mathbf{A}}V < 0$ and global maximum occurs at the point $\tilde{\mathbf{x}}$ on the boundary.

Case(2): $\frac{1}{2} \leq \beta < (\frac{27}{64})^{\frac{1}{4}}$

Let $\tilde{\mathbf{x}} = (0, 0)$ and $\bar{\mathbf{x}} = (1, 0)$. Then with $\psi(0) = \mathbf{x}$ for all $\mathbf{x} \in (\tilde{\mathbf{x}}, \bar{\mathbf{x}})$ we have $D_{\mathbf{A}}V < 0$ and maximum occurs at $(1 - \frac{1}{4\beta^2}, 0)$ on the boundary.

Case(3): $\beta \geq (\frac{27}{64})^{\frac{1}{4}}$. In this case maximum occurs at $x^* \in \text{int}\Gamma$.

What we observe here is that when β is low ($\beta < \frac{1}{2}$), the producer chooses to consume all he has in Period 0, that is, does not hold currency to be used for the next period. As β rises, Agent 2 discounts future consumption less and we observe a consumption smoothing behaviour. For $\beta \geq \beta_{critical}$ we have $C_0, C_1, C_2 > 0$.

2.2.3 Unique Solution For Logarithmic Utility

It is straightforward to check that the objective function V satisfies Condition (*) if we choose the instantaneous utility function $U(C)$ from the logarithmic family. Then Proposition 2 says that there exists an interior global maximum x^* of V . For the uniqueness of x^* we will use the following lemma, proof of which can easily be done by using ordinary calculus.

Lemma 4 *Let $g_1, g_2 : [x_0, \infty) \rightarrow \mathbb{R}$, $g_1, g_2 \in C^2[x_0, \infty)$ be two functions satisfying the following conditions:*

- (i) $g_1(x_0) \leq g_2(x_0)$
- (ii) $g_1'(x_0) < g_2'(x_0)$
- (iii) $g_1''(x) > g_2''(x) \quad \forall x > x_0$
- (iv) $\exists \bar{x} > x_0$ such that $g_1(\bar{x}) > g_2(\bar{x})$

Then there exists a unique point $\tilde{x} \in (x_0, \bar{x})$ such that $g_1(\tilde{x}) = g_2(\tilde{x})$.

Proposition 5 *For $U(C) = \ln(C)$ and $f_2(L) = L^2$, the solution to the producers optimal money demand problem is unique.*

Proof: For $U(C) = \ln C$ equations (1) and (2) become

$$\left(-\frac{1}{p_0}\right) \frac{1}{\left(\frac{M_0^2}{w_0^2} - \frac{M_1}{p_0}\right)} + \left(\frac{\beta}{w_1}\right) \frac{1}{\left(\frac{M_1^2}{w_1^2} - \frac{M_2}{p_1}\right)} \left(\frac{2M_1}{w_1}\right) = 0 \quad (6)$$

$$\left(-\frac{\beta}{p_1}\right) \frac{1}{\left(\frac{M_1^2}{w_1^2} - \frac{M_2}{p_1}\right)} + \left(\frac{\beta^2}{w_2}\right) \frac{1}{\left(\frac{M_2^2}{w_2^2}\right)} \left(\frac{2M_2}{w_2}\right) = 0 \quad (7)$$

Let $M_1 \equiv x$ and solve for M_2 in terms of M_1 in (6) and (7) to get

$$g_1(x) = \frac{p_1(1+2\beta)}{w_1^2}x^2 - \frac{2\beta p_0 p_1 M_0^2}{w_0^2 w_1^2}x$$

$$g_2(x) = \frac{2\beta p_1}{(1+2\beta)w_1^2}x^2$$

It is easy to check that with $x_0 = 0$ and $\bar{x} = p_0 f(\frac{M_0}{w_0}) = p_0 \frac{M_0^2}{w_0^2}$, conditions (i – iv) of Lemma 4 are satisfied. This means that equations (6) and (7) can be solved to find the unique solution $x^* = (M_1^*, M_2^*)$ where

$$M_1^* = \frac{2\beta(1+2\beta)}{(1+2\beta+4\beta^2)}p_0 \frac{M_0^2}{w_0^2}$$

$$M_2^* = \frac{(2\beta)^3(1+2\beta)}{(1+2\beta+4\beta^2)^2} \frac{p_0^2 p_1}{w_1^2 w_0^4} M_0^4$$

which is the unique global maximum of our optimization problem with $U(C) = \ln C$. \square

3 General equilibrium

3.1 Existence of Competitive Equilibrium

Let $M_0 \equiv \bar{M} > 0$ be the total money stock and $\bar{L} > 0$ be the labor endowment of Agent 1.

Proposition 6 *An equilibrium exists in this economy with $U(C) = \ln C$ and $f(L) = L^2$ and is given by*

$$w_t = \frac{\bar{M}}{\bar{L}} \quad \forall t$$

$$p_0 = \frac{\bar{M}}{\bar{L}^2} \frac{(1+2\beta+4\beta^2)}{2\beta(1+2\beta)}$$

$$p_1 = \frac{\bar{M}}{\bar{L}^2} \frac{(1+2\beta)}{2\beta}$$

$$\begin{aligned}
p_2 &= \bar{p}_2 \in \left(\frac{\bar{M}}{f(\bar{L})}, \infty \right) \\
L_t^d &= L_t^s = \bar{L} \quad \forall t \\
q_t^d &= q_t^s = \frac{\bar{M}}{p_t} \quad \text{for } t = 0, 1 \\
M_{1,t+1} &= 0 \quad M_{2,t+1} = \bar{M} \quad \text{for } t = 0, 1 \\
C_{1,t} &= q_t^d = \frac{\bar{M}}{p_t} \quad \forall t \\
C_{2,t} &= f(\bar{L}) - \frac{\bar{M}}{p_t} \quad \text{for } t = 0, 1 \\
C_{2,t} &= f(\bar{L}) \quad t = 2
\end{aligned}$$

Proof: For money market clearing we must have $M_{2,t+1} = \bar{M}$ for $t = 0, 1$ in equilibrium since $M_{1,t+1} = 0$ for $t = 0, 1$. Note that such a money holding plan is feasible and optimal for Agent 2 by Proposition 2. It is feasible because $(M_1^*, M_2^*) \in \text{int}\Gamma$ hence budget constraints are satisfied at (M_1^*, M_2^*) . It is optimal because (M_1^*, M_2^*) maximizes the discounted sum of utilities, V . Therefore from the money market clearing we have the following two equations:

$$\bar{M} = \frac{2\beta(1+2\beta)}{(1+2\beta+4\beta^2)} p_0 \frac{\bar{M}^2}{w_0^2} \quad (8)$$

$$\bar{M} = \frac{(2\beta)^3(1+2\beta)}{(1+2\beta+4\beta^2)^2} \frac{p_0^2 p_1}{w_1^2 w_0^4} \bar{M}^4 \quad (9)$$

Labor market clearing conditions $L_t^s = \bar{L} = L_t^d = \frac{\bar{M}}{w_t} \quad \forall t$ can be used to find the money wages w_t :

$$w_0 = w_1 = w_2 = \frac{\bar{M}}{\bar{L}} \quad (10)$$

Substituting (10) into (8) and (9) prices p_t can be solved:

$$p_0 = \frac{\bar{M} (1+2\beta+4\beta^2)}{\bar{L}^2 2\beta(1+2\beta)}$$

$$p_1 = \frac{\bar{M}(1+2\beta)}{\bar{L}^2 2\beta}$$

Optimality For Agent 1:

(i) Supplying $\bar{L} > 0$ for all t is always optimal for Agent 1 because his utility is strictly increasing in L_t^s . To see this consider the Agent 1's optimization problem:

$$\max \sum_{t=0}^{T=2} \beta^t U\left(\frac{M_t - M_{t+1}}{p_t} + \frac{w_t}{p_t} L_t^s\right)$$

subject to for all t

$$\left(\frac{M_t - M_{t+1}}{p_t} + \frac{w_t}{p_t} L_t^s\right) = q_t^d$$

$$L_t^s \leq \bar{L}$$

$$M_{1,t+1} = M_{1,t} + w_t L_t^s - p_t q_t^d$$

where $M_t, L_t^s \geq 0$ and $M_{1,0} = 0$ is given. Since $\frac{w_t}{p_t} > 0$ and $U'(\cdot) > 0$, $U\left(\frac{w_t}{p_t} L_t^s\right)$ increases if L_t^s increases. Therefore supplying \bar{L} is optimal for Agent 1.

(ii) Holding zero currency at each period is optimal for Agent 1, when the following condition is satisfied at each period:

$$U'(C_t) > \frac{p_t}{p_{t+1}} \beta U'(C_{t+1})$$

In Period 0, with $U(C) = \ln C$ and $C_t = \frac{\bar{M}}{p_0}$, $p_t = p_0$, $p_{t+1} = p_1$, $C_{t+1} = \frac{\bar{M}}{p_1}$ above condition becomes $\beta < 1$ therefore it is automatically satisfied. For the other periods same argument applies.

Optimality For Agent 2:

Last period deserves attention. Agent 2 has two choices:

(i) He does not produce apples and uses his money to purchase apples from the government at \bar{p}_2 .

(ii) he hires labor (\bar{L}), produces ($f(\frac{\bar{M}}{w_2}) = f(\bar{L})$) and consumes all.

We want to make Agent 2 to hire labor and produce apples so \bar{p}_2 must be set to satisfy

$$U\left(\frac{\bar{M}}{\bar{p}_2}\right) < U\left(f\left(\frac{\bar{M}}{w_2}\right)\right)$$

Since $U'(\cdot) > 0$ this means $\bar{p}_2 > \frac{\bar{M}}{f(\bar{L})}$. So with the last periods price p_2 is set at \bar{p}_2 , sufficiently high, Agent 2 will hire labor \bar{L} , produce $f(\bar{L})$ and consumes all. Government sells apples to Agent 1 at \bar{p}_2 , thus agent1 consumes $c_{1,2} = \frac{\bar{M}}{\bar{p}_2}$ which is clearly decreasing in \bar{p}_2 .

Remark 7 *Since we have $\bar{p}_2 \in (\frac{\bar{M}}{f(\bar{L})}, \infty)$ the government can use arbitrarily small amount of resources to back the currency by setting \bar{p}_2 as high as she wishes.*

Indeed the condition $\bar{p}_2 > \frac{\bar{M}}{f(\bar{L})}$ can be taught of as a *profitability condition* and should hold at each period. But when we look at the equilibrium prices p_0, p_1 we see that this condition is automatically satisfied for the other periods.

It is now clear that Agent 1 and Agent 2 are maximized at the described equilibrium. This completes the proof. \square

Remark 8 *We see that equilibrium prices p_0 and p_1 decrease with β_2 so with an impatient firm type (low β) p_0 and p_1 will be higher at the equilibrium reducing the real value of apples.*

3.2 Uniqueness of Competitive Equilibrium

In section 2.2.3 we showed that optimal allocations (M_1^*, M_2^*) are unique for $U(C) = \ln C$ and $f(L) = L^2$. Money and labor market clearing conditions were used to find equilibrium prices p_0, p_1 .

Proposition 9 *Prices p_0 and p_1 are the unique prices that support the competitive equilibrium which is given by Proposition 6.*

Proof: At the equilibrium we have $w_t = \frac{\bar{M}}{\bar{L}}$ with $M_1, M_2 = \bar{M}$.

Taking the total differentials in equations (1) and (2) we get

$$\left\{ \frac{\beta}{p_1^2} U'(C_1) + \frac{\bar{M}}{p_1^2} U''(C_1) \left(\frac{-\beta}{p_1} \right) \right\} dp_1 = 0 \quad (11)$$

$$\left\{ \frac{1}{p_0^2} U'(C_0) - \frac{\bar{M}}{p_0^3} U''(C_0) \right\} dp_0 + \frac{\beta \bar{L}}{p_1^2} f'(\bar{L}) U''(C_1) dp_1 = 0 \quad (12)$$

From (11) and (12) we have

$$dp_1 = 0 \text{ or } p_1 = \frac{\bar{M} U''(C_1)}{U'(C_1)} < 0 \quad (13)$$

$$\frac{dp_0}{dp_1} = - \frac{\frac{\beta \bar{L}}{p_1^2} f'(\bar{L}) U''(C_1)}{\frac{1}{p_0^2} U'(C_0) - \frac{\bar{M}}{p_0^3} U''(C_0)} \quad (14)$$

Looking at (14) we see that $\frac{dp_0}{dp_1} < 0$ iff $p_0 < \frac{\bar{M} U''(C_0)}{U'(C_0)} < 0$ which is impossible by the assumption of strictly positive prices. Again by (14) we have $\frac{dp_0}{dp_1} > 0$ iff $p_0 > \frac{\bar{M} U''(C_0)}{U'(C_0)}$ which is trivial. For the same reason (13) implies $dp_1 = 0$, that is, p_1 is constant. Therefore equations (11) and (12) can be solved to determine the *unique* solution (p_0, p_1) which is given by Proposition 6 at the beginning of this Section. \square

4 Conclusion

Concerning the structure of the model used here, our existence results are not directly comparable with the ones in the existing literature. Almost all of the general equilibrium papers on increasing returns use standard assumptions of the classical complete markets setup of the Arrow-Debreu model except for convexity of the production set. Existence issue is analyzed in this framework and results are obtained when firms follow special pricing rules without necessarily maximizing profits. Moreover, important part of the theory is devoted to the efficiency considerations (in the context of the second welfare theorem) which is not studied here. We, in a competitive setup with incomplete markets, show the existence of equilibrium under increasing returns with firm type agent making positive profits.

The basic explanation for this nonstandard result is as follows. We assume that factor payments must be paid in cash and producer can not use the money earned from selling output in the goods market within the same period, to pay for factor services. This limits the demand for labor. Therefore producer does not face unbounded profit opportunities because there is an upper bound on the labor input to be used in production. This limited participation assumption as well as the finite time horizons are responsible for our existence result.

If one looks at the equilibrium prices, one will see that the last period's price \bar{p}_2 can be set arbitrarily large without distorting the equilibrium. This means that we have an equilibrium, in a finite horizon economy, with valued fiat money. There are many infinite horizon examples in the literature on the existence of equilibria (even without the cash-in-advance constraints) with valued fiat money (e.g., Benveniste and Cass, 1986). But it is well known that the finite horizon assumption makes the value of money zero because agents do not want to hold money near the horizon. For this reason it is interesting to see in our model the possibility that even if the horizon is finite, individuals may want to hold money, under a negligible cost to the government of backing currency.

It would be a natural extension to search for the competitive equilibrium with infinitely lived agents. Unfortunately, non-concavity of objective function causes problems in the application of dynamic optimization techniques. It is not impossible to overcome this technical difficulty but one should not expect to get the the existence result easily. Indeed Sotomayor (1987) claims that, under certain restrictions, the value function for the dynamic optimization problem (resulting from a discrete time one-good model of optimal accumulation) is concave and the optimal stationary policy exhibits properties similar to that obtained in the model where the technology is assumed to be convex. However, Roy (1993) shows that the conditions on the utility and production function functions imposed in Sotomayor's paper are insufficient to ensure the results claimed about the concavity of the value function and other classical

properties. These findings suggests that existence issue in infinite horizon models still deserves further investigation and it may very well be the case to have indeterminacy with infinite horizon. Nevertheless concerning the structure of the model, the solution technique introduced and results obtained, our work is a new contribution to the literature when horizon is finite.

There are some papers dealing with existence of equilibrium under increasing returns but they are different in one important aspect; in the assumption on the type of increasing returns. They allow either an initial face of increasing returns or an aggregate increasing returns with individual firms having CRS technology (external economies of scale). For example Majumdar and Mitra (1993) have some existence results for a dynamic optimization example with a non-convex technology in the case of a linear objective function but the non-convexity is such that production function exhibits an initial phase of increasing returns.

As mentioned in Section 2, our existence result easily extends to an $n - period$ economy. However there are technical difficulties in solving the variables M_1, \dots, M_n explicitly. A potential future research, in spite of the technical difficulties mentioned above, would be carry out the analysis in the infinite horizon case and see its implications on the nature of the Bellman's value function.

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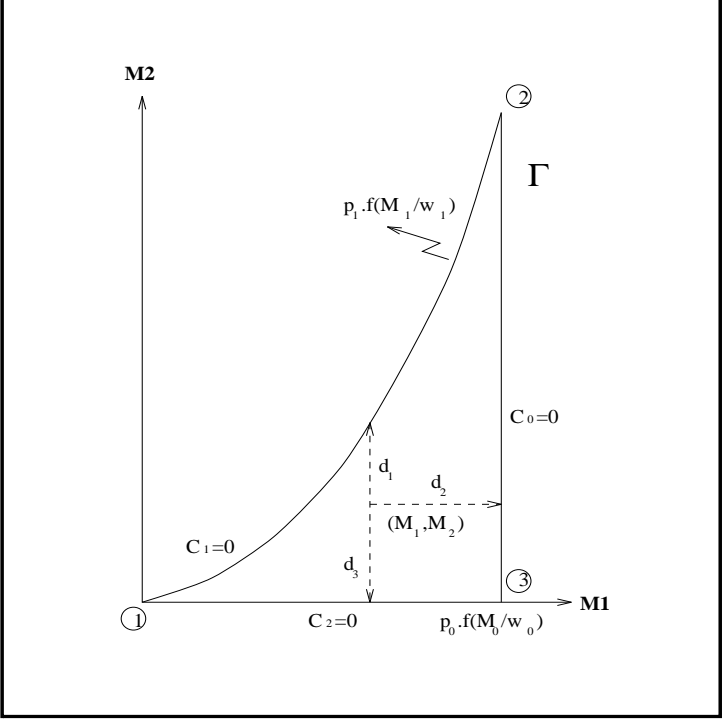


Figure 1: Constraint Set

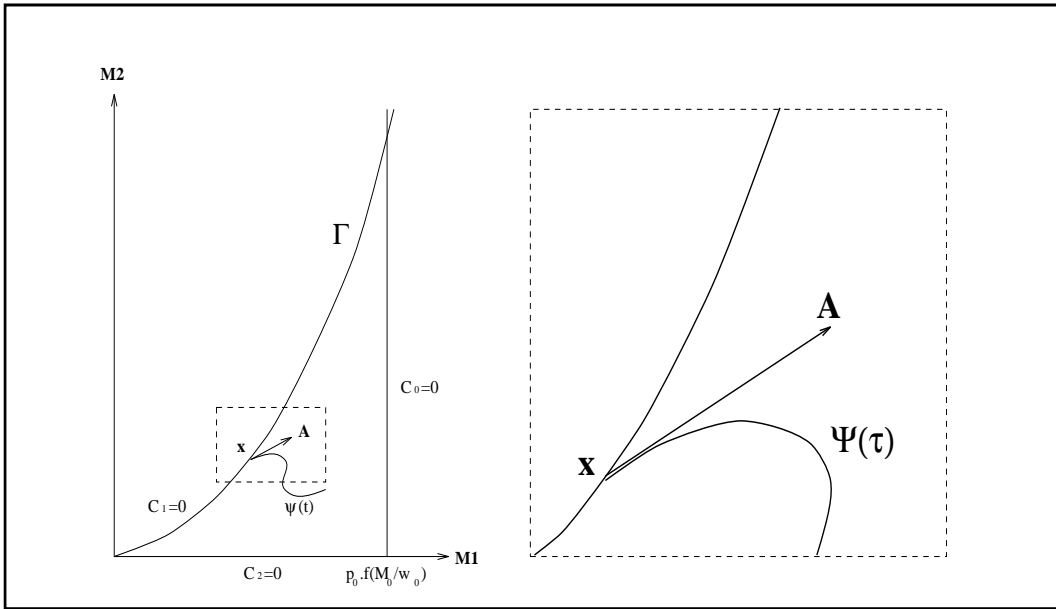


Figure 2: Illustration of **Condition***

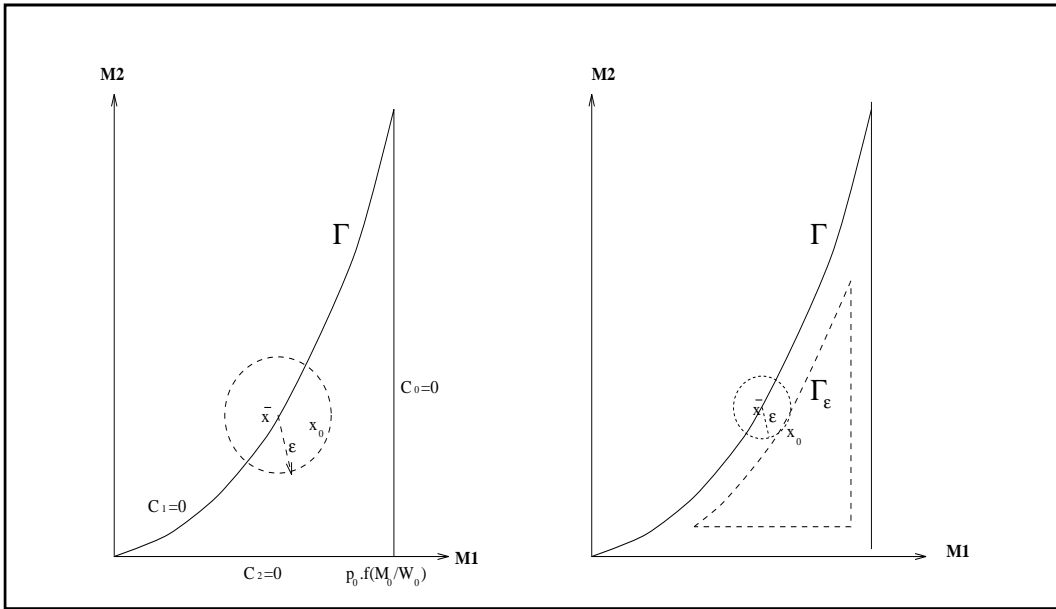


Figure 3: Illustration of the proof of proposition 2