Implications of Unequal Rates of Population Growth for Trade

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Abstract

We study the closed-form solutions obtained from a two-sector overlapping generations model to explore the possible implications of population growth for the patterns of commodity and factor flows between economies that are identical except for the population growth rates.

Keywords: Population growth rate; Overlapping-generations; General equilibrium

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1 Introduction

By the static 2x2x2 Heckscher-Ohlin (HO) model of international trade, differences in relative factor endowments across countries suffice to render trade Pareto-superior to autarky, as long as the factor intensity of production is different for each commodity. This standard model, however, fails to describe the long-run trade equilibrium, since trade itself would, in the long-run, eliminate the initial differences between relative factor endowments of trading countries that are assumed to be identical in every other respect. This implies that for trade to continue to occur in the long-run, additional differences would be needed to make factor proportions evolve over time. Differential speed of population growth as actually observed between different regions in the world provides an empirically relevant example to factors that could lead to such an evolution in relative factor endowments, guaranteeing the continuation of trade in the long-run.[1]

This paper looks into the role that the differences in the population growth rates across nations could play as a determinant of long-run comparative advantages, and discusses the validity of welfare predictions of the static HO model in the long-run. For this purpose, we consider a world that is made up of two economies each producing two commodities, using two factors of production and are identical in every other respect than the population growth rates. This enables us to focus on the dynamic autarky equilibrium in one country, and project the results on the sensitivity of steady-state values of endogenous variables to changes in the population growth rate into a two-country set up. The economies we consider are populated by individuals that live for two periods, and the population in each is allowed to grow at a constant rate that is different than the other. Such an

[1] United Nations projections indicate that the existing gap between the population growth rates in the developing and developed parts of the world will continue to diminish over time, but is likely to remain visible even beyond the year 2050. Changes in relative factor endowments due to the differential speed of demographic transition in these two areas gradually become a major factor to affect future patterns of trade, even though the dynamic trade literature is just beginning to address this issue [6].

overlapping generations (OLG) structure implicitly allows saving behaviour of the individuals to change over the working and retirement phases of the life cycle, thereby letting relative factor endowments to evolve, due not only to the changes in labour supply, but also to the changes in capital accumulation resulting from the adjustments of savings to the changing age profile of population.

The organization of the paper is as follows. The next section describes the model. Section 3 presents the closed-form solutions, and discusses the implications of population growth. Section 4 concludes the paper.

2 The Model

2.1 Consumption and Saving

At every period t, a generation made up of N_t individuals is born. Population grows at the rate n so that $N_t = (1+n)N_{t-1}$. For all periods t, individuals born and living the first period of their lives at time t inelastically supply a fixed amount of labour, $\bar{l}$, earn labour income at the competitive wage rate, w_t, and decide on how to allocate it between first period consumption of goods 1 and 2 (c_{1yt}, c_{2yt}), and savings, s_t, which bring interest earnings at the rate of r_{t+1} the next period. In the second period, they retire and consume c_{1ot+1} units of good 1, and c_{2ot+1} units of good 2 by spending all their capital income from previous period's savings. Given the price, p_t, of the consumption good (good 2) in terms of the investment-consumption good (good 1) at time t, each individual solves the following problem:

$$\begin{aligned} &max && (c_{1yt}^{\theta}c_{2yt}^{1-\theta})^{\mu}(c_{1ot+1}^{\theta}c_{2ot+1}^{1-\theta})^{1-\mu} && \\ &\text{subject to} && c_{1yt}+p_tc_{2yt}+\tfrac{1}{1+r_{t+1}}(c_{1ot+1}+p_{t+1}c_{2ot+1}) && = w_t\bar{l}, \\ & && c_{1yt}, c_{2yt}, c_{1ot+1}, c_{2ot+1} && \geq 0, \end{aligned} \tag{1}$$

where $0 < \theta < 1$ and $0 < \mu < 1$.

The solution to this problem results in the following consumption decisions:

$$\begin{aligned} c_{1yt} &= \mu\theta w_t \bar{l}, \\ c_{2yt} &= \mu(1-\theta)\tfrac{w_t\bar{l}}{p_t}, \\ c_{1ot+1} &= (1-\mu)\theta(1+r_{t+1})w_t\bar{l}, \\ c_{2ot+1} &= (1-\mu)(1-\theta)(1+r_{t+1})\tfrac{w_t\bar{l}}{p_{t+1}}. \end{aligned} \tag{2}$$

2.2 Production

Both the investment-consumption good and the consumption good are produced according to constant returns to scale Cobb-Douglas production technologies by using capital, K, and labour, L, such that sectoral outputs X_{1t} and X_{2t} are given in per capita terms by

$$x_{1t} = k_{1t}^{\alpha} l_{1t}^{1-\alpha}, \tag{3}$$

$$x_{2t} = k_{2t}^{\beta} l_{2t}^{1-\beta}, \tag{4}$$

for $0 < \alpha < 1$, $0 < \beta < 1$ and $\alpha \neq \beta$ where $x_{it} = \frac{X_{it}}{N_t}$, $k_{it} = \frac{K_{it}}{N_t}$, $l_{it} = \frac{L_{it}}{N_t}$, for $i = 1, 2$ and total labour supply at time t, $L_t = N_t\bar{l}$. Factor market equilibrium requires that

$$k_{1t} + k_{2t} = k_t, \tag{5}$$

$$l_{1t} + l_{2t} = \bar{l}. \tag{6}$$

The demands for labour and capital in each sector are characterized by the first order conditions for profit maximization. If labour and capital are perfectly mobile across sectors and if both goods are produced, then

$$r_t = \alpha k_{1t}^{\alpha-1} l_{1t}^{1-\alpha} = p_t \beta k_{2t}^{\beta-1} l_{2t}^{1-\beta}, \tag{7}$$

$$w_t = (1-\alpha) k_{1t}^{\alpha} l_{1t}^{-\alpha} = p_t(1-\beta) k_{2t}^{\beta} l_{2t}^{-\beta}. \tag{8}$$

Solution of the producers' problem gives

$$l_{1t} = \frac{\delta_t \bar{l} - k_t}{\delta_t - \epsilon_t}, \tag{9}$$

$$l_{2t} = \frac{k_t - \epsilon_t \bar{l}}{\delta_t - \epsilon_t}, \tag{10}$$

$$k_{1t} = \epsilon_t \frac{\delta_t \bar{l} - k_t}{\delta_t - \epsilon_t}, \tag{11}$$

$$k_{2t} = \delta_t \frac{k_t - \epsilon_t \bar{l}}{\delta_t - \epsilon_t}, \tag{12}$$

where

$$\epsilon_t = p_t^{\frac{1}{\alpha-\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\beta}{\alpha-\beta}} \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1-\beta}{\alpha-\beta}}, \tag{13}$$

$$\delta_t = p_t^{\frac{1}{\alpha-\beta}} \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{1-\beta}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha-\beta}}. \tag{14}$$

Hence,

$$r_t = \alpha \epsilon_t^{\alpha-1} = p_t \beta \delta_t^{\beta-1}, \tag{15}$$

$$w_t = (1-\alpha)\epsilon_t^{\alpha} = p_t(1-\beta)\delta_t^{\beta}, \tag{16}$$

and per capita outputs can now be written as

$$x_{1t} = l_{1t}\epsilon_t^{\alpha}, \tag{17}$$

$$x_{2t} = l_{2t}\delta_t^{\beta}. \tag{18}$$

2.3 The Autarky Equilibrium

A perfect-foresight equilibrium is a sequence $\{k_t, p_t\}_{t=0}^{\infty}$ that clears the goods' markets at every period t, while satisfying the dynamics of the capital stock at time $t+1$. The fraction of income saved during the first period of life is $1-\mu$. Thus, the evolution of per capita capital is governed by

$$k_{t+1} = \frac{(1-\mu)w_t\bar{l}}{(1+n)}. \tag{19}$$

The clearance of the goods' market in period t requires that the per capita supply of each good be equal to its respective per capita demand. Hence,

$$x_{1t} = c_{1yt} + \frac{1}{(1+n)}c_{1ot} + (1+n)k_{t+1} - k_t, \tag{20}$$

$$x_{2t} = c_{2yt} + \frac{1}{(1+n)}c_{2ot}. \tag{21}$$

Walras' law allows us to focus on the market clearance condition for the consumption good (good 2) alone. Substituting c_{2yt} and c_{2ot} from (2) into (21), using (10), (14), (15), (16), and remembering that $x_{2t} = l_{2t}\delta_t^{\beta}$ from (18), one obtains

$$k_t = \phi_1 p_t^{\frac{1}{\alpha-\beta}} + \phi_2 p_{t-1}^{\frac{\alpha}{\alpha-\beta}} p_t^{\frac{1-\alpha}{\alpha-\beta}} + \phi_3 p_{t-1}^{\frac{\alpha}{\alpha-\beta}}, \tag{22}$$

where

$$\begin{aligned}
\phi_1 &= \mu(1-\theta)(1-\beta)\bar{l}\Psi + \gamma^{\frac{\beta}{\alpha-\beta}} \eta^{\frac{1-\beta}{\alpha-\beta}}, &(23)\\
\phi_2 &= \frac{(1-\mu)(1-\theta)(1-\beta)\bar{l}}{1+n}\Psi, &(24)\\
\phi_3 &= \frac{(1-\mu)(1-\theta)(1-\beta)\beta\bar{l}}{1+n}\Psi\gamma^{\frac{\alpha(\beta-1)}{\alpha-\beta}} \eta^{\frac{(1-\beta)(\alpha-1)}{\alpha-\beta}}, &(25)\\
\Psi &= (\gamma^{\frac{\alpha}{\alpha-\beta}} \eta^{\frac{1-\alpha}{\alpha-\beta}} - \gamma^{\frac{\beta}{\alpha-\beta}} \eta^{\frac{1-\beta}{\alpha-\beta}}), &(26)\\
\gamma &= \frac{\beta}{\alpha}, &(27)\\
\eta &= \frac{1-\beta}{1-\alpha}. &(28)
\end{aligned}$$

Now, combining (14) and (16), and substituting into (19), per capita capital dynamics equation can simply be written as

$$k_{t+1} = \phi_4 p_t^{\frac{\alpha}{\alpha-\beta}}, \tag{29}$$

where

$$\phi_4 = \frac{(1-\mu)(1-\beta)\bar{l}}{1+n}\gamma^{\frac{\alpha\beta}{\alpha-\beta}} \eta^{\frac{\beta(1-\alpha)}{\alpha-\beta}}. \tag{30}$$

Remembering (22), and using (29) one can obtain a nonlinear difference equation in terms of price ratios only. This equation characterizes the dynamics of our model economy and is given by

$$(\phi_4 - \phi_3) p_t^{\frac{\alpha}{\alpha-\beta}} = \phi_1 p_{t+1}^{\frac{1}{\alpha-\beta}} + \phi_2 p_t^{\frac{\alpha}{\alpha-\beta}} p_{t+1}^{\frac{1-\alpha}{\alpha-\beta}}. \tag{31}$$

3 Closed Form Solutions

3.1 Steady-State Values of Key Variables

The equilibrium steady state value of p_s satisfies (31) with $p_{t+1} = p_t = p_s$. Ruling out $p_s = 0$, p_s is given by

$$p_s = \Phi^{\frac{\alpha-\beta}{1-\alpha}} \tag{32}$$

where

$$\Phi = \left(\frac{\phi_4 - \phi_3}{\phi_1 + \phi_2}\right)^{\frac{\alpha-\beta}{1-\alpha}}.$$

Proposition 1 *The equilibrium price ratio, p_s, for this perfect foresight overlapping-generations general equilibrium model with constant returns to scale production exists and is unique for any given values of* $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$, $0 < \bar{l} < 1$, *and* $n > 0$.[2]

Proof:

The analytical solution of the steady state price ratio p_s is easily obtained by rearranging terms of (31) into

$$\phi_4 - \phi_3 = \left(\phi_1 \left(\frac{p_{t+1}}{p_t}\right)^{\frac{\alpha}{\alpha-\beta}} + \phi_2\right) p_{t+1}^{\frac{1-\alpha}{\alpha-\beta}}. \tag{33}$$

Hence,

$$p_{t+1} = \left(\frac{\phi_4 - \phi_3}{\phi_1 \left(\frac{p_{t+1}}{p_t}\right)^{\frac{\alpha}{\alpha-\beta}} + \phi_2}\right)^{\frac{\alpha-\beta}{1-\alpha}}. \tag{34}$$

Since the steady state value of p_s is the one satisfying $p_{t+1} = p_t = p_s$ and (31), that

$$p_s = \Phi^{\frac{\alpha-\beta}{1-\alpha}}, \quad \text{where} \quad \Phi = \left[\frac{\phi_4-\phi_3}{\phi_1+\phi_2}\right]^{\frac{\alpha-\beta}{1-\alpha}}. \tag{35}$$

Now, (35) shows that p_s is unique for any given set of parameter values. For existence, we also need to show that p_s is positive for any $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$, $0 < \bar{l} < 1$ and $n > 0$.

[2]This result is broadly consistent with Galor [3].

So, $\Phi > 0$ if and only if $\phi_4 - \phi_3 > 0$ and $\phi_1 + \phi_2 > 0$ or $\phi_4 - \phi_3 < 0$ and $\phi_1 + \phi_2 < 0$.

Now $\phi_4 - \phi_3 > 0 \Leftrightarrow$

$$\frac{(1-\mu)(1-\beta)\bar{l}}{(1+n)}\gamma^{\frac{\alpha\beta}{\alpha-\beta}}\eta^{\frac{\beta(1-\alpha)}{\alpha-\beta}} \quad > \quad \frac{(1-\mu)(1-\theta)(1-\beta)\beta\bar{l}}{(1+n)}\Psi\gamma^{\frac{\alpha(\beta-1)}{\alpha-\beta}}\eta^{\frac{(1-\beta)(\alpha-1)}{\alpha-\beta}}.$$

Substituting Ψ and simplifying, we get

$$1 > (1-\theta)\beta - (1-\theta)\beta\frac{\eta}{\gamma}.$$

Substituting γ and η in the above expression and rearranging terms, we obtain

$$\frac{1-\alpha\theta}{1-\theta} > \beta.$$

Since $\frac{1-\alpha\theta}{1-\theta} > 1$ and $0 < \beta < 1$, $\frac{1-\alpha\theta}{1-\theta} > \beta$ holds for any given α, β and θ.

Thus $\phi_4 - \phi_3 > 0$ for any given $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$, $0 < \bar{l} < 1$ and $n > 0$.

Similarly, $\phi_1 + \phi_2 > 0 \Leftrightarrow$

$$\mu(1-\theta)(1-\beta)\bar{l}\Psi + \gamma^{\frac{\beta}{\alpha-\beta}}\eta^{\frac{1-\beta}{\alpha-\beta}} + \frac{(1-\mu)(1-\theta)(1-\beta)\bar{l}}{1+n}\Psi \quad > \quad 0. \tag{36}$$

Substituting Ψ in the above expression, simplifying with the expressions for γ and η in mind, and rearranging terms, we get

$$1 - \frac{1+n}{(1+\mu n)(1-\theta)\bar{l}} < \frac{\beta}{\alpha}.$$

Since

$$\frac{1+n}{(1+\mu n)(1-\theta)\bar{l}} > 1,$$

it follows that

$$1 - \frac{1+n}{(1+\mu n)(1-\theta)\bar{l}} < 0.$$

But

$$\frac{\beta}{\alpha} > 0,$$

and hence,

$$\frac{\beta}{\alpha} > 1 - \frac{1+n}{(1+\mu n)(1-\theta)\bar{l}}$$

holds for any given values of α, β, μ, θ, $\bar{l}$ and n. Thus, $\phi_1 + \phi_2 > 0$ for any given α, β, μ, θ, $\bar{l}$ and n. Therefore $p_s > 0$ for any given α, β, μ, θ, $\bar{l}$ and n, where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < \mu < 1$, $0 < \theta < 1$, $0 < \bar{l} < 1$, and $n > 0$. ∎

Consequently, the closed form solutions for the steady state per capita values are obtained as

$$
\begin{aligned}
k_s &= \phi_4 \Phi^{\frac{\alpha}{1-\alpha}}, && (37)\\
w_s &= \frac{(1+n)}{(1-\mu)\bar{l}} \phi_4 \Phi^{\frac{\alpha}{1-\alpha}}, && (38)\\
r_s &= \alpha \gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \frac{1}{\Phi}, && (39)\\
c_{1ys} &= \frac{\mu\theta(1+n)}{(1-\mu)} \phi_4 \Phi^{\frac{\alpha}{1-\alpha}}, && (40)\\
c_{2ys} &= \frac{\mu(1-\theta)(1+n)}{(1-\mu)} \phi_4 \Phi^{\frac{\beta}{1-\alpha}}, && (41)\\
c_{1os} &= (1+n)\theta\phi_4 \Phi^{\frac{\alpha}{1-\alpha}} (1+r_s), && (42)\\
c_{2os} &= (1+n)(1-\theta)\phi_4 \Phi^{\frac{\beta}{1-\alpha}} (1+r_s). && (43)
\end{aligned}
$$

3.2 The Effect of the Population Growth Rate

In order to analytically examine the effect of population growth rate, we need to identify the signs of $\frac{\partial \Phi}{\partial n}$ and $\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right)$. Using (23), (24), (25) and (30), we have

$$
\frac{\phi_4 - \phi_3}{\phi_1 + \phi_2} = \frac{(1-\mu)(1-\beta)\bar{l}(\gamma^{\frac{\alpha\beta}{\alpha-\beta}} \eta^{\frac{\beta(1-\alpha)}{\alpha-\beta}} - (1-\theta)\beta\Psi\gamma^{\frac{\alpha(\beta-1)}{\alpha-\beta)}} \eta^{\frac{(1-\beta)(\alpha-1)}{\alpha-\beta}})}{(1-\theta)(1-\beta)\bar{l}\Psi[(1+n)\mu + (1-\mu)] + (1+n)\gamma^{\frac{\beta}{\alpha-\beta}} \eta^{\frac{1-\beta}{\alpha-\beta}}}.
$$

Given that Ψ can be expressed as

$$
\begin{aligned}
\Psi &= \gamma^{\frac{\alpha}{\alpha-\beta}} \eta^{\frac{1-\alpha}{\alpha-\beta}} \frac{(\beta-\alpha)}{(1-\alpha)\beta} \quad \text{or} && (44)\\
\Psi &= \gamma^{\frac{\beta}{\alpha-\beta}} \eta^{\frac{1-\beta}{(\alpha-\beta)}} \frac{(\beta-\alpha)}{(1-\beta)\alpha}, && (45)
\end{aligned}
$$

plugging (44) into the numerator and (45) into the denominator gives, after rearranging terms

$$
\frac{\phi_4 - \phi_3}{\phi_1 + \phi_2} = \frac{(1-\mu)(1-\beta)\bar{l}\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{\beta(2-\alpha)-1}{\alpha-\beta}} \left(1 - \frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}\right)}{(1-\theta)(\beta-\alpha)\frac{\bar{l}}{\alpha}(1+\mu n) + 1 + n}.
$$

So,

$$\frac{\partial \Phi}{\partial n} = -\frac{(1-\mu)(1-\beta)\bar{l}\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{\beta(2-\alpha)-1}{\alpha-\beta}}\left(1-\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}\right)\left((1-\theta)(\beta-\alpha)\bar{l}\frac{\mu}{\alpha}+1\right)}{[(1-\theta)(\beta-\alpha)\frac{\bar{l}}{\alpha}(1+\mu n)+1+n]^2}.$$

Since $0 < \beta < 1$ and $0 < \alpha < 1$, $-\alpha < \beta - \alpha < 1 - \alpha$.

We also have

$$0 < (1-\theta)\bar{l}\frac{\mu}{\alpha}.$$

So,

$$-\alpha(1-\theta)\bar{l}\frac{\mu}{\alpha} < (\beta-\alpha)(1-\theta)\bar{l}\frac{\mu}{\alpha}.$$

Thus,

$$-(1-\theta)\bar{l}\mu < (1-\theta)(\beta-\alpha)\bar{l}\frac{\mu}{\alpha}.$$

But

$$(1-\theta)\bar{l}\mu < 1.$$

Hence,

$$0 < (1-\theta)(\beta-\alpha)\bar{l}\frac{\mu}{\alpha} + 1.$$

Similarly, we have $\frac{\beta-\alpha}{1-\alpha} < 1$, and $0 < 1-\theta < 1$, so $\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)} < (1-\theta) < 1$.

Thus,

$$0 < 1 - \frac{(1-\theta)(\beta-\alpha)}{1-\alpha}.$$

Therefore,

$$\frac{\partial \Phi}{\partial n} < 0.$$

As for the sign of $\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right)$, (23), (24), (25) and (30) together imply

$$\frac{1}{\Phi} = \frac{\phi_1+\phi_2}{\phi_4-\phi_3} = \frac{(1-\theta)(\beta-\alpha)\frac{\bar{l}}{\alpha}(1+\mu n)+1+n}{(1-\mu)(1-\beta)\bar{l}\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{\beta(2-\alpha)-1}{\alpha-\beta}}\left(1-\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}\right)}. \quad (46)$$

Hence,

$$\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right) = \frac{(1-\theta)(\beta-\alpha)\frac{\bar{l}}{\alpha}\mu+1}{(1-\mu)(1-\beta)\bar{l}\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{\beta(2-\alpha)-1}{\alpha-\beta}}\left(1-\frac{(1-\theta)(\beta-\alpha)}{(1-\alpha)}\right)}. \quad (47)$$

It was previously shown that $(1-\theta)(\beta-\alpha)\frac{\bar{l}}{\alpha}\mu+1>0$ and $1-\frac{(1-\theta)(\beta-\alpha)}{1-\alpha}>0$. In fact,

$$\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right) = -\frac{1}{\Phi^2}\frac{\partial \Phi}{\partial n}. \tag{48}$$

Thus,

$$\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right) > 0.$$

Corollary 1 *The equilibrium price ratio, p_s, is decreasing in the population growth rate, n, if the relatively capital-intensive sector is sector 1, and is increasing in the population growth rate, if it is sector 2.*

The effect of the population growth rate, n, on the steady state price ratio is given by

$$\frac{\partial p_s}{\partial n} = \left(\frac{\alpha-\beta}{1-\alpha}\right)\Phi^{\frac{\alpha-\beta}{1-\alpha}-1}\frac{\partial \Phi}{\partial n}. \tag{49}$$

Since $\frac{\partial \Phi}{\partial n}<0$,

$$\frac{\partial p_s}{\partial n}\begin{cases} <0 & \text{for } \alpha>\beta \\ >0 & \text{for } \alpha<\beta. \end{cases} \tag{50}$$

Thus, the equilibrium price of a good decreases with n, if the production of that good is relatively labour-intensive, and increases, if the production of that good is relatively capital-intensive. This implies that countries with a rapidly growing population will have a relative cost advantage in the production of labour-intensive commodities, whereas countries with slowly growing population will have a relative cost advantage in the production of capital-intensive commodities. In other words, if we start with two countries/regions that are identical in every respect except the population growth rates, the high-(low-)population growth country will become labour-(capital-)abundant over time, and have a comparative advantage/specialise in the production of labour-(capital) intensive commodity, just as predicted by the HO model.

Corollary 2 *The steady state values of per capita capital, k_s, and the wage rate,*

w_s, are decreasing in the population growth rate n, whereas that of the rental rate, r_s, is increasing in the population growth rate n.

The effect of the population growth rate on the steady state value of per capita capital can be seen from

$$\frac{\partial k_s}{\partial n} = \Phi^{\frac{\alpha}{1-\alpha}} \frac{\partial \phi_4}{\partial n} + \phi_4 \left(\frac{\alpha}{1-\alpha}\right) \Phi^{\frac{\alpha}{1-\alpha}-1} \frac{\partial \Phi}{\partial n}. \tag{51}$$

Since $\phi_4 > 0$, $\Phi > 0$ and $\frac{\partial \phi_4}{\partial n} = -\frac{\phi_4}{1+n} < 0$, and $\frac{\partial \Phi}{\partial n} < 0$, $\frac{\partial k_s}{\partial n} < 0$. So, the prediction of the neoclassical economic growth models by Solow [7], Swan [8] and in particular the one-sector OLG model by Diamond [1] concerning the population growth rate is also captured by our two-sector model.

The effect of the population growth rate on the steady state wage rate, w_s, depends on the sign of

$$\frac{\partial w_s}{\partial n} = \frac{1+n}{(1-\mu)\bar{l}} \phi_4 \left(\frac{\alpha}{1-\alpha}\right) \Phi^{\frac{\alpha}{1-\alpha}-1} \frac{\partial \Phi}{\partial n}, \tag{52}$$

since $\frac{\partial \phi_4}{\partial n} = -\frac{\phi_4}{1+n}$. Hence, $\frac{\partial w_s}{\partial n} < 0$. Thus, low-population growth countries tend to have a higher wage rate than high-population growth countries, explaining why they would have a comparative disadvantage in the production of labour-intensive commodities.[3]

The effect of the population growth rate on the steady state rental rate, r_s, can be observed through

$$\frac{\partial r_s}{\partial n} = \alpha \gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \frac{\partial}{\partial n} \left(\frac{1}{\Phi}\right), \tag{53}$$

which is always positive, since $\frac{\partial}{\partial n}\left(\frac{1}{\Phi}\right) > 0$. Hence, countries with a slowly growing population tend to have a lower rental rate on capital than countries with a rapidly growing population. This is what gives these countries a comparative advantage in the production of capital-intensive commodities, and, in the absence of restrictions to capital mobility, would encourage flows of capital from capital-abundant countries to labour-abundant countries.

[3] This also implies that unequal population growth rates could induce labour-migration from high- to low-population growth nations in the absence of barriers to labour mobility.

Corollary 3 *The equilibrium per capita consumptions by youngs of good* 1*,* c_{1ys}*, and good* 2*,* c_{2ys}*, are decreasing in the population growth rate* n*, whereas the equilibrium per capita consumptions by the elderly of both goods are ambiguous in the population growth rate,* n*.*

The first period equilibrium consumptions of both goods decrease in the population growth rate. This inverse relationship between n and equilibrium values of young generation's consumption follows from the negative relationship between the wage rate and n in the case of good 1, and from the fact that the population growth rate elasticity of the price ratio is higher than the population growth rate elasticity of the wage rate in the case of good 2.

Proof:

The effect of n on the first period equilibrium consumption of good 1 is captured by

$$\frac{\partial c_{1ys}}{\partial n} = \mu\theta\bar{l}\frac{\partial w_s}{\partial n}.$$

Since

$$\frac{\partial w_s}{\partial n} < 0,$$

as shown before, we have

$$\frac{\partial c_{1ys}}{\partial n} < 0.$$

The first period equilibrium per capita consumption of good 2 is

$$c_{2ys} = \mu(1-\theta)\bar{l}\frac{w_s}{p_s},$$

and plugging in the closed form solutions for w_s and p_s in the above expression gives

$$c_{2ys} = \frac{\mu(1-\theta)}{1-\mu}(1+n)\phi_4\Phi^{\frac{\beta}{1-\alpha}}.$$

Taking the derivative of this with respect to the population growth rate n gives

$$\frac{\partial c_{2ys}}{\partial n} = \frac{\mu(1-\theta)}{1-\mu} \Bigg[\phi_4\Phi^{\frac{\beta}{1-\alpha}}$$

$$+(1+n)\Phi^{\frac{\beta}{1-\alpha}}\frac{\partial \phi_4}{\partial n}$$

$$+(1+n)\phi_4\left(\frac{\beta}{1-\alpha}\right)\Phi^{\frac{\beta}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}\Bigg].$$

However, since $\frac{\partial\phi_4}{\partial n}=-\frac{\phi_4}{1+n}$, the above expression reduces to

$$\frac{\partial c_{2ys}}{\partial n} = \frac{\mu(1-\theta)}{1-\mu}(1+n)\phi_4\left(\frac{\beta}{1-\alpha}\right)\Phi^{\frac{\beta}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}.$$

Remembering that

$$\frac{\partial\Phi}{\partial n}<0,$$

one concludes

$$\frac{\partial c_{2ys}}{\partial n}<0.$$

Thus,

$$\frac{\partial c_{2ys}}{\partial n}=\mu(1-\theta)\bar{l}\frac{1}{p_s^2}\left(p_s\frac{\partial w_s}{\partial n}-w_s\frac{\partial p_s}{\partial n}\right)<0,$$

implying that

$$\frac{n}{w_s}\frac{\partial w_s}{\partial n}<\frac{n}{p_s}\frac{\partial p_s}{\partial n}.$$

Consequently, the population growth rate elasticity of the wage rate is less than the population growth rate elasticity of the price ratio. ∎

The second period equilibrium per capita consumption of good 1 is decreasing in n, if $\frac{1}{r_s}>\frac{1-2\alpha}{\alpha}$, and is increasing in n, if $\frac{1}{r_s}<\frac{1-2\alpha}{\alpha}$, where r_s is the steady state rental rate. Similarly, the second period per capita consumption of good 2 is decreasing in n, if $\frac{1}{r_s}>\frac{1-(\alpha+\beta)}{\beta}$, and is increasing in n, if $\frac{1}{r_s}<\frac{1-(\alpha+\beta)}{\beta}$.

Proof:

The second period equilibrium consumption of good 1 is given by

$$c_{1os}=(1-\mu)\theta\bar{l}w_s(1+r_s).$$

Sustituting the closed form solutions of w_s and r_s in the above expression results in

$$c_{1os}=(1+n)\theta\phi_4\Phi^{\frac{\alpha}{1-\alpha}}\left(1+\alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}}\frac{1}{\Phi}\right).$$

Taking the derivative of the above expression with respect to the population growth rate leads to

$$\begin{aligned}
\frac{\partial c_{1os}}{\partial n} &= \theta\phi_4\Phi^{\frac{\alpha}{1-\alpha}} \\
&+ (1+n)\theta\Phi^{\frac{\alpha}{1-\alpha}}\frac{\partial\phi_4}{\partial n} \\
&+ (1+n)\theta\phi_4\left(\frac{\alpha}{1-\alpha}\right)\Phi^{\frac{\alpha}{1-\alpha}-1}\frac{\partial\Phi}{\partial n} \\
&+ \theta\phi_4\Phi^{\frac{\alpha}{1-\alpha}-1}\alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \\
&+ (1+n)\theta\Phi^{\frac{\alpha}{1-\alpha}-1}\alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}}\frac{\partial\phi_4}{\partial n} \\
&+ (1+n)\theta\phi_4\alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}}\left(\frac{\alpha}{1-\alpha}-1\right)\Phi^{\frac{\alpha}{1-\alpha}-2}\frac{\partial\Phi}{\partial n}.
\end{aligned}$$

Plugging in $\frac{\partial\phi_4}{\partial n} = -\frac{\phi_4}{1+n}$ in the above expression yields

$$\begin{aligned}
\frac{\partial c_{1os}}{\partial n} &= (1+n)\theta\phi_4\left(\frac{\alpha}{1-\alpha}\right)\Phi^{\frac{\alpha}{1-\alpha}-1}\frac{\partial\Phi}{\partial n} \\
&+ (1+n)\theta\phi_4\alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}}\left(\frac{\alpha}{1-\alpha}-1\right)\Phi^{\frac{\alpha}{1-\alpha}-2}\frac{\partial\Phi}{\partial n}.
\end{aligned}$$

So, rewriting the above expression we obtain

$$\frac{\partial c_{1os}}{\partial n} = (1+n)\theta\phi_4\left(\frac{\alpha}{1-\alpha}\right)\Phi^{\frac{\alpha}{1-\alpha}-1}\frac{\partial\Phi}{\partial n}(\alpha+(2\alpha-1)r_s),$$

where

$$r_s = \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}}\eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}}\left(\frac{1}{\Phi}\right).$$

Since

$$\frac{\partial\Phi}{\partial n} < 0,$$

$$\frac{\partial c_{1os}}{\partial n}\begin{cases} < 0 & \text{if } \alpha+(2\alpha-1)r_s > 0 \\ > 0 & \text{if } \alpha+(2\alpha-1)r_s < 0, \end{cases} \tag{54}$$

or,

$$\frac{\partial c_{1os}}{\partial n}\begin{cases} < 0 & \text{if } \frac{1}{r_s} > \frac{1-2\alpha}{\alpha} \\ > 0 & \text{if } \frac{1}{r_s} < \frac{1-2\alpha}{\alpha}. \end{cases} \tag{55}$$

Similarly, the second period equilibrium consumption of good 2 is given by

$$c_{2os} = (1-\mu)(1-\theta)\bar{l}\frac{w_s}{p_s}(1+r_s).$$

Sustituting the closed form solutions of w_s, p_s and r_s in the above expression results in

$$c_{2os} = (1+n)(1-\theta)\phi_4 \Phi^{\frac{\beta}{1-\alpha}} \left(1 + \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \frac{1}{\Phi}\right).$$

The derivative of the above expression with respect to the population growth rate is given by,

$$\begin{aligned}
\frac{\partial c_{2os}}{\partial n} &= (1-\theta)\phi_4 \Phi^{\frac{\beta}{1-\alpha}} \\
&+ (1+n)(1-\theta)\Phi^{\frac{\beta}{1-\alpha}} \frac{\partial \phi_4}{\partial n} \\
&+ (1+n)(1-\theta)\phi_4 \left(\frac{\beta}{1-\alpha}\right) \Phi^{\frac{\beta}{1-\alpha}-1} \frac{\partial \Phi}{\partial n} \\
&+ (1-\theta)\phi_4 \Phi^{\frac{\beta}{1-\alpha}-1} \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \\
&+ (1+n)(1-\theta)\Phi^{\frac{\beta}{1-\alpha}-1} \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \frac{\partial \phi_4}{\partial n} \\
&+ (1+n)(1-\theta)\phi_4 \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \left(\frac{\beta}{1-\alpha} - 1\right) \Phi^{\frac{\beta}{1-\alpha}-2} \frac{\partial \Phi}{\partial n}.
\end{aligned}$$

Plugging in $\frac{\partial \phi_4}{\partial n} = -\frac{\phi_4}{1+n}$ in the above expression results in

$$\begin{aligned}
\frac{\partial c_{2os}}{\partial n} &= (1+n)(1-\theta)\phi_4 \left(\frac{\beta}{1-\alpha}\right) \Phi^{\frac{\beta}{1-\alpha}-1} \frac{\partial \Phi}{\partial n} \\
&+ (1+n)(1-\theta)\phi_4 \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \left(\frac{\beta}{1-\alpha} - 1\right) \Phi^{\frac{\beta}{1-\alpha}-2} \frac{\partial \Phi}{\partial n}.
\end{aligned}$$

So, rewriting the above expression, we obtain

$$\frac{\partial c_{2os}}{\partial n} = (1+n)(1-\theta)\phi_4 \left(\frac{1}{1-\alpha}\right) \Phi^{\frac{\beta}{1-\alpha}-1} \frac{\partial \Phi}{\partial n} (\beta + (\beta+\alpha-1)r_s),$$

where

$$r_s = \alpha\gamma^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \eta^{\frac{(\alpha-1)(1-\beta)}{\alpha-\beta}} \left(\frac{1}{\Phi}\right).$$

Since

$$\frac{\partial \Phi}{\partial n} < 0,$$

$$\frac{\partial c_{2os}}{\partial n} \begin{cases} < 0 & \text{if } \beta + (\beta+\alpha-1)r_s > 0 \\ > 0 & \text{if } \beta + (\beta+\alpha-1)r_s < 0, \end{cases} \tag{56}$$

or,

$$\frac{\partial c_{2os}}{\partial n} \begin{cases} < 0 & \text{if } \frac{1}{r_s} > \frac{1-(\alpha+\beta)}{\beta} \\ > 0 & \text{if } \frac{1}{r_s} < \frac{1-(\alpha+\beta)}{\beta}. \end{cases} \tag{57}$$

$\blacksquare$

This implies that the welfare effects of trade between two countries would not be as straightforward to predict as in the case of the static HO model. The welfare results under autarky are driven by the relationship between the values of production parameters and the value of r_s which itself varies with n as shown by (53). Given that trade will lead to the establishment of a common rental rate between the autarky rentals of trading nations distinguished solely by their population growth rates, the welfare level of each nation may change in either direction, depending on how different consumption variables are affected by the trade-induced change in rentals on capital.[4]

4 Conclusion

Our discussion of the closed-form solutions to the 2x2 OLG model in the paper has shown that of the two countries/regions that are identical in every respect except the population growth rates, the high-(low-) population growth country will become labour-(capital-) abundant over time, and must be expected to have a comparative advantage in the production of labour-(capital-) intensive commodity, as suggested by the static HO model. The welfare effects of trade between two countries, however, were found to depend on the values of system parameters, and hence, were ambiguous, adding unequal population growth rates to the list of previously suggested reasons explaining why trade may not improve welfare for both parties in a dynamic, OLG set-up.

[4]Results indicating that trade would not necessarily lead to welfare gains for both countries and might not even be Pareto-superior to autarky have been presented in previous studies based on OLG models with stationary populations (see, for example, Fried, [2], Galor, [4], Mountford, [5]).

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