Integration of Pumped Hydro Energy Storage and Wind Energy Generation: A Structural Analysis

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(1) Problem definition: We study the energy generation and storage problem for a hybrid energy system that includes a wind farm and a pumped hydro energy storage (PHES) facility with two connected reservoirs fed by a natural inflow. The operator decides in real-time how much water to pump or release in the PHES facility, how much energy to generate in the wind farm, and how much energy to buy or sell. **(2) Methodology/results:** We model this problem as a Markov decision process (MDP) under uncertainty in streamflow rate, wind speed, and electricity price. We prove the optimality of a state-dependent threshold policy under positive prices: The state space can be partitioned into several disjoint domains, each associated with a different action type, such that it is optimal to bring the water level of the upper reservoir to a different state-dependent target level in each domain. Once the optimal amount of water that should be pumped or released is found, we can immediately derive the optimal amount of wind energy that should be generated. **(3) Managerial implications:** The existence of natural inflow in the PHES facility – the major source of structural complexity – improves the profits by 19.9% on average in our data-calibrated instances with possibly negative prices. Leveraging our structural results, we develop a policy-approximation algorithm as a heuristic solution method for such realistic instances. This algorithm yields near-optimal solutions up to 23 times faster than the standard dynamic programming algorithm. It also significantly outperforms profit-approximation and rolling-horizon approaches adapted from the literature with respect to objective value.

Key words: pumped hydro energy storage; streamflow; wind; Markov decision processes; dynamic programming

1. Introduction

PHES is the most mature large-scale energy storage technology with a long history that can be traced back to the 1890s (Rehman et al. 2015). The earliest PHES facilities were installed by state-owned utilities to support base load power plants such as coal-fired and nuclear power systems. The number of installed PHES facilities reached a saturation level in the 1990s when the popularity of nuclear and fossil-fuel based power plants has declined due to environmental and safety concerns. More recently, however, PHES has received growing attention from private companies as a result of emerging deregulated electricity markets and increasing share of intermittent renewable sources in power generation (Deane et al. 2010). PHES accounted for more than 90% of global utility-scale energy storage with an installed capacity of 160 GW in 2020 (IHA 2021), and this capacity is expected to reach 325 GW by 2050 (IRENA 2020).

A typical PHES facility consists of two reservoirs with an elevation difference and stores energy in the form of hydraulic potential energy in the upper reservoir. Some amount of water in the lower reservoir can be pumped to the upper reservoir if the operator wants to store energy during off-peak periods (i.e., when there is excess energy supply and thus the electricity prices are low). Some amount of water in the upper reservoir can be released to the lower reservoir if the operator wants to generate energy during peak periods. With its quick start-up and bulk storage capabilities, PHES is an attractive storage option for use in conjunction with intermittent renewable energy sources such as wind and solar (Rehman et al. 2015 and IRENA 2020). It provides an energy arbitrage opportunity as well as a hedge against the intermittency of the renewable energy generation. It also helps achieve a more effective utilization of the renewable sources by reducing the amount of curtailment (i.e., the difference between the potentially available energy and the actually generated energy) (Lew et al. 2013 and Wu and Kapuscinski 2013).

The PHES facilities can be designed in different configurations depending on the specific geologic and hydrologic conditions. They can be categorized into the following two broad classes: closed-loop and open-loop. The closed-loop facilities are off-stream and have no continuous natural inflow to either reservoir (Lu et al. 2018). The open-loop facilities, however, are on-stream and have natural inflows to the upper and/or lower reservoirs (Rogner and Troja 2018). In this paper, we study the energy generation and storage problem for a hybrid energy system that consists of an open-loop PHES facility co-located with a wind farm. Considering a wholesale-market framework with no advance commitment decisions, we model the decision-making process of the system operator as an MDP under uncertainty in the energy sources as well as electricity prices.

In the literature dealing with the operational planning problem for PHES facilities integrated with other renewable sources, many papers focus on hybrid systems that include a closed-loop PHES facility and a wind farm by taking a scenario-based approach to model the wind speed and/or electricity price uncertainties (e.g., Castronuovo and Lopes 2004, Garcia-Gonzalez et al. 2008, Duque et al. 2011, Ding et al. 2014, and Al-Swaiti et al. 2017). In this study, we consider an open-loop PHES facility that reduces to a closed-loop one when there is no incoming streamflow. The open-loop configuration adds nontrivial features to the energy generation and storage problem. Our analysis involves the joint optimization of two energy sources – the wind speed and the streamflow rate – under uncertainty. Our MDP framework enables us to establish the optimal policy structure for this complex problem. To our knowledge, there is no extant characterization of the optimal policy structure for the PHES facilities.

As far as we are aware, Löhndorf et al. (2013) and Toufani et al. (2022) are the only papers with MDP formulations in the PHES literature. Löhndorf et al. (2013) optimize the commitment and storage decisions for a PHES facility that participates in a day-ahead electricity market. They formulate a multi-stage stochastic program for the intraday decisions and an MDP for the interday decisions. They develop an efficient solution approach by combining the methods of stochastic dual dynamic programming and approximate dynamic programming (ADP). Unlike Löhndorf et al. (2013), we consider a hybrid system that consists of a PHES facility and a wind farm in an electricity market with no commitment decisions, and characterize the structure of the optimal energy generation and storage policy for this system. Toufani et al. (2022) evaluate the potential benefit of transforming existing cascade hydropower stations into PHES systems (with no renewable source other than the natural inflow), but present no optimal policy structure in their setting.

Several papers study the MDP representations of the operational planning problem for energy systems with storage units other than PHES facilities (e.g., Kim and Powell 2011, van de Ven et al. 2013, Harsha and Dahleh 2014, Jiang and Powell 2015a,b, and Zhou et al. 2016, 2019). In this research stream, the closest paper to ours is that of Zhou et al. (2019). They consider a hybrid system that consists of a wind farm co-located with an industrial battery. Similarly to ours, their hybrid system operates in a wholesale-market framework with a limited transmission capacity. They show that the optimal energy generation and storage policy is a partial-state-dependent threshold policy under positive electricity prices. In our study, the existence of an open-loop PHES facility rather than an industrial battery makes the problem significantly more challenging: (i) With the streamflow incoming to the upper reservoir, the PHES facility behaves like not only a storage unit but also a generator; the operator aims to jointly optimize two energy sources – the wind speed and the streamflow rate – under uncertainty. (ii) For a hybrid system with an industrial battery, it is sufficient to keep track of the amount of energy stored via a single endogenous state variable. For our hybrid system, however, we need to keep track of the water levels in both reservoirs via two endogenous state variables. (iii) Some amount of water may spill from our PHES facility when there is excess streamflow incoming to the upper reservoir or when the operator wants to pump too much water to benefit from negative electricity prices (see Zhou et al. 2016 for an explanation of why the electricity prices can be negative). The operator thus decides how much water to dispose of in the PHES facility as well as how much energy to curtail in the wind farm. Consequently, we obtain the optimal policy structure for our hybrid system by establishing several *multi-dimensional* properties of the optimal profit function that are not required in Zhou et al. (2019).

Despite the widespread use of PHES in practice, and despite a vast literature on energy storage facilities (see Parker et al. 2019 for a comprehensive review), the structural properties of the optimal energy generation and storage policy are still unknown for the PHES facilities. Our study is the first attempt to fill this gap in the literature. We characterize the optimal policy structure for our hybrid system in the presence of limited transmission capacity when the electricity price is always positive. Specifically, we prove that the state space of our MDP can be partitioned into several disjoint domains – each associated with a different optimal action type that can be 'pump & purchase,' 'pump & sell,' 'release & sell,' 'curtail & sell,' or 'keep unchanged' – such that it is optimal to bring the water level in the upper reservoir to a different state-dependent threshold level in each domain. We also show that the system becomes more profitable as the water level in either reservoir grows, while an increment in the water level in the upper reservoir improves the profit more than that in the lower reservoir. The optimal amount of wind energy that should be generated can be easily derived from the optimal amount of water pumped or released.

Inspired by these structural results, we construct a policy-approximation algorithm as a heuristic solution method for our problem when the electricity price can also be negative. In this approach, we implement the optimal policy structure available under positive prices into a backward induction algorithm that calculates the state-dependent threshold levels for the upper reservoir in each period. The actions in states with positive prices are determined by these threshold levels and the actions in states with negative prices are determined by the myopically optimal solutions. We numerically test the performance of this algorithm, comparing it to two other heuristic methods adapted from the literature that are based on profit-approximation and rolling-horizon approaches, respectively.

We conduct a data-calibrated numerical study based on the observed data sets collected from spatially close locations around Albany in the State of New York. We construct time series models that can predict the electricity price (which can be negative), wind speed, and streamflow rate with acceptable accuracy levels. For the electricity price, taking a similar path to that in Zhou et al. (2019), we model the seasonality component via linear regression, the mean-reversion component via an autoregressive of order one, AR(1), process, and the spike component via an empirical distribution. For the wind speed, we model the seasonality component via dynamic harmonic regression and the random component via an AR(1) process. For the streamflow rate, taking a similar path to those in Wang et al. (2004, 2006), we develop a periodic autoregressive (PAR) model by identifying three distinct seasons within a year and fitting a different AR(1) process to each season. We incorporate these models into our MDP with the help of exogenous state variables.

Our key findings from this numerical study are as follows:

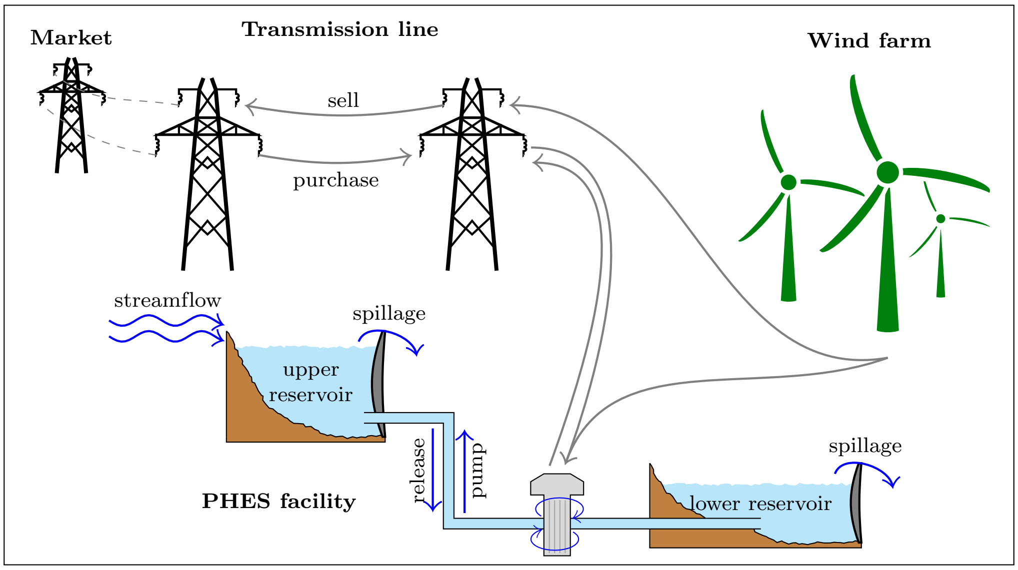
- Our policy-approximation method provides near-optimal solutions (with an average distance of 0.31% and a maximum distance of 1.19% from the optimal profit) up to 23 times faster than the standard dynamic programming algorithm. These findings highlight the practical importance of taking into account our structural results in decision-making.
- The rolling-horizon method yields instantaneous solutions that often deviate substantially from the optimal profit (with an average distance of 3.12% and a maximum distance of 8.28%), while the profit-approximation method fails to ensure convergence to the optimal profit within the solution times of our policy-approximation method.
- The existence of natural inflow (i.e., the open-loop configuration) improves the profits by 19.9% on average, providing a greater benefit when the PHES facility is integrated with a small wind farm. The systems with limited energy supply better exploit the arbitrage opportunity when the negative prices occur more frequently. Finally, increasing the capacity of the upper reservoir is more profitable than increasing the capacity of the lower reservoir.

The rest of this study is organized as follows: Section 2 formulates the energy generation and storage problem for our hybrid system. Section 3 establishes the optimal policy structure when the electricity price is always positive. Section 4 describes the time series models that we embed into our MDP formulation. Section 5 offers the heuristic solution method that we construct based on our structural results. Section 6 describes the benchmark solution methods from the related literature. Section 7 presents the numerical results when the price can also be negative. Section 8 offers a summary and conclusion. Proofs of the analytical results are contained in an online appendix.

2. Problem Formulation

We consider a hybrid energy system that consists of a wind farm and an open-loop PHES facility. See Figure 1 for an illustration. A typical open-loop PHES facility requires quite specific site conditions such as access to natural water inflows and favorable topography. It can be constructed on a river by either building two new reservoirs at different altitudes or replacing the turbines of a conventional hydropower station with reversible ones. Currently, many investors operating in liberalized markets prefer to build PHES facilities by the rehabilitation of conventional hydropower stations (Ardizzon et al. 2014). The PHES facility that we consider is referred to as pumped-back facility (Deane et al. 2010) and is the same as the one studied in Kocaman and Modi (2017), in which the natural streamflow feeds the upper reservoir and the excess streamflow leads to a water spillage from the facility. This system reduces to a closed-loop PHES facility when the streamflow is absent and a conventional hydropower station when the pumping mode is disabled. In this facility,

Figure 1 Illustration of the hybrid energy system.



energy can be generated by releasing water from the upper reservoir to the lower reservoir and can be stored by pumping water from the lower reservoir to the upper reservoir. The water flow between the reservoirs does not change the total amount of water in the facility unless the water level exceeds the capacity of either reservoir and the excess water spills from the facility.

The hybrid system participates in a wholesale market that accepts the dispatch and purchase amounts determined by the hybrid system operator (unlike forward markets that conduct electricity trading through commitments submitted by participants). Concentrating on energy generation and storage decisions that are free of interactions with commitment decisions enables us to better capture the structural dynamics and stochastic nature of the problem. Similar approaches also appear in many related papers; see, for example, Castronuovo and Lopes (2004), Abbaspour et al. (2013), Shu and Jirutitijaroen (2013), Harsha and Dahleh (2014), Steffen and Weber (2016), and Zhou et al. (2016, 2019). In addition, the hybrid system makes only a very limited contribution to the overall energy supply in the market so that the operator can be viewed as a price-taker. This assumption has also been made in many papers that study the operational planning problem for PHES facilities; see, for example, Garcia-Gonzalez et al. (2008), Vespucci et al. (2012), Löhndorf et al. (2013), Ding et al. (2014), Al-Swaiti et al. (2017), and Kusakana (2018). Finally, the hybrid system is connected to the market with a single transmission line.

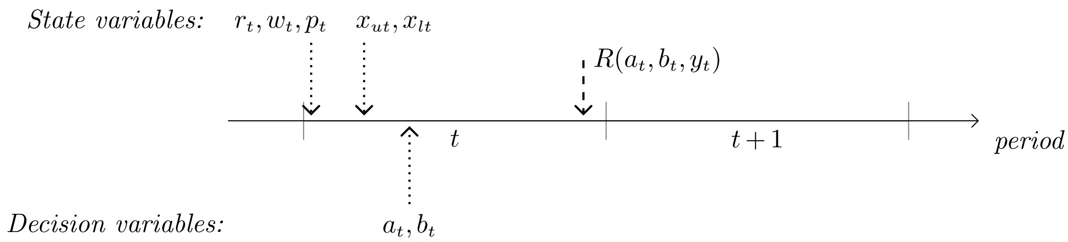
The amount of energy that can be generated in the PHES facility by releasing a unit volume of water from the upper reservoir equals the multiplication of the gravitational constant, the difference in elevation between the reservoirs, and the water density (if the facility is perfectly efficient). With this conversion, we express the water levels in the upper and lower reservoirs in energy terms.

The PHES facility has finite energy and power capacities. The energy capacity of the upper (or lower) reservoir is the maximum amount of water (in energy units) that can be stored in the upper (or lower) reservoir. We denote the capacities of the upper and lower reservoirs by C_U and C_L, respectively, in energy units. The power capacity is the maximum amount of water (in energy units) that can be released from the upper reservoir or pumped from the lower reservoir in a single time period of length Δt. We denote these capacities by K_R and K_P, respectively, in power units. The transmission line also has a finite power capacity. We denote this capacity by K_T in power units. For notational convenience, we define $C_R = K_R \Delta t$, $C_P = K_P \Delta t$, and $C_T = K_T \Delta t$. Notice that the power capacities can play an active role in the optimization of the hybrid system only when $C_R \leq C_U$ and $C_P \leq C_L$. We assume that the PHES facility has the same efficiency in both the releasing and pumping modes. We denote this efficiency by $\theta \in (0, 1]$. We also assume that the transmission line has the same efficiency in both the selling and purchasing modes. We denote this efficiency by $\tau \in (0, 1]$. These efficiencies represent the ratio of energy output to energy input. (Our analytical results in Section 3 continue to hold, with only minor modifications, when the PHES facility and/or the transmission line has different efficiencies in different modes.)

We study the energy generation and storage problem in this hybrid system via a dynamic model over a finite planning horizon of T periods. Let $\mathcal{T} := \{1, 2, \ldots, T\}$ denote the set of periods. We define x_{ut} and x_{lt} as the amounts of water accumulated (in energy units) at the beginning of period t in the upper and lower reservoirs, respectively. Note that $x_{ut} \in [0, C_U]$ and $x_{lt} \in [0, C_L]$. We include x_{ut} and x_{lt} in our state description. These state variables evolve over time according to the energy generation and storage decisions as well as the streamflow rate. We define r_t as the amount of water runoff to the upper reservoir (in energy units) at the beginning of period t. We also define w_t as the wind speed in period t and $g(w_t)$ as the maximum amount of wind energy that can be generated in period t. We derive $g(w_t)$ from the multiplication of the power output of a wind turbine when the wind speed is w_t, the number of turbines in the wind farm W, and the period length Δt. Lastly, we define p_t as the electricity price in period t. We also include the tuple $y_t := (r_\eta, w_\eta, p_\eta)_{\eta \leq t}$ in our state description. The state tuple y_t follows an exogenous stochastic process.

At the beginning of period $t \in \mathcal{T}$, the operator observes first the exogenous state variables (r_t, w_t, and p_t) and then the accumulated amounts of water in the reservoirs (x_{ut} and x_{lt}). Thus, the accumulated amount of water in the upper reservoir x_{ut} includes the amount of water runoff r_t (as long as the capacity allows). With these observations, the operator determines the amount of water that will be released or pumped $a_t \in \mathbb{R}$ (in energy units) and the amount of wind energy that will be generated $b_t \in \mathbb{R}_+$. If $a_t > 0$, the water is released from the upper reservoir. If $a_t \leq 0$, it is pumped

Figure 2 Sequence of events in each period.



from the lower reservoir. See Figure 2 for an illustration of the sequence of events. Let $\mathbb{U}(x_{ut}, x_{lt}, y_t)$ denote the set of action pairs (a_t, b_t) that are admissible in state (x_{ut}, x_{lt}, y_t). For any action pair $(a_t, b_t) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, the following conditions must hold: The energy and power capacities of the PHES facility imply that $-\min\{x_{lt}, C_P\} \le a_t \le \min\{x_{ut}, C_R\}$. In addition, the observed wind speed limits the amount of wind energy that can be generated in the form of $0 \le b_t \le g(w_t)$. Finally, the power capacity of the transmission line implies that $\theta a_t + b_t \le C_T$ if $a_t > 0$ and $-\tau C_T \le a_t/\theta + b_t \le C_T$ if $a_t \le 0$. Since the excess water spilling from each reservoir is lost, the state variables x_{ut} and x_{lt} evolve over time as follows: $x_{u(t+1)} = \min\left\{\min\{x_{ut} - a_t, C_U\} + r_{t+1}, C_U\right\} = \min\{x_{ut} - a_t + r_{t+1}, C_U\}$ and $x_{l(t+1)} = \min\{x_{lt} + a_t, C_L\}$.

The objective is to maximize the expected total cash flow that accrues from selling or purchasing energy over the finite horizon. There are three different types of decisions that we need to consider in our payoff formulation in any period t: (i) A certain amount of water is released from the upper reservoir to generate energy ($a_t > 0$). The resulting energy together with the generated wind energy is sold in the market. We label this type of decision RS (the initials of 'release' and 'sell'). (ii) A certain amount of water is pumped from the lower reservoir to store energy ($a_t \le 0$). If the generated wind energy is sufficient to pump the water ($a_t/\theta \ge -b_t$), the excess wind energy is sold in the market. We label this type of decision PS (the initials of 'pump' and 'sell'). (iii) If the generated wind energy is not sufficient to pump the water ($a_t/\theta < -b_t$), the required additional energy is purchased from the market. We label this type of decision PP (the initials of 'pump' and 'purchase'). Hence, the payoff in period t as a function of action pair (a_t, b_t) and exogenous state tuple y_t can be formulated as

$$R(a_t, b_t, y_t) = \begin{cases} p_t\big(\theta a_t + b_t\big)\tau & \text{if } a_t > 0 \text{ (RS)}, \\ p_t\left(a_t/\theta + b_t\right)\tau & \text{if } 0 \ge a_t/\theta \ge -b_t \text{ (PS), and} \\ p_t\left(a_t/\theta + b_t\right)/\tau & \text{if } 0 \ge -b_t > a_t/\theta \text{ (PP)}. \end{cases}$$

Although we define τ as the efficiency parameter for the transmission line, we note that the terms $(1-\tau)p_t$ and $((1-\tau)/\tau)p_t$ can also be interpreted as the grid usage fees per unit energy sold and per unit energy purchased, respectively, in period t.

A control policy π can be characterized by a sequence of decision rules $\{A_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t)\}_{t\in\mathcal{T}}$, where $A_t^\pi(\cdot)$ maps the state $(x_{ut}^\pi, x_{lt}^\pi, y_t)$ to a feasible action pair $(a_t^\pi(\cdot), b_t^\pi(\cdot))$, and x_{ut}^π and x_{lt}^π denote the random state variables governed by policy π, $\forall t \in \mathcal{T}\backslash\{1\}$. We define Π as the set of all admissible control policies. For any initial state (x_{u1}, x_{l1}, y_1), the optimal expected total cash flow over the finite horizon is given by

$$\max_{\pi\in\Pi} \mathbb{E}\left[\sum_{t\in\mathcal{T}} R(A_t^\pi(x_{ut}^\pi, x_{lt}^\pi, y_t), y_t)\middle| x_{u1}, x_{l1}, y_1\right].$$

For each period $t \in \mathcal{T}$ and each state (x_{ut}, x_{lt}, y_t), the optimal profit function $v_t^*(x_{ut}, x_{lt}, y_t)$ can be calculated with the following dynamic programming (DP) recursion:

$$v_t^*(x_{ut}, x_{lt}, y_t) = \max_{(a_t,b_t)\in\mathbb{U}(x_{ut},x_{lt},y_t)} \left\{R(a_t, b_t, y_t) + \mathbb{E}_{y_{t+1}|y_t}\left[v_{t+1}^*\left(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}\right)\right]\right\} \tag{1}$$

where $v_T^*(x_{uT}, x_{lT}, y_T) = 0$. Note that $v_1^*(x_{u1}, x_{l1}, y_1)$ is the optimal expected total cash flow for the initial state (x_{u1}, x_{l1}, y_1) over the finite horizon. We denote by $(a_t^*(x_{ut}, x_{lt}, y_t), b_t^*(x_{ut}, x_{lt}, y_t))$ the optimal action pair for the optimization problem in equation (1).

3. Structural Analysis

In this section, we establish several structural properties of our optimal profit function and use these properties to characterize the structure of the optimal energy generation and storage policy. To this end, we assume that the electricity price is strictly positive throughout the finite horizon:

ASSUMPTION 1. *$p_t > 0$, $\forall t \in \mathcal{T}$.*

We require Assumption 1 to show that the payoff in period t is jointly concave in action pair (a_t, b_t):

LEMMA 1. *Under Assumption 1, $R(a_t, b_t, y_t)$ is jointly concave in a_t and b_t for all y_t.*

We first establish the following structural property of our optimal profit function.

LEMMA 2. *$v_t^*(x_{ut}, x_{lt}, y_t) \le v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$ and $v_t^*(x_{ut}, x_{lt}, y_t) \le v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$ for $\alpha > 0$, $\forall t \in \mathcal{T}$. Moreover, under Assumption 1, $v_t^*(x_{ut}, x_{lt}, y_t) = v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$ for $x_{ut} + x_{lt} \ge C_U$ and $\alpha > 0$, $\forall t \in \mathcal{T}$.*

Lemma 2 states that the system becomes more profitable as the amount of water in the upper or lower reservoir grows. This is because the system is capable of generating (or storing) more energy when the amount of water in the upper (or lower) reservoir is larger. Lemma 2 also states that if the electricity price is always positive and the total amount of water in the system is larger than the upper reservoir capacity, the system becomes no more profitable as the amount of water in the lower reservoir grows.

Using Lemma 2, we introduce an upper bound on the optimal amount of water that should be pumped in any period. We also formulate the optimal amount of wind energy that should be generated in terms of the optimal amount of water that should be released or pumped.

LEMMA 3. *Under Assumption 1, for each* $t \in \mathcal{T}$*,* $x_{ut} - C_U \leq a_t^*(x_{ut}, x_{lt}, y_t)$ *and* $b_t^*(x_{ut}, x_{lt}, y_t) = \min\{g(w_t), \max\{C_T - a_t^*(x_{ut}, x_{lt}, y_t)/\theta, C_T - \theta a_t^*(x_{ut}, x_{lt}, y_t)\}\}$*.*

A key implication of Lemma 3 is that, if the price can only be positive, the amount of water pumped can be restricted to take values that do not cause any water spillage from the upper reservoir without loss of optimality. This is because the water spillage due to pumping induces a loss of water in the PHES facility that brings no benefit according to Lemma 2. Another key implication of Lemma 3 is that one can easily determine the optimal amount of wind energy that should be generated once the optimal amount of water that should be released or pumped is found. This result allows us to restrict our structural analysis to the characterization of the optimal energy generation and storage policy in the PHES facility. We note that the curtailed amount of wind energy is $g(w_t) - b_t^*(x_{ut}, x_{lt}, y_t)$ (if $b_t^*(x_{ut}, x_{lt}, y_t) < g(w_t)$). We label this type of decision CS (the initials of 'curtail' and 'sell').

Using Lemmas 1–3, we establish several other structural properties of our optimal profit function:

PROPOSITION 1. *Under Assumption 1, the following structural properties hold for* $\alpha > 0$ *and* $\beta > 0$*:*

(a) $v_t^*(x_{ut} + \alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha + \beta, y_t)$*,* $\forall t$*.*

(b) $v_t^*(x_{ut} + \alpha + \beta, x_{lt}, y_t) - v_t^*(x_{ut} + \beta, x_{lt} + \alpha, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$*,* $\forall t$*.*

(c) $v_t^*(x_{ut} + \alpha, x_{lt} + \beta, y_t) - v_t^*(x_{ut}, x_{lt} + \beta, y_t) \leq v_t^*(x_{ut} + \alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}, y_t)$*,* $\forall t$*.*

We discuss below the implications of Proposition 1:

- Point (a) of Proposition 1 says that it becomes less desirable to release a certain amount of water from the upper reservoir as the amount of water in the lower reservoir grows. The intuition behind this result can be seen when the consequences of having too much water in the lower reservoir are considered: Holding a large amount of water in the lower reservoir in any period increases the risk of losing some water in the PHES facility in future periods with high electricity prices in which it is beneficial to sell energy by releasing too much water from the upper reservoir. On the other hand, pumping a large amount of water from the lower reservoir in any period limits the capacity of the PHES facility to sell energy to the market in this period. This may even entail purchasing energy from the market. Hence, increasing the amount of water in the lower reservoir exhibits diminishing returns.

- Point (b) of Proposition 1 says that it becomes more desirable to release a certain amount of water from the upper reservoir as the amount of water in the upper reservoir grows. This is because holding some amount of water in the upper reservoir may be beneficial in anticipation of high electricity prices in future periods while holding a large amount of water in the upper reservoir increases the risk of underutilizing the water runoff in elevating the total amount of water in the PHES facility. Hence, increasing the amount of water in the upper reservoir exhibits diminishing returns. The summation of the properties in parts (a) and (b) implies that $v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \leq v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, $\forall y_t$. Thus, if $\alpha = \beta$, it becomes less desirable to release water as the water flows from the upper reservoir to the lower reservoir at any particular rate.
- Point (c) of Proposition 1 says that it is more desirable to have an extra amount of water in the upper reservoir when the amount of water in the lower reservoir is smaller. Likewise, it is more desirable to have an extra amount of water in the lower reservoir when the amount of water in the upper reservoir is smaller. It is important to note that the summation of the properties in parts (a) and (c) implies the concavity of $v_t^*(x_{ut}, \cdot, y_t)$, i.e., $v_t^*(x_{ut}, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) \leq v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, $\forall x_{ut}, y_t$. Similarly, the summation of the properties in parts (b) and (c) implies the concavity of $v_t^*(\cdot, x_{lt}, y_t)$, i.e., $v_t^*(x_{ut}, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t) \leq v_t^*(x_{ut}+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, $\forall x_{lt}, y_t$. Finally, we note that the property in part (c) can be viewed as Topkis' (1998) submodularity property in x_{ut} and x_{lt}, $\forall y_t$.

We now introduce optimal state-dependent target levels that are associated with the water level of the upper reservoir for each of the four different action types: For $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$,

$$S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) := \underset{z_{ut}\in[0,C_U]}{\arg\max}\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)\},$$

where

$$V_t(z_{ut}, x_{ut}, x_{lt}, y_t) := \mathbb{E}_{y_{t+1}|y_t}\Big[v_{t+1}^*\Big(\min\big\{z_{ut} + r_{t+1}, C_U\big\}, \min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, y_{t+1}\Big)\Big],$$

$$R_t^{(\nu)}(x_{ut} - z_{ut}, y_t) = \begin{cases} p_t[(x_{ut} - z_{ut})/\theta + g(w_t)]/\tau & \text{if } \nu = \mathsf{PP}, \\ p_t[(x_{ut} - z_{ut})/\theta + g(w_t)]\tau & \text{if } \nu = \mathsf{PS}, \\ p_t[\theta(x_{ut} - z_{ut}) + g(w_t)]\tau & \text{if } \nu = \mathsf{RS}, \\ p_t\tau C_T & \text{if } \nu = \mathsf{CS}, \end{cases}$$

and $z_{ut} := x_{ut} - a_t$ is the water level of the upper reservoir at the end of period t if the action a_t is taken in period t. Recall that x_{ut} is the water level of the upper reservoir at the beginning of period t, including the amount of water runoff r_t. For notational convenience, we often suppress the dependency of $S_t^{(\nu)}$ on (x_{ut}, x_{lt}, y_t) in the remainder of the paper. Recall from Lemma 3 that the amount of water pumped can be limited to prevent water spillage from the upper reservoir. Hence, in any period, one can easily determine the optimal amount of water that should remain in the lower reservoir at the end of this period by bringing the amount of water in the upper reservoir to the optimal target level in this period. It is thus sufficient to define the optimal target level for only the upper reservoir in our optimal policy characterization. Using Lemma 2 and Proposition 1, Lemma 4 proves that $V_t(z_{ut}, x_{ut}, x_{lt}, y_t)$ is concave in z_{ut}. For each $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$, since $R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)$ is linear in z_{ut}, $V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\nu)}(x_{ut} - z_{ut}, y_t)$ is also concave in z_{ut}.

Lemma 4. *Under Assumption 1, $V_t(z_{ut}, x_{ut}, x_{lt}, y_t)$ is concave in z_{ut}.*

Let Ω denote the domain of (x_{ut}, x_{lt}, w_t), i.e., $\Omega := [0, C_U] \times [0, C_L] \times [0, \infty)$. We define the set $\Psi^0 := \{(x_{ut}, x_{lt}, w_t) \in \Omega : g(w_t) > C_T + \min\{x_{lt}, C_P, C_U - x_{ut}\}/\theta\}$ as the subdomain of Ω where the maximum amount of wind energy that can be generated in period t is greater than the maximum total amount of energy that can be used for selling and storing in period t at optimality, the set $\Psi^1 := \{(x_{ut}, x_{lt}, w_t) \in \Omega : C_T < g(w_t) \leq C_T + \min\{x_{lt}, C_P, C_U - x_{ut}\}/\theta\}$ as the subdomain of Ω where the maximum amount of wind energy that can be generated in period t is greater than the transmission line capacity but less than the maximum total amount of energy that can be used for selling and storing in period t at optimality, and the set $\Psi^2 := \{(x_{ut}, x_{lt}, w_t) \in \Omega : 0 \leq g(w_t) \leq C_T\}$ as the subdomain of Ω where the maximum amount of wind energy that can be generated in period t is less than the transmission line capacity. With this notation, and leveraging the above analytical results, we are now ready to state the main result of this section:

Theorem 1. *Under Assumption 1, the structure of the optimal energy generation and storage policy in the PHES facility can be specified as follows. In any period t, if $(x_{ut}, x_{lt}, w_t) \in \Psi^0$, it is optimal to*

- *pump up to get as close as possible to $S_t^{(\mathsf{CS})}$ if $x_{ut} \leq S_t^{(\mathsf{CS})}$ and*
- *release down to get as close as possible to $S_t^{(\mathsf{CS})}$ if $S_t^{(\mathsf{CS})} < x_{ut}$.*

If $(x_{ut}, x_{lt}, w_t) \in \Psi^1$, it is optimal to

- *pump up to get as close as possible to $S_t^{(\mathsf{PP})}$ if $x_{ut} \leq S_t^{(\mathsf{PP})} - \theta g(w_t)$,*
- *pump up to get as close as possible to $S_t^{(\mathsf{PS})}$ if $S_t^{(\mathsf{PP})} - \theta g(w_t) < x_{ut} \leq S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T)$,*
- *pump up to get as close as possible to $S_t^{(\mathsf{CS})}$ if $S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T) < x_{ut} \leq S_t^{(\mathsf{CS})}$, and*
- *release down to get as close as possible to $S_t^{(\mathsf{CS})}$ if $S_t^{(\mathsf{CS})} < x_{ut}$.*

If $(x_{ut}, x_{lt}, w_t) \in \Psi^2$, it is optimal to

- *pump up to get as close as possible to $S_t^{(\mathsf{PP})}$ if $x_{ut} \leq S_t^{(\mathsf{PP})} - \theta g(w_t)$,*
- *pump up to get as close as possible to $S_t^{(\mathsf{PS})}$ if $S_t^{(\mathsf{PP})} - \theta g(w_t) < x_{ut} \leq S_t^{(\mathsf{PS})}$,*

- *keep unchanged if* $S_t^{(\mathsf{PS})} < x_{ut} \leq S_t^{(\mathsf{RS})}$,
- *release down to get as close as possible to* $S_t^{(\mathsf{RS})}$ *if* $S_t^{(\mathsf{RS})} < x_{ut} \leq S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta$, *and*
- *release down to get as close as possible to* $S_t^{(\mathsf{CS})}$ *if* $S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta < x_{ut}$.

Furthermore, the optimal state-dependent target levels obey (i) $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\mathsf{PS})}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\mathsf{CS})}(x_{ut}, x_{lt}, y_t)$, *(ii)* $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) = S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t)$ *if* $x_{ut} + x_{lt} \geq C_U$, *and (iii)* $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t) \leq S_t^{(\nu)}(x_{ut}, x_{lt} + \alpha, y_t) = S_t^{(\nu)}(x_{ut} + \alpha, x_{lt}, y_t)$, *for each* $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$ *and* $\alpha > 0$.

We discuss below the implications of Theorem 1:

- If the maximum amount of wind energy that can be generated in period t is high enough (i.e., if $(x_{ut}, x_{lt}, w_t) \in \Psi^0$), it is optimal to bring the water level in the upper reservoir as close to $S_t^{(\mathsf{CS})}$ as possible, by selling the maximum amount of energy that can be transmitted and by curtailing the excess amount of wind energy.
- If the maximum amount of wind energy that can be generated in period t is in the medium range (i.e., if $(x_{ut}, x_{lt}, w_t) \in \Psi^1$), it is optimal to bring the water level in the upper reservoir as close to $S_t^{(\mathsf{CS})}$ as possible only if it is large enough (i.e., only if $x_{ut} > S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T)$). Notice that raising a very low water level to $S_t^{(\mathsf{CS})}$ (which is greater than the other target levels) would consume a large amount of energy in period t, restricting the amount of energy sold and inducing a lower payoff. Thus the optimal target levels should be different for lower water levels. If the water level in the upper reservoir is not large enough but not too small (i.e., if $S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T) \geq x_{ut} > S_t^{(\mathsf{PP})} - \theta g(w_t)$), it is optimal to bring it as close to $S_t^{(\mathsf{PS})}$ as possible, by pumping water and selling energy (without any curtailment of wind energy). If the water level in the upper reservoir is too small (i.e., if $S_t^{(\mathsf{PP})} - \theta g(w_t) \geq x_{ut}$), it is optimal to bring it as close to $S_t^{(\mathsf{PP})}$ as possible, by pumping water and purchasing energy.
- If the maximum amount of wind energy that can be generated in period t is low (i.e., if $(x_{ut}, x_{lt}, w_t) \in \Psi^2$), unlike the previous scenario, it is never optimal to pump water and also curtail wind energy. If the water level in the upper reservoir is large but insufficient for the optimality of releasing water and curtailing wind energy (i.e., if $S_t^{(\mathsf{PS})} < x_{ut} \leq S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta$), it is optimal to drop the water level in the upper reservoir as close to $S_t^{(\mathsf{RS})}$ as possible if it is above $S_t^{(\mathsf{RS})}$ and keep it unchanged otherwise, by selling energy in both cases. The optimal actions are similar to those in the previous scenario if the water level in the upper reservoir is small (i.e., if $x_{ut} \leq S_t^{(\mathsf{PS})}$).
- The target water level in the upper reservoir is highest if it is optimal to curtail some amount of wind energy, that is, if there is excess energy supply. If it is optimal to fully utilize the

available wind energy, the target water level is highest if the optimal action type is RS and lowest if it is PP: The PHES facility inefficiency leads to different marginal payoffs in the energy generation and storage modes. Thus $S_t^{(\mathsf{PS})} \leq S_t^{(\mathsf{RS})}$. Note that $S_t^{(\mathsf{PS})} = S_t^{(\mathsf{RS})}$ if $\theta = 1$. Likewise, the transmission line inefficiency leads to different marginal payoffs in the energy selling and purchasing modes. Thus $S_t^{(\mathsf{PP})} \leq S_t^{(\mathsf{PS})}$. Note $S_t^{(\mathsf{PP})} = S_t^{(\mathsf{PS})}$ if $\tau = 1$. Finally, for each action type, the target level increases with the total amount of water in the PHES facility as long as the total amount of water is no larger than the upper reservoir capacity. However, the target level is independent of how the available water is distributed between the two reservoirs.

The optimal policy structure in Theorem 1 involves scenarios where it is optimal to curtail some wind energy by bringing the water level in the upper reservoir as close to the target level $S_t^{(\mathsf{CS})}$ as possible. However, without loss of optimality, the target level $S_t^{(\mathsf{CS})}$ can be restricted to take any value from the interval $[C_U - \underline{r}_{t+1}, C_U]$ where $\underline{r}_{t+1}$ is a lower bound on the amount of water runoff in period $t+1$. This is because any target value in this interval (if accessible) ensures that the upper reservoir will be full at the beginning of the next period, and the available water in the lower reservoir will provide no additional benefit in this case according to Lemma 2. The existence of this optimal solution interval (or multiple optimal solutions in the discrete-state version of our MDP) for $S_t^{(\mathsf{CS})}$ indicates the substitutability of the two renewable energy sources in our hybrid system: the curtailed amount of wind energy is larger and the amount of water spilling from the upper reservoir is lower when $S_t^{(\mathsf{CS})} < C_U$, and the reverse is true when $S_t^{(\mathsf{CS})} = C_U$.

To our knowledge, we are the first to characterize the optimal policy structure for energy systems in which different types of renewable energy sources are jointly optimized under uncertainty. In the literature on energy systems planning in regards to optimal policy characterization, the closest paper to ours is that of Zhou et al. (2019): They study the energy generation and storage problem for a wind farm co-located with an industrial battery that has no renewable energy input other than the wind energy. Our hybrid system includes that of Zhou et al. (2019) as a special case when our PHES facility is forced to act like the battery. Specifically, this case arises when (i) the reservoirs have the same capacity ($C_U = C_L$), (ii) the total amount of water in the PHES facility initially equals this capacity ($x_{u1} + x_{l1} = C_U$), and (iii) there is no natural inflow to the upper reservoir throughout the entire horizon ($r_t = 0$, $\forall t$). The existence of the open-loop PHES facility (with a natural inflow in the upper reservoir as well as possible water spillages from both reservoirs) makes the structural analysis much more challenging. Our optimal policy structure differs significantly from that in Zhou et al. (2019) by including two endogenous state variables for the PHES facility (rather than only one state variable for the industrial battery) and one exogenous state variable for the streamflow rate (in addition to the state variables for the price and wind speed), and by explaining the complex interplay between the state variables through our structural properties in Lemmas 2-3 and Proposition 1 (that are not available in Zhou et al. 2019). The structural knowledge derived in this section has inspired us to develop the heuristic solution method in Section 5 that can be usefully employed in a more general setting where the electricity price can also be negative.

4. Time Series Models and Discretization

In this section, using the historical data available from the State of New York, we develop three distinct time series models for the electricity price, streamflow rate, and wind speed, respectively (Sections 4.1–4.3). We incorporate these parametric models into our MDP by utilizing the exogenous state variables. We then discretize the continuous space of the exogenous state variables for numerical calculations (Section 4.4). We set the period length to be one hour in our experiments; we use t as the index for one-hour length periods in the remainder of the paper. This assumption is common in the PHES literature; see, for example, Brown et al. (2008), Connolly et al. (2011), Duque et al. (2011), Kusakana (2016), Kocaman and Modi (2017), and Jurasz et al. (2018).

4.1. Time Series Model for the Electricity Price

We consider the electricity price data available for Albany, in the State of New York, in which the price is set every five minutes between the years 2007 and 2019. We retrieve this data from NYISO (2020); the average, median, minimum, and maximum values of the price are \$45.14, \$34.09, −\$3678.02, and \$3393.33, per MWh, respectively. We define $\bar{t}$ as the index for five-minute length periods and $\overline{\mathcal{T}}$ as the set of these periods.

We construct our five-minute price model by adopting the iterative time series modeling approach of Zhou et al. (2019). This approach decomposes the time series into the components of seasonality ($s_{\bar{t}}$), mean reversion ($\rho_{\bar{t}}$), and spike ($j_{\bar{t}}$), by taking into account the possible occurrences of negative prices. Specifically, this approach sets the initial values of the estimated spikes $\{\hat{j}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ to zero and takes the following four steps in each iteration performed until the parameter estimates of the model converge: The first step calculates the estimated despiked prices $\{\hat{p}'_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ by subtracting the estimated spikes from the observed prices (i.e., $\hat{p}'_{\bar{t}} = p_{\bar{t}} - \hat{j}_{\bar{t}}$). The second step accommodates *negative prices* by applying an inverse hyperbolic transformation to the despiked prices; the transformed estimated despiked price in period $\bar{t}$ is $\sinh^{-1}(\hat{p}'_{\bar{t}}/\ell)$, where $\sinh^{-1}$ is the inverse hyperbolic sine function and ℓ is the scale parameter that is chosen to be 30. It then eliminates the *seasonality* effect by deseasonalizing the transformed despiked prices. To obtain the estimated seasonality model $\{\hat{s}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$, we fit the following linear regression to the transformed despiked prices:

$$s_{\bar{t}} = \gamma_1^{(p)} + \sum_{i=1}^{11} \gamma_{2i}^{(p)} D_t^{2i} + \sum_{j=1}^{6} \gamma_{3j}^{(p)} D_t^{3j} + \sum_{h=1}^{23} \gamma_{4h}^{(p)} D_t^{4h},\ \forall \bar{t}, \tag{2}$$

where $\gamma_1^{(p)}$ is a constant and $\gamma_{2i}^{(p)}$, $\gamma_{3j}^{(p)}$, and $\gamma_{4h}^{(p)}$ are the respective coefficients of the dummy variables D_t^{2i}, D_t^{3j}, and D_t^{4h}, that are equal to one if the corresponding hourly period $t = \lceil \bar{t}/12 \rceil$ is in month i, week day j, and hour h, respectively, and zero otherwise. It calculates the estimated mean-reverting component $\{\hat{\rho}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ by removing the estimated seasonality effect from the transformed despiked prices (i.e., $\hat{\rho}_{\bar{t}} = \sinh^{-1}(\hat{p}'_{\bar{t}}/\ell) - \hat{s}_{\bar{t}}$). The third step captures the *mean-reverting* behavior via an AR(1) model. Assuming that error terms $\{\epsilon_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ are independent standard normal random variables, the AR(1) process is formulated as follows:

$$\rho_{\bar{t}} = \left(1 - \kappa^{(\bar{p})}\right) \rho_{\bar{t}-1} + \sigma^{(\bar{p})} \epsilon_{\bar{t}},\ \forall \bar{t}, \tag{3}$$

where $\kappa^{(\bar{p})}$ is the speed of mean reversion and $\sigma^{(\bar{p})}$ is the volatility of white noise. The last step identifies the *spikes* for each $\bar{t} \in \overline{\mathcal{T}}$: The price $p_{\bar{t}+1}$ contains a spike if $\hat{j}_{\bar{t}+1}$ is zero and $\left|\hat{p}'_{\bar{t}+1} - \mathbb{E}[p'_{\bar{t}+1}|\hat{\rho}_{\bar{t}}]\right|$ is no less than 50, where $p'_{\bar{t}+1}|\rho_{\bar{t}}$ is assumed to follow a Johnson SU distribution (Johnson 1949) with mean $-\ell \exp\left[0.5(\sigma^{(\bar{p})})^2\right] \sinh\left[-\rho_{\bar{t}}(1-\kappa^{(\bar{p})}) - s_{\bar{t}+1}\right]$. If the price $p_{\bar{t}+1}$ contains a spike, $\hat{j}_{\bar{t}+1}$ is updated to $\hat{p}'_{\bar{t}+1} - \mathbb{E}[p'_{\bar{t}+1}|\hat{\rho}_{\bar{t}}]$. Finally, $\hat{p}'_{\bar{t}+1}$ is replaced with $\mathbb{E}[p'_{\bar{t}+1}|\hat{\rho}_{\bar{t}}]$ and $\hat{\rho}_{\bar{t}+1}$ is replaced with $\sinh^{-1}(\mathbb{E}[p'_{\bar{t}+1}|\hat{\rho}_{\bar{t}}]/\ell) - s_{\bar{t}+1}$. We have found that the mean absolute error (MAE) of our calibration is only \$4.75/MWh for the despiked price.

As we assume hourly periods for our MDP, we convert the five-minute price model to an hourly price model that again consists of the components of seasonality (s_t), mean reversion (ρ_t), and spike (j_t): We estimate the hourly seasonality effect, s_t, from the right hand side of the equation in (2). We model the hourly mean-reverting component, ρ_t, as the AR(1) process:

$$\rho_t = \left(1 - \kappa^{(p)}\right) \rho_{t-1} + \sigma^{(p)} \epsilon_t, \tag{4}$$

where $\kappa^{(p)}$ is the speed of mean reversion and $\sigma^{(p)}$ is the volatility of white noise. We obtain these parameters by recursively iterating the equation in (3):

$$\rho_{\bar{t}+12} = \left(1 - \kappa^{(\bar{p})}\right)^{12} \rho_{\bar{t}} + \sum_{\eta=1}^{12} \left(1 - \kappa^{(\bar{p})}\right)^{12-\eta} \sigma^{(\bar{p})} \epsilon_{\bar{t}+\eta}. \tag{5}$$

Equating the mean and standard deviation of the right-hand side of (5) with those of (4) yields $\kappa^{(p)} = 1 - \left(1 - \kappa^{(\bar{p})}\right)^{12}$ and $\sigma^{(p)} = \sigma^{(\bar{p})} \sqrt{\left[1 - (1-\kappa^{(\bar{p})})^{2\times 12}\right] / \left[1 - (1-\kappa^{(\bar{p})})^2\right]}$. Lastly, we model the hourly spike component, j_t, as a compound Bernoulli process; a spike occurs with probability λ and its size follows an empirical distribution. The probability estimate $\hat{\lambda}$ is the ratio of the number of identified spikes to the number of five-minute length periods, while the empirical distribution is based on the estimated spikes $\{\hat{j}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ in the five-minute price model.

4.2. Time Series Model for the Streamflow Rate

We consider the streamflow data available for the Hudson River at North Creek, in the State of New York, in which the streamflow rate is recorded every fifteen minutes between the years 2007 and 2019. We retrieve this data from USGS (2020); the average, median, minimum, and maximum values of the streamflow rate are 55.46, 39.01, 2.18, and 988.25, in m^3/s, respectively. We redefine $\bar{t}$ as the index for fifteen-minute length periods, and define $f_{\bar{t}}$ as the streamflow rate in period $\bar{t}$.

The most widely used time series models for the streamflow fall within the family of autoregressive integrated moving average (ARIMA) models (Modarres 2007 and Wang et al. 2015). The PAR model from this family seems to be an appropriate choice for our streamflow data since it can successfully capture the high seasonality in the mean, variance, and serial dependence structure of the series (McLeod 1994 and Wang et al. 2004): We fit the PAR model to our streamflow data by partitioning the 365 days of the year into three disjoint clusters with the fuzzy clustering approach of Wang et al. (2006). In this approach, we consider the log-transformation of daily average streamflow rates and the autocorrelation values at different lag times (from one day to ten days). The three clusters that we have found correspond to the 'normal', 'drought', and 'flood' flow conditions that can be observed throughout the year. See Figure 3 for these clusters. We model the streamflow rate in each cluster $i \in \{\text{normal, drought, flood}\}$ as a different AR(1) process:

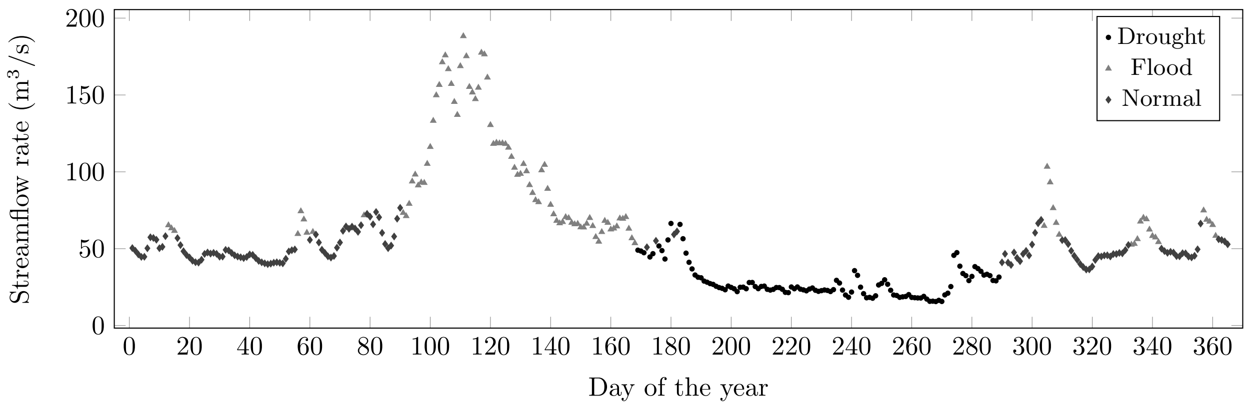
$$f_{\bar{t}} = \delta_i^{(\bar{r})} + \phi_i^{(\bar{r})} f_{\bar{t}-1} + \sigma_i^{(\bar{r})} \epsilon_{\bar{t}}, \ \forall \bar{t} \in \overline{\mathcal{T}}_i, \tag{6}$$

where $\delta_i^{(\bar{r})}$ is a constant, $\phi_i^{(\bar{r})}$ is the autoregressive coefficient, $\sigma_i^{(\bar{r})}$ is the volatility of white noise, and $\overline{\mathcal{T}}_i$ is the set of fifteen-minute length periods that belong to cluster i. We have found that the MAE of our calibration is only 0.35 m^3/s for the streamflow rate.

Our frequency spectrum analysis indicates the significance of daily patterns, while there is no substantial fluctuation within each day. Therefore, we convert the above fifteen-minute model to a daily model. We define $\underline{t}$ as the index for daily periods and $f_{\underline{t}}$ as the streamflow rate in period $\underline{t}$. For the daily streamflow model, we formulate the AR(1) process in each cluster $i \in \{\text{normal, drought, flood}\}$ as follows:

$$f_{\underline{t}} = \delta_i^{(\underline{r})} + \phi_i^{(\underline{r})} f_{\underline{t}-1} + \sigma_i^{(\underline{r})} \epsilon_{\underline{t}}, \ \forall \underline{t} \in \underline{\mathcal{T}}_i, \tag{7}$$

Figure 3 Three clusters for 365 days of the year and daily average streamflow rates over the years 2007-2019.



where $\delta_i^{(r)}$ is a constant, $\phi_i^{(r)}$ is the autoregressive coefficient, $\sigma_i^{(r)}$ is the volatility of white noise, and $\underline{\mathcal{T}}_i$ is the set of daily periods that belong to cluster i. Following similar steps to those in Section 4.1, we obtain $\delta_i^{(\underline{r})} = \delta_i^{(\bar{r})}$, $\phi_i^{(\underline{r})} = \left(\phi_i^{(\bar{r})}\right)^{96}$, and $\sigma_i^{(\underline{r})} = \sigma_i^{(\bar{r})}\sqrt{\left[1-\left(\phi_i^{(\bar{r})}\right)^{2\times 96}\right] / \left[1-\left(\phi_i^{(\bar{r})}\right)^{2}\right]}$, by iterating the expression in (6) recursively and equating the mean and standard deviation of the right hand side of the resulting expression with those of (7). As we assume hourly periods for our MDP, we incorporate the daily model by restricting the intraday hourly streamflow rates to stay constant.

We note that the fifteen-minute model can also be converted to an hourly model. However, the volatility of white noise is too low for hourly streamflow rates so that our discrete-state approximations for the AR(1) processes (required for numerical calculations) would induce deterministic streamflow rates. We also note that the accuracy of our PAR model can be improved via higher-order autoregressive processes. Such a relaxation can reduce the MAE by at most a few percent for our streamflow data. Since higher-order autoregressive processes lead to an exponential growth of the state space in our MDP, we choose to implement the AR(1) process into our PAR model. Finally, our further investigations have shown that fitting the PAR model to our data with a different number of clusters (other than three) provides no significant improvement in accuracy.

4.3. Time Series Model for the Wind Speed

We consider the wind speed data available for Albany, in the State of New York, in which the wind speed is recorded every five minutes between the years 2007 and 2012. We retrieve this data from NOAA (2020); the average, median, minimum, and maximum values of the wind speed are 8.52, 8.21, 0.05, and 28.67, in m/s, respectively. We redefine $\bar{t}$ as the index for five-minute length periods and $\overline{\mathcal{T}}$ as the set of these periods.

We construct our five-minute wind speed model by adopting the dynamic harmonic regression with ARIMA (DHR+ARIMA) in which DHR captures the seasonality component $q_{\bar{t}}$ and ARIMA models the deseasonalized component $\xi_{\bar{t}}$ (Chen and Rabiti 2017 and Zhou et al. 2019): Since our frequency spectrum analysis shows the significance of hourly and daily patterns, we fit the following linear regression with Fourier terms to our data to obtain the estimated seasonality model $\{\hat{q}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$:

$$q_{\bar{t}} = \gamma_0^{(w)} + \gamma_1^{(w)} \cos\left(\frac{2\pi\left(t+\omega_1^{(w)}\right)}{24}\right) + \gamma_2^{(w)} \cos\left(\frac{2\pi\left(\lceil t/24\rceil + \omega_2^{(w)}\right)}{365}\right), \tag{8}$$

where $\lceil . \rceil$ is the ceiling function, $t = \lceil \bar{t}/12 \rceil$ is the corresponding hourly period, $\gamma_0^{(w)}$ is a constant, $\gamma_1^{(w)}$ and $\omega_1^{(w)}$ are the hourly magnitude and phase-shift parameters, and $\gamma_2^{(w)}$ and $\omega_2^{(w)}$ are the daily magnitude and phase-shift parameters, respectively. We calculate the deseasonalized wind speeds $\{\hat{\xi}_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ by removing the seasonality effect from the observed wind speeds $\{w_{\bar{t}}\}_{\bar{t}\in\overline{\mathcal{T}}}$ (i.e., $\hat{\xi}_{\bar{t}} = w_{\bar{t}} - \hat{q}_{\bar{t}}$). We model the deseasonalized wind speed as an AR(1) process: $\xi_{\bar{t}} = \phi^{(\bar{w})}\xi_{\bar{t}-1} + \sigma^{(\bar{w})}\epsilon_{\bar{t}},\ \forall \bar{t}$, where $\phi^{(\bar{w})}$ is the autoregressive coefficient and $\sigma^{(\bar{w})}$ is the volatility of white noise. We have found that the MAE of our calibration is 0.17 m/s for the deseasonalized wind speed.

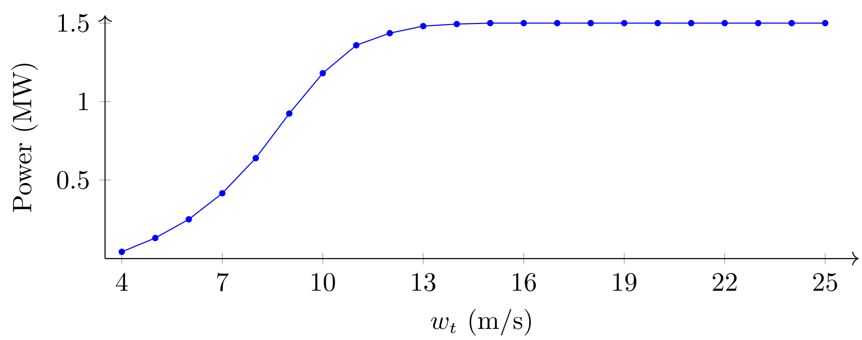
As we assume hourly periods for our MDP, we convert the above five-minute model to an hourly model by following similar steps to those in Section 4.1: We estimate the hourly seasonality effect, q_t, from the right hand side of the expression in (8). We model the hourly deseasonalized wind speed, ξ_t, as the AR(1) process: $\xi_t = \phi^{(w)}\xi_{t-1} + \sigma^{(w)}\epsilon_t,\ \forall t$, where $\phi^{(w)} = \left(\phi^{(\bar{w})}\right)^{12}$ is the autoregressive coefficient and $\sigma^{(w)} = \sigma^{(\bar{w})}\sqrt{\left[1-\left(\phi^{(\bar{w})}\right)^{2\times 12}\right] / \left[1-\left(\phi^{(\bar{w})}\right)^{2}\right]}$ is the volatility of white noise.

4.4. Discretization for the Numerical Study

Our time series models in Sections 4.1–4.3 formulate the stochastic component of each exogenous state variable in our MDP as an AR(1) process. This allows us to reduce the computational burden of our MDP by redefining the exogenous state tuple in period t as $y_t = (f_t, \xi_t, \rho_t, j_t)$. Although the spike component of the price is state-independent, we include it in the tuple y_t for calculation of the effective price in period t. For notational convenience, we also define $\bar{y}_t = (f_t, \xi_t, \rho_t)$ as the exogenous partial-state tuple in period t. For numerical calculations, we now provide discrete-state approximations for the continuous-state AR(1) processes embedded in the tuple $\bar{y}_t$.

For the electricity price, we employ the trinomial lattice method of Hull and White (1994) to characterize the AR(1) process as a finite-state Markov chain. Following the suggestions of Hull and White (1994) and Jaillet et al. (2004) for the number of time steps that should be iterated, we construct a three-hour trinomial lattice for our AR(1) process. The Markov chain obtained from this lattice has the state space $\mathcal{P} := \{-0.57, 0, 0.57\}$. We also restrict the spikes to take values from the set $\mathcal{J} := \{-300, -250, \ldots, 550\}$. The spike occurrence probability is 9.3% in each period.

Figure 4 Power curve of a single GE 1.5-77 wind turbine (General Electric 2019).



For the streamflow rate, after we remove the lowest and highest ten percent of the data in each cluster, the streamflow rates vary between 21 and 88 in the normal flow cluster, between 8 and 57 in the drought flow cluster, and between 24 and 196 in the flood flow cluster. We thus restrict the streamflow rate in our MDP to take values from the sets $\mathcal{R}^n := \{20, 40, 60, 80\}$, $\mathcal{R}^d := \{10, 30, 50\}$, and $\mathcal{R}^f := \{30, 50, \ldots, 190\}$ in these three clusters, respectively. We characterize the AR(1) process in each cluster as a different Markov chain for which we calculate the transition probabilities by following Tauchen (1986). We denote by $\mathcal{R}_t \in \{\mathcal{R}^n, \mathcal{R}^d, \mathcal{R}^f\}$ the state space of the AR(1) process in period t. If the system transitions to a different cluster in period t ($\mathcal{R}_{t-1} \neq \mathcal{R}_t$) and the AR(1) process defined on $\mathcal{R}_{t-1}$ in period $t-1$ moves to a state in period t that is not in $\mathcal{R}_t$, we take the closest state in $\mathcal{R}_t$ as the streamflow rate in period t.

For the wind speed, we note that the maximum wind speed observed between 2007 and 2012 is 28.67 m/s. We thus characterize the AR(1) process as a Markov chain with the state space $\{0, 1, \ldots, 28\}$ for which we calculate the transition probabilities by again following Tauchen (1986). In our experiments, we consider a wind farm with General Electric (GE) 1.5–77 wind turbines; the power output of each such turbine is based on the power curve in Figure 4. The stochastic component values greater than nine, combined with the seasonality component that we have found to be always larger than five for our data, yield the same power output. Hence, following Sheskin (1985), we reformulate our Markov chain by reducing its state space to the set $\mathcal{W} := \{0, 1, \ldots, 10\}$.

With the above modifications, $y_t = (f_t, \xi_t, \rho_t, j_t) \in \mathcal{Y}_t := \mathcal{R}_t \times \mathcal{W} \times \mathcal{P} \times \mathcal{J}$ and $\bar{y}_t = (f_t, \xi_t, \rho_t) \in \overline{\mathcal{Y}}_t := \mathcal{R}_t \times \mathcal{W} \times \mathcal{P}$. We restrict the water levels in the upper and lower reservoirs to take values from the sets $\mathcal{X}_u := \{n\zeta_a \in [0, C_U] : n \in \mathbb{Z}\}$ and $\mathcal{X}_l := \{n\zeta_a \in [0, C_L] : n \in \mathbb{Z}\}$, respectively, where ζ_a is a prespecified constant. For the discrete-state version of our MDP, let $\mathbb{U}^D(x_{ut}, x_{lt}, y_t)$ denote the set of action pairs (a_t, b_t) that are admissible in state $(x_{ut}, x_{lt}, y_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \mathcal{Y}_t$. The set $\mathbb{U}^D(x_{ut}, x_{lt}, y_t)$ consists of the set $\{(n\zeta_a, m\zeta_b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t) : n \in \mathbb{Z}, m \in \mathbb{Z}_+\}$, where ζ_b is a prespecified constant, as well as the extreme points of $\mathbb{U}(x_{ut}, x_{lt}, y_t)$. Notice that the water level in the upper reservoir may take a value $\tilde{x}_{ut} \notin \mathcal{X}_u$ in any period t since the amount of water runoff need not be a multiple of ζ_a. In such cases, in order to solve the recursion in (1) for period $t-1$, we linearly interpolate the optimal profits in period t for the two states in $\mathcal{X}_u$ that are adjacent to $\tilde{x}_{ut}$. To speed up computation and save memory, we calculate first the optimal profit function $\bar{v}^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1}) := \mathbb{E}_{j_{t+1}}\left[v^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})\right]$ in state $(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1})$ in period $t+1$ and then the optimal action pair in state (x_{ut}, x_{lt}, y_t) in period t, using the recursion in (1) with $v^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$ replaced by $\bar{v}^*_{t+1}(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1})$ and the expectation taken with respect to $\bar{y}_{t+1}|\bar{y}_t$. Finally, we note that the number of states in each period is of order $O(|\mathcal{X}_u||\mathcal{X}_l||\mathcal{Y}|)$ and the number of feasible action pairs in each state is of order $O(|\mathcal{X}_u|W/\zeta_b)$: the total number of operations required to exhaustively search the optimal action pairs is of order $O(TW|\mathcal{X}_u|^2|\mathcal{X}_l||\mathcal{Y}|/\zeta_b)$.

5. The Value of Structural Knowledge: Heuristic Algorithm Development

Threshold policies are often easy-to-implement heuristics and, if constructed with the structural knowledge, can provide comparable performances to those of the optimal policies (e.g., Benjaafar and ElHafsi 2006, Nadar et al. 2016, and Zhou et al. 2019). Recall that Theorem 1 establishes the optimality of a state-dependent threshold policy when the electricity price is always positive. We now implement this policy structure into a heuristic solution method for the more general problem with possibly negative prices. Our heuristic method determines the action pair in each state with a positive price via the target water levels (described in Theorem 1) associated with that state, while it implements the myopically optimal action pair in each state with a negative price. Such a myopic approach, although not necessarily optimal, yields instantaneous decisions without a significant drain on the total profit, because the negative price occurrence frequency (NPF) is quite small in our time series data. We present below two variants of this heuristic method.

5.1. Policy Approximation

In this method, for each period $t \in \mathcal{T}$ and each state (x_{ut}, x_{lt}, y_t), we define $v^{\mathsf{PA}}_t(x_{ut}, x_{lt}, y_t)$ as the profit function and $S^{(\nu),\mathsf{PA}}_t(x_{ut}, x_{lt}, y_t)$ as the state-dependent target level for each $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$. We compute these profit functions and target levels, as well as the corresponding action pairs $(a^{\mathsf{PA}}_t, b^{\mathsf{PA}}_t)$, as outlined by Theorem 1. See Algorithm 1 for the resulting backward induction algorithm. We incorporate the properties of the target levels in points (i)–(iii) of Theorem 1 into this algorithm: the target levels increase with the total amount of water in the PHES facility (if less than C_U) and with the decision type (from PP to CS), and the variables $S^{(\nu),lower}_t$ and S^{lower}_t reduce the search space for the target levels in each iteration. See steps 9–16 of Algorithm 1. We label this method PA (the initials of policy approximation).

Algorithm 1 Policy approximation.

$\bar{v}_T^{\mathsf{PA}}(x_{uT}, x_{lT}, \bar{y}_T) \leftarrow 0,\ \forall (x_{uT}, x_{lT}, \bar{y}_T) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_T$.
for $t = T-1, \ldots, 1$ **do**
  for $y_t \in \mathcal{Y}_t$ such that $p_t > 0$ **do**
    $S_t^{(\nu),lower} \leftarrow 0,\ \forall \nu$.
    for $x = x_{ut} + x_{lt} \in \{0, \zeta_a, ..., \lfloor C_U/\zeta_a \rfloor \zeta_a, C_U\}$ **do**
      $S_t^{lower} \leftarrow 0$.
      for $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$ **do**
        $S_t \leftarrow \arg\max_{z_{ut} \in [\max\{S_t^{(\nu),lower}, S_t^{lower}\}, C_U]} \Big\{ R_t^{(\nu)}(x_{ut} - z_{ut}, y_t) + \mathbb{E}_{\bar{y}_{t+1}|\bar{y}_t} \Big[\bar{v}_{t+1}^{\mathsf{PA}}(\min\{z_{ut} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, \bar{y}_{t+1}) \Big] \Big\}$.
        $S_t^{lower} \leftarrow S_t$. ▷ See point (i) of Theorem 1.
        $S_t^{(\nu),lower} \leftarrow S_t$. ▷ See point (iii) of Theorem 1.
        for $(x_{ut}, x_{lt}) \in \mathcal{X}_u \times \mathcal{X}_l$ such that $x_{ut} + x_{lt} = x$ **do**
          $S_t^{(\nu),\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow S_t$. ▷ See point (iii) of Theorem 1.
        end for
        if $x = C_U$ **then**
          for $(x_{ut}, x_{lt}) \in \mathcal{X}_u \times \mathcal{X}_l$ such that $x_{ut} + x_{lt} \geq C_U$ **do**
            $S_t^{(\nu),\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow S_t$. ▷ See point (ii) of Theorem 1.
          end for
        end if
      end for
    end for
  end for
  for $(x_{ut}, x_{lt}, y_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \mathcal{Y}_t$ **do**
    if $p_t > 0$ **then** ▷ The price is positive.
      Compute $(a_t^{\mathsf{PA}}, b_t^{\mathsf{PA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by $S_t^{(\nu),\mathsf{PA}}$, $\forall \nu$.
    else ▷ The price is negative.
      $a_t^{\mathsf{PA}} = -\min\{x_{lt}, C_P\}$ and $b_t^{\mathsf{PA}} = 0$.
    end if
    $v_t^{\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \leftarrow R(a_t^{\mathsf{PA}}, b_t^{\mathsf{PA}}, y_t) + \mathbb{E}_{\bar{y}_{t+1}|\bar{y}_t} \left[\bar{v}_{t+1}^{\mathsf{PA}}(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1}) \right]$.
  end for
  for $(x_{ut}, x_{lt}, \bar{y}_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_t$ **do**
    $\bar{v}_t^{\mathsf{PA}}(x_{ut}, x_{lt}, \bar{y}_t) \leftarrow \mathbb{E}_{j_t} \left[v_t^{\mathsf{PA}}(x_{ut}, x_{lt}, y_t) \right]$.
  end for
end for

This heuristic method accelerates the standard DP algorithm of our problem, without loss of optimality, if the price is always positive. We note that the number of states in which we need to compute the target levels in each period is of order $O\big((|\mathcal{X}_u| + |\mathcal{X}_l|)|\mathcal{Y}|\big)$ and the number of feasible action pairs that we need to consider for target level computation in each state is of order $O(|\mathcal{X}_u|)$: the total number of operations required to exhaustively search the action pairs is of order $O\big(T(|\mathcal{X}_u| + |\mathcal{X}_l|)|\mathcal{X}_u||\mathcal{Y}|\big)$. Recall that the computational complexity of the standard DP algorithm

Algorithm 2 Cash flow calculation for reduced-state-space policy approximation.

$\bar{v}_T^{\mathsf{RPA}}(x_{uT}, x_{lT}, \bar{y}_T) \leftarrow 0$, $\forall (x_{uT}, x_{lT}, \bar{y}_T) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_T$.
for $t = T-1, \ldots, 1$ **do**
  for $(x_{ut}, x_{lt}, y_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \mathcal{Y}_t$ **do**
    if $p_t > 0$ and $j_t = 0$ **then** ▷ The price is positive with no spike.
      Compute $(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by $S_t^{(\nu),\mathsf{RPA}}$, $\forall \nu$.
    else if $p_t > 0$ and $j_t > 0$ **then** ▷ The price is positive with positive spike.
      Compute $(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by 0, $\forall \nu$.
    else if $p_t > 0$ and $j_t < 0$ **then** ▷ The price is positive with negative spike.
      Compute $(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by C_U, $\forall \nu$.
    else ▷ The price is negative.
      $a_t^{\mathsf{RPA}} = -\min\{x_{lt}, C_P\}$ and $b_t^{\mathsf{RPA}} = 0$.
    end if
    $v_t^{\mathsf{RPA}}(x_{ut}, x_{lt}, y_t) \leftarrow R(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}}, y_t) + \mathbb{E}_{\bar{y}_{t+1}|\bar{y}_t}\left[\bar{v}_{t+1}^{\mathsf{RPA}}(x_{u(t+1)}, x_{l(t+1)}, \bar{y}_{t+1})\right]$.
  end for
  for $(x_{ut}, x_{lt}, \bar{y}_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \overline{\mathcal{Y}}_t$ **do**
    $\bar{v}_t^{\mathsf{RPA}}(x_{ut}, x_{lt}, \bar{y}_t) \leftarrow \mathbb{E}_{j_t}\left[v_t^{\mathsf{RPA}}(x_{ut}, x_{lt}, y_t)\right]$.
  end for
end for

is of order $O(TW|\mathcal{X}_u|^2|\mathcal{X}_l||\mathcal{Y}|/\zeta_b)$. Hence, for realistic parameter values, our heuristic method can be shown to have a distinct computational advantage over the optimal algorithm.

5.2. Reduced-State-Space Policy Approximation

Our PA method calculates the target levels in period t for each exogenous state tuple in the set $\mathcal{Y}_t$; see step 3 of Algorithm 1. If the spike component of the price is assumed to be always zero, the set $\mathcal{Y}_t$ can be reduced to the set $\overline{\mathcal{Y}}_t$ in each period t. Note $|\overline{\mathcal{Y}}_t| = |\mathcal{Y}_t|/|\mathcal{J}|$. This assumption can thus significantly reduce computations of Algorithm 1. Since the nonzero spike values represent the extreme price conditions, we can speculate that the myopic decisions lead to no loss of optimality in most periods with nonzero spike values. Therefore, we consider a reduced-state-space version of our PA method that calculates the target levels via Algorithm 1 with $\mathcal{J}$ replaced by $\mathcal{J}^{\mathsf{RPA}} := \{0\}$, uses these target levels (denoted by $S_t^{(\nu),\mathsf{RPA}}$) to determine the action pairs in zero-spike states, and adopts the myopically optimal action pairs in nonzero-spike states. We label this method RPA (the initials of reduced-state-space policy approximation). We calculate the expected total cash flow of the resulting heuristic policy in Algorithm 2: In each state with a positive price and a zero spike value, we compute the action pair $(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by $S_t^{(\nu),\mathsf{RPA}}$. In each state with a positive price and a nonzero spike value, we compute the action pair $(a_t^{\mathsf{RPA}}, b_t^{\mathsf{RPA}})$ from Theorem 1 with $S_t^{(\nu)}$ replaced by myopically optimal target levels. Lastly, in each state with a negative price, we implement the myopically optimal action pair.

6. Benchmark Solution Methods

We next adapt two heuristic solution methods from the literature to our problem when the price can also be negative. The first method uses the monotonicity properties of the optimal profit function. The second method solves the deterministic version of our problem in an online fashion.

6.1. Profit-Function Approximation

ADP methods are widely used in the literature for value-function approximations when the state space is immense; see Powell (2007) and Bertsekas (2012) for detailed discussions. Recall from Lemma 2 that the optimal value function of our MDP is monotone in the endogenous state variables. For such value functions, the monotonicity structure can be exploited to speed up the ADP algorithms; see Papadaki and Powell (2002) and Jiang and Powell (2015a,b). Taking a similar approach, we develop a monotone-ADP algorithm tailored to our structural results.

Although monotone-ADP algorithms are applicable to the *pre-decision state* formulation of our MDP in (1), we will use an alternative equivalent formulation for easy implementation. We consider the *post-decision state* formulation of our MDP with post-decision state x_t^a in period t and post-decision state space $\mathcal{X}$. Specifically, x_t^a denotes the state after the action in period t is taken, but before the exogenous state y_{t+1} in period $t+1$ is observed: $x_t^a = (x_{ut}^a, x_{lt}^a, y_t)$ where $x_{ut}^a = \min\{x_{ut} - a_t, C_U\}$ and $x_{lt}^a = \min\{x_{lt} + a_t, C_L\}$. The recursion in (1) can thus be reformulated as

$$v_t^*(x_{ut}, x_{lt}, y_t) = \max_{(a_t,b_t)\in\mathbb{U}(x_{ut},x_{lt},y_t)} \{R(a_t, b_t, y_t) + v_t^a(x_t^a)\},$$

where $v_t^a(x_t^a) = \mathbb{E}_{y_{t+1}|y_t}\left[v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \middle| x_{ut}^a, x_{lt}^a, y_t\right]$ is the post-decision profit function. The above recursion can be rewritten as follows:

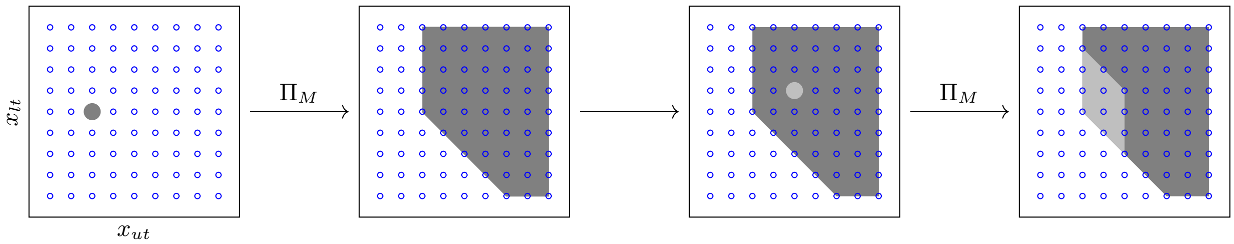
$$v_{t-1}^a(x_{t-1}^a) = \mathbb{E}_{y_t|y_{t-1}}\left[\max_{(a_t,b_t)\in\mathbb{U}(x_{ut},x_{lt},y_t)} \{R(a_t, b_t, y_t) + v_t^a(x_t^a)\} \middle| x_{t-1}^a\right].$$

For each iteration n of the algorithm, we define $v_t^{a,n}(x)$ as the estimate of the post-decision profit function in post-decision state $x \in \mathcal{X}$, and $x_t^n = (x_{ut}^n, x_{lt}^n, y_t^n)$ and $x_t^{a,n} = (x_{ut}^{a,n}, x_{lt}^{a,n}, y_t^n)$ as the pre-decision and post-decision states, respectively, in period t. We also define $\alpha_t^n(x_t^{a,n})$ as the stepsize used to smooth new estimates with previous ones in state $x_t^{a,n}$. We choose $\alpha_t^n(x_t^{a,n}) := 1/N(\tilde{x}_t^{a,n}, n)$ where $N(\tilde{x}_t^{a,n}, n) = \sum_{m=1}^n \mathbf{1}_{\{\tilde{x}_t^{a,m} = \tilde{x}_t^{a,n}\}}$ and $\tilde{x}_t^{a,m} \in \mathcal{X}$ is the closest state to $x_t^{a,m}$, $m = 1, \ldots, n$.

In addition to our monotonicity results in Lemma 2, under a mild condition, we show that the system becomes more profitable as the amount of water in the upper reservoir grows while the total amount of water in the two reservoirs remains constant:

Assumption 2. $r_t \le \min\{C_R, C_T/\theta\}$, $\forall t \in \mathcal{T}$.

Figure 5 Illustration of the projection operator Π_M.



Note. Darker colors indicate larger values: ● and ● are observed values where ● < ●.

LEMMA 5. *Under Assumptions 1 and 2,* $v_t^*(x_{ut}, x_{lt} + \alpha, y_t) \le v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$*,* $\alpha > 0$*,* $\forall t \in \mathcal{T}$*.*

We require Assumption 2 to prove Lemma 5. Unless Assumption 2 holds, the excess streamflow would inevitably result in a water spillage in some states, regardless of the control policy employed. For instance, suppose that the upper reservoir is full at the beginning of any particular period and the maximum possible amount of water is released from the upper reservoir in this period. If the amount of water runoff in the next period is larger than the releasing capacity of the PHES facility or the transmission line capacity, both limiting the amount of water released in any period, some amount of water spills from the facility. We note that Assumption 2 holds in all of our data-calibrated numerical instances in Section 7. For any two states $x = (x_u, x_l, y)$ and $x' = (x'_u, x'_l, y)$, we say that $x \preccurlyeq x'$ if and only if $x_u \le x'_u$ and $x_u + x_l \le x'_u + x'_l$. Lemmas 2 and 5 imply that the optimal profit in state x is no larger than that in state x' if $x \preccurlyeq x'$ (under Assumptions 1 and 2). We implement this result into the algorithm via the monotonicity preserving projection operator:

$$\Pi_M(x_t^{a,n}, z_t^{a,n}, x, v) = \begin{cases} z_t^{a,n} & \text{if } x = x_t^{a,n}, \\ \max\{z_t^{a,n}, v\} & \text{if } x_t^{a,n} \preccurlyeq x \text{ and } x \ne x_t^{a,n}, \\ \min\{z_t^{a,n}, v\} & \text{if } x \preccurlyeq x_t^{a,n} \text{ and } x \ne x_t^{a,n}, \\ v & \text{otherwise,} \end{cases}$$

where v is the previous estimate of the optimal profit in state x and $z_t^{a,n}$ is the new observation of the optimal profit in the currently visited state $x_t^{a,n}$. See Figure 5 for an illustration.

With these formulations, the monotone-ADP algorithm calculates an approximate (post-decision) profit function v_t^a via Monte Carlo simulation; see Algorithm 3. To speed up computation, the state tuple y_t^n can be replaced with the partial-state tuple $\bar{y}_t^n$ everywhere except in the function $R(a_{t+1}, b_{t+1}, y_{t+1}^n)$ in step 5, and the expectation of the right-hand side in step 5 can be taken with respect to the spike component of the price. We construct an admissible heuristic policy for our MDP by computing the action pair $(a_t^{\mathsf{B1}}, b_t^{\mathsf{B1}})$ in each state (x_{ut}, x_{lt}, y_t) in each period t as follows: $(a_t^{\mathsf{B1}}, b_t^{\mathsf{B1}}) = \arg\max_{(a_t, b_t) \in \mathbb{U}^D(x_{ut}, x_{lt}, y_t)} \{R(a_t, b_t, y_t) + v_t^a(x_t^a)\}$. To evaluate the expected total cash flow of the resulting policy, we compute the profit function $v_t^{\mathsf{B1}}(x_{ut}, x_{lt}, y_t)$ in each state $(x_{ut}, x_{lt}, y_t) \in \mathcal{X}_u \times \mathcal{X}_l \times \mathcal{Y}_t$ in each period t by backward induction. We label this method B1.

Algorithm 3 Profit-function approximation.

$v_t^{a,0}(x) \leftarrow 0$, $\forall x \in \mathcal{X}, t \in \mathcal{T}$; and $v_T^{a,n}(x) \leftarrow 0$, $\forall x \in \mathcal{X}, n \leq N$.

$x_0^{a,n} \leftarrow (x_{u0}^{a,n}, x_{l0}^{a,n}, y_0^n)$ such that $\mathbb{P}\{(x_{u1}^n, x_{l1}^n, y_1^n) = (x_{u1}, x_{l1}, y_1) | x_0^{a,n}\} = 1$, $\forall n \leq N$.

for $n = 1, \ldots, N$ **do**

  for $t = 0, \ldots, T-1$ **do**

    Sample $(x_{u(t+1)}^n, x_{l(t+1)}^n, y_{t+1}^n)$ and get a noisy observation:

$$\hat{v} \leftarrow \max_{(a_{t+1}, b_{t+1}) \in \mathbb{U}^D(x_{u(t+1)}^n, x_{l(t+1)}^n, y_{t+1}^n)} \{R(a_{t+1}, b_{t+1}, y_{t+1}^n) + v_{t+1}^{a,n-1}(x_{t+1}^{a,n})\}.$$

    Smooth new observation with previous value:

$$z_t^{a,n} \leftarrow (1 - \alpha_t^n(x_t^{a,n}))\ v_t^{a,n-1}(x_t^{a,n}) + \alpha_t^n(x_t^{a,n})\ \hat{v}.$$

    Enforce monotonicity. For each $x \in \mathcal{X}$:

$$v_t^{a,n}(x) \leftarrow \Pi_M(x_t^{a,n}, z_t^{a,n}, x, v_t^{a,n-1}(x)).$$

    Choose the next state $x_{t+1}^{a,n}$.

  end for

end for

6.2. Rolling-Horizon Approach

We now consider a heuristic method that solves a small number of tractable optimization problems in an online fashion. In each period, it determines the action pair by solving a deterministic version of our problem starting from the current period and state if the current price is positive, while it implements the myopically optimal action pair if the current price is negative. The deterministic version of our problem replaces the random components with their expected values based on the current state information. The deterministic problem in state (x_{ut}, x_{lt}, y_t) in period t is given by

$$\max_{\{(a_\eta, b_\eta, x'_{u\eta}, x'_{l\eta})\}_{\eta \in \mathcal{T}: \eta \geq t}} \sum_{\eta \in \mathcal{T}: \eta \geq t} R(a_\eta, b_\eta, y_{t,\eta}) \tag{9}$$

subject to

$$x'_{u\eta} = \min\{x'_{u(\eta-1)} - a_{\eta-1} + r_{t,\eta}, C_U\}, \quad \forall \eta \in \mathcal{T} : \eta > t, \tag{10}$$

$$x'_{l\eta} = \min\{x'_{l(\eta-1)} + a_{\eta-1}, C_L\}, \quad \forall \eta \in \mathcal{T} : \eta > t, \tag{11}$$

$$(a_\eta, b_\eta) \in \mathbb{U}(x'_{u\eta}, x'_{l\eta}, y_{t,\eta}), \quad \forall \eta \in \mathcal{T} : \eta \geq t, \tag{12}$$

where $(x'_{ut}, x'_{lt}) = (x_{ut}, x_{lt})$ and $y_{t,\eta} := (r_{t,\eta}, w_{t,\eta}, p_{t,\eta})$ is the expected exogenous state in period η based on the exogenous state information y_t in period t. The decision variables are the actions a_η and b_η as well as the water levels of the upper and lower reservoirs, $x'_{u\eta}$ and $x'_{l\eta}$, respectively. The objective function is the total cash flow from period t through T. Constraints (10)-(11) represent the water level balance equations and constraints (12) ensure the feasibility of the action pairs. See Secomandi (2008), Lai et al. (2010), and Nadarajah and Secomandi (2018) for similar approaches.

This deterministic problem is difficult to solve due to the nonlinear structure of the objective function and constraints (10)-(11). However, since Lemma 2 implies that having more water in the system is more profitable, each minimization operator in constraints (10)-(11) can be replaced with two inequalities without loss of optimality. Assuming that $p_{t,\eta} \geq 0$, $\forall \eta \geq t$, it is also possible to linearize the objective function since the payoff function in each period can be shown to be the minimum of affine functions. Therefore, in order to reduce computations via a linear program (LP), we set negative expected values of future prices to zero if the current price is positive. This heuristic method thus solves the LP formulation of the deterministic problem in periods with positive prices: the actions in period t are $a_t^{\mathsf{B2}} = a_t^{LP}$ and $b_t^{\mathsf{B2}} = b_t^{LP}$ if $p_t \geq 0$ where a_t^{LP} and b_t^{LP} denote the LP solutions, and $a_t^{\mathsf{B2}} = -\min\{x_{lt}, C_P\}$ and $b_t^{\mathsf{B2}} = 0$ if $p_t < 0$. To estimate the expected total cash flow of the resulting policy, we generate 1,000 sample paths via Monte Carlo simulation, compute the total cash flow for each path, and take the empirical mean. We label this method B2.

7. Numerical Experiments

We conduct numerical experiments to examine the performances of our PA and RPA methods in comparison with the B1 and B2 methods (Section 7.1) and gain insights into the hybrid system operations in different environments (Section 7.2). We consider instances in which the planning horizon spans the first week of January, April, or August ($T = 168$ hours); the number of GE 1.5–77 wind turbines (W) is 0, 50, 100, or 150; the vertical distance between the reservoirs is 30 meters; the PHES facility is open-loop ($r_t \geq 0$, $\forall t$) or closed-loop ($r_t = 0$, $\forall t$); NPF$\in \{0.80\%, 1.57\%, 2.32\%\}$ in January, and NPF$\in \{1.26\%, 2.48\%, 3.67\%\}$ in April and August; $C_U \in \{1000, 1250, 1500\}$ and $C_L \in \{500, 750, 1000\}$ (in MWh); $C_R = C_P = 100$ MWh; $C_T = 200$ MWh; $\theta = 0.88$; and $\tau = 0.95$. Our parameter values are realistic for small to medium-sized PHES facilities (Hayes 2009, Catalão 2017, and Kocaman and Modi 2017). The electricity price can be negative in our time series model: the NPF is 0.80% in January, and 1.26% in April and August. We obtain the other two values of NPF by raising the numbers of negative spike occurrences in our time series model. The initial water levels x_{u1} and x_{l1} are the closest states to $C_U/2$ and $C_L/2$, respectively, in all instances. The initial exogenous state $y_1 = (f_1, \xi_1, \rho_1, j_1)$ is $(60, 5, 0, 0)$ in January, $(110, 5, 0, 0)$ in April, and $(30, 5, 0, 0)$ in August, in open-loop instances. We take $y_1 = (0, 5, 0, 0)$ in closed-loop instances.

Table 1 Optimality gaps and computation times (in CPU minutes) for the solution methods.

Season	W	v_1^*	Optimality gaps				Computation times		
			PA	RPA	B1	B2	Optimal policy	PA	RPA
January	50	1.038	0.01%	0.20%	22.40%	2.73%	112.3	33.9	9.3
	100	1.634	0.01%	1.15%	11.40%	2.69%	169.6	39.6	9.6
	150	1.906	0.00%	0.25%	0.95%	0.75%	211.6	34.6	9.4
April	50	0.851	0.02%	0.15%	20.38%	2.03%	252.1	76.3	21.6
	100	1.243	0.02%	0.75%	10.66%	3.70%	381.3	88.9	22.3
	150	1.365	0.01%	0.45%	4.05%	1.23%	474.3	77.4	22.1
August	50	0.609	0.02%	0.38%	33.66%	8.28%	78.0	25.0	6.8
	100	0.912	0.01%	0.80%	21.72%	4.19%	113.5	28.2	7.0
	150	1.131	0.07%	1.19%	13.19%	2.46%	140.7	26.3	6.9

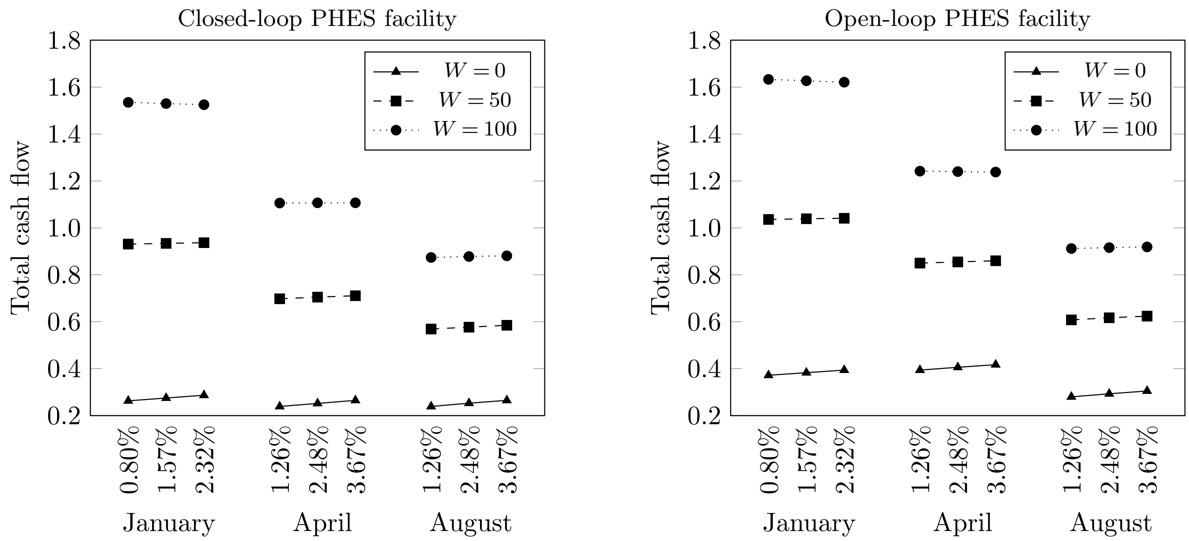
Note. v_1^* is the optimal expected total cash flow (in million dollars).

7.1. Comparison of the Heuristics

We evaluate the performances of our heuristic solution methods in the nine instances in which the PHES facility is open-loop; $W \in \{50, 100, 150\}$; NPF $= 0.80\%$ in January, and NPF $= 1.26\%$ in April and August; $C_U = C_L = 1000$ MWh; and $\zeta_a = \zeta_b = 25$ MWh. All computations were executed on a dual 3.7 GHz Intel Xeon W-2255 CPU server with 96 GB of RAM. Table 1 exhibits the optimality gaps and computation times of our heuristic methods.

Our PA method yields near-optimal solutions with a maximum distance of only 0.07% from the optimal profit, and reduces the computation time of the optimal algorithm by 76.1% on average and by up to 83.7%. Our RPA method performs only slightly worse than our PA method: it yields solutions with a maximum distance of only 1.19% and an average distance of 0.59% from the optimal profit. Our RPA method, however, provides a further significant advantage in computations: it reduces the computation time of the optimal algorithm by 93.6% on average and by up to 95.6%. We run Algorithm 3 of the B1 method no longer than the solution times of our PA method. The B1 method fails to ensure convergence to the optimal profit within these time limits, yielding solutions with an average distance of 15.38% from the optimal profit. The B2 method provides instantaneous solutions with an average distance of 3.12% from the optimal profit. Hence, our PA and RPA methods have a better solution quality than the B1 and B2 methods from the literature. The B2 method is computationally very simple, but it performs substantially worse with respect to objective value. All these results underscore the practical value of our structural analysis.

Figure 6 Total cash flow (in million dollars) vs. NPF when $C_U = C_L = 1000$ MWh.



7.2. Discussion of the Numerical Results

We solved the recursion of our MDP to optimality when $W \in \{0, 50, 100\}$, $\zeta_a = 50$ MWh, and $\zeta_b = 25$ MWh. Figure 6 compares the open-loop and closed-loop PHES configurations in terms of the total cash flow when $C_U = C_L = 1000$ MWh: the existence of a natural inflow in the upper reservoir improves the profit by 19.9% on average in these instances. We observe that the open-loop configuration has the greatest benefit when W is 0 and the lowest benefit when W is 100: A large wind farm negates the need for releasing water to generate energy in the PHES facility. The role of the PHES facility as an energy generator is thus less critical when W is large, reducing the benefit of having a natural inflow. We also observe that increasing the number of wind turbines improves the profits more in January than in April and August, because the wind energy potential is greater in January than in April and August. Another important observation is that the PHES facilities with no wind farm benefit from an increment in NPF more than the other system configurations: the systems with limited energy supply better exploits the arbitrage opportunity. However, the PHES facilities with large wind farms suffer from an increment in NPF in seasons with high energy potential (January and April): the transmission line capacity limits the arbitrage capability.

Figure 7 compares the closed-loop and open-loop PHES configurations in terms of the amount of energy consumed to pump water (ECP) for the same instances. The values of ECP are smaller in open-loop facilities than in closed-loop facilities: the existence of a natural inflow provides an additional energy input to the system, reducing the need for pumping water to store wind energy. For both PHES configurations, ECP tends to increase with NPF: the operator pumps more water to purchase more energy when the negative prices occur more frequently. The values of ECP grow

Figure 7 Energy consumed to pump water (in MWh) vs. NPF when $C_U = C_L = 1000$ MWh.

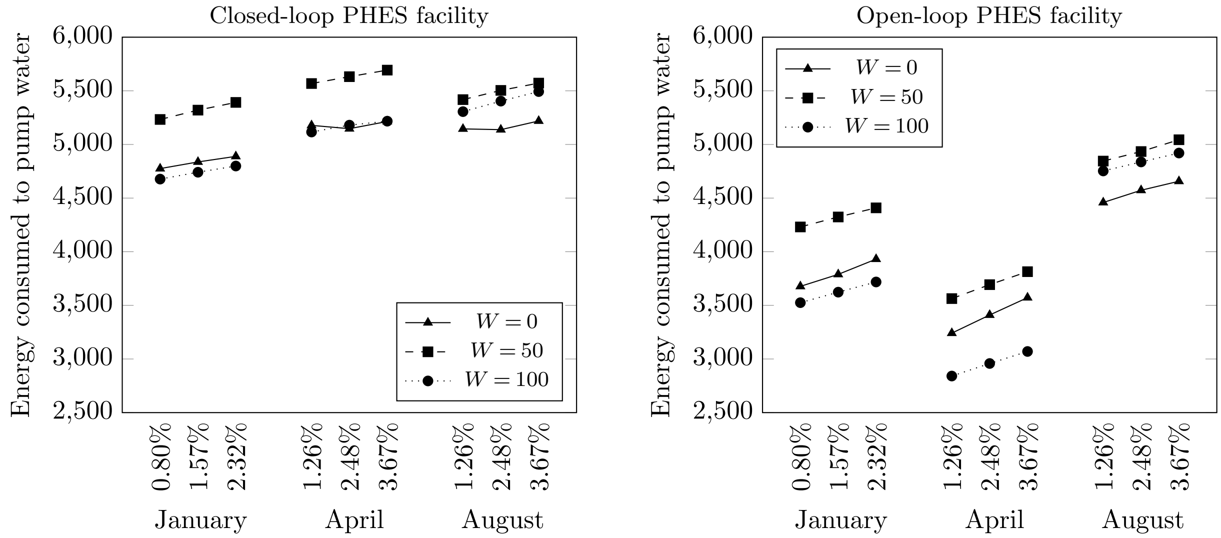
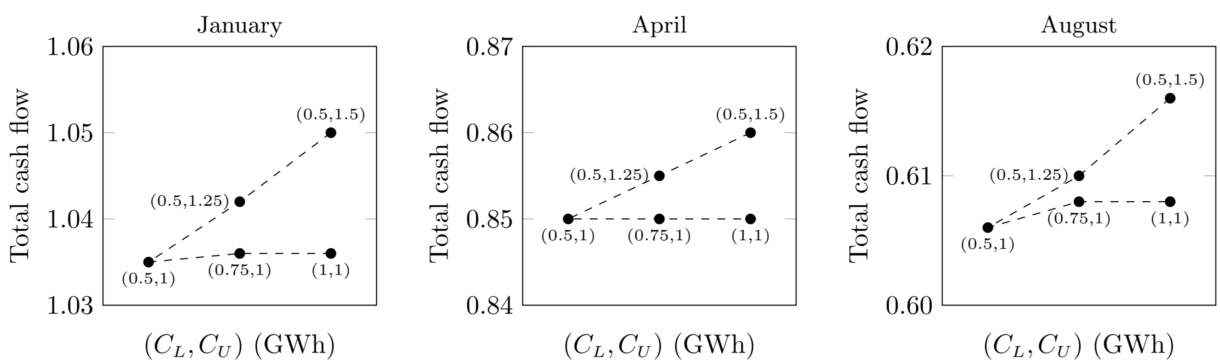


Figure 8 Total cash flow (in million dollars) for open-loop PHES facilities when $W = 50$, NPF $= 0.80\%$ in January, and NPF $= 1.26\%$ in April and August.



as W increases from 0 to 50 and drops as W increases from 50 to 100: The PHES facility can pump water by only purchasing energy from the market when $W = 0$, while it can pump more water by also using the available wind energy when $W = 50$. However, when $W = 100$, the transmission capacity limits the need for storing wind energy as well as generating energy in the PHES facility.

Figure 8 exhibits the total cash flow when the PHES facility is open-loop, $W = 50$, NPF $= 0.80\%$ in January, and NPF $= 1.26\%$ in April and August. We observe that increasing the capacity of the upper reservoir seems to be more profitable (although not very significant) than increasing the capacity of the lower reservoir: A larger upper reservoir increases the energy generation potential by allowing the PHES facility to better utilize the natural inflow in the upper reservoir. Such a potential can provide an immediate positive return in any period with a positive price. A larger lower reservoir, on the other hand, increases the energy storage potential, which can provide a positive return in future periods with positive prices after it is converted to the generation potential.

8. Concluding Remarks

We have studied the energy generation and storage problem for a hybrid system that consists of an open-loop PHES facility and a wind farm and participates in a wholesale-market framework with a limited transmission capacity. Modeling this problem as an MDP, we characterize the optimal policy structure when the price is assumed to be positive throughout the finite horizon. Leveraging our structural results, we construct two policy-approximation algorithms as heuristic methods for solving the more general problem in which the price can also be negative. Numerical experiments in this general problem reveal that our policy-approximation algorithms enable substantial computational savings with no significant drain on profits. Numerical results also imply that the open-loop PHES facilities provide 19.9% more profit on average than the closed-loop PHES facilities.

Future research may extend our analysis to more complex PHES facilities that can still be modeled as MDPs with potentially larger state and/or action spaces. Examples include PHES facilities in which there are multiple connected reservoirs with or without natural inflows, energy can also be generated by releasing water from the lower reservoir, or excess water spilling from the upper reservoir feeds the lower reservoir. Future extensions of this study may also consider PHES facilities integrated with a different renewable energy source (e.g., solar, geothermal, and biomass) or multiple renewable energy sources. Lastly, future research could study the energy generation and storage problem for hybrid systems participating in forward electricity markets by incorporating the energy commitment decisions in different time frames. Such an extension may entail developing multi-stage stochastic programming approaches in addition to MDPs.

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Online Appendix. Proofs of the Analytical Results

Proof of Lemma 1. Under Assumption 1, we have $p_t > 0, \forall t \in \mathcal{T}$. Recall that $\theta \in (0,1]$ and $\tau \in (0,1]$. Then, for $a_t > 0$, $R(a_t,b_t,y_t) = p_t(\theta a_t + b_t)\tau \le p_t(a_t/\theta + b_t)\tau \le p_t(a_t/\theta + b_t)/\tau$. Similarly, for $0 \ge a_t/\theta \ge -b_t$, $R(a_t,b_t,y_t) = p_t(a_t/\theta + b_t)\tau \le p_t(a_t/\theta + b_t)/\tau$ and $R(a_t,b_t,y_t) = p_t(a_t/\theta + b_t)\tau \le p_t(\theta a_t + b_t)\tau$. Lastly, for $0 \ge -b_t > a_t/\theta$, $R(a_t,b_t,y_t) = p_t(a_t/\theta + b_t)/\tau \le p_t(a_t/\theta + b_t)\tau \le p_t(\theta a_t + b_t)\tau$. Thus, $R(a_t,b_t,y_t) = \min\{p_t(\theta a_t + b_t)\tau, p_t(a_t/\theta + b_t)\tau, p_t(a_t/\theta + b_t)/\tau\}$. Since the minimum of affine functions is concave, $R(a_t,b_t,y_t)$ is jointly concave in a_t and b_t. □

Proof of Lemma 2. Note that $v_T^*(x_{uT}, x_{lT}, y_T) = v_T^*(x_{uT}, x_{lT} + \alpha, y_T) = v_T^*(x_{uT} + \alpha, x_{lT}, y_T) = 0$. Assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \le v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)} + \alpha, y_{t+1})$, we show $v_t^*(x_{ut}, x_{lt}, y_t) \le v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Note that $(a,b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$. Thus, $v_t^*(x_{ut}, x_{lt}, y_t) = R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a + \alpha, C_L\}, y_{t+1}\Big)\Big] \le v_t^*(x_{ut}, x_{lt} + \alpha, y_t)$. Similarly, assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1}) \le v_{t+1}^*(x_{u(t+1)} + \alpha, x_{l(t+1)}, y_{t+1})$, we show $v_t^*(x_{ut}, x_{lt}, y_t) \le v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$. Note that $(a,b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus, $v_t^*(x_{ut}, x_{lt}, y_t) = R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + \alpha + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \le v_t^*(x_{ut} + \alpha, x_{lt}, y_t)$.

Assuming $v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)} + \alpha, y_{t+1}) \le v_{t+1}^*(x_{u(t+1)}, x_{l(t+1)}, y_{t+1})$ for $x_{u(t+1)} + x_{l(t+1)} \ge C_U$, we show $v_t^*(x_{ut}, x_{lt} + \alpha, y_t) \le v_t^*(x_{ut}, x_{lt}, y_t)$ for $x_{ut} + x_{lt} \ge C_U$. Let $a = a_t^*(x_{ut}, x_{lt} + \alpha, y_t)$ and $b = b_t^*(x_{ut}, x_{lt} + \alpha, y_t)$. We consider the following two scenarios to prove the statement:

- Suppose that $a > -x_{lt}$: Since $(a,b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$ and $a > -x_{lt}$, note that $(a,b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus, $v_t^*(x_{ut}, x_{lt} + \alpha, y_t) = R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + \alpha + a, C_L\}, y_{t+1}\Big)\Big] \le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \le v_t^*(x_{ut}, x_{lt}, y_t)$. The first inequality holds in each of the following two cases: (1) If $C_L \ge x_{lt} + a$, since $x_{ut} + x_{lt} \ge C_U$, $v_{t+1}^*(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + \alpha + a, C_L\}, y_{t+1}) \le v_{t+1}^*(\min\{x_{ut} - a + r_{t+1}, C_U\}, x_{lt} + a, y_{t+1})$ from the induction assumption. (2) If $x_{lt} + a > C_L$, both sides become $v_{t+1}^*(\min\{x_{ut} - a + r_{t+1}, C_U\}, C_L, y_{t+1})$.
- Suppose that $a \le -x_{lt}$: Let $\hat{a} = -x_{lt}$ and $\hat{b} = \min\{b, C_T + x_{lt}/\theta\}$. We show $(\hat{a}, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$: Since $(a,b) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$, note that $0 \le \hat{b} \le b \le g(w_t)$. If $\hat{b} = b$, then $-\tau C_T \le a/\theta + b \le \hat{a}/\theta + \hat{b} \le C_T$. If $\hat{b} = C_T + x_{lt}/\theta$, then $\hat{a}/\theta + \hat{b} = C_T$. Hence $(\hat{a}, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. If $\hat{b} = b$, since $R(\cdot, b, y_t)$ is a non-decreasing function, $R(\hat{a}, \hat{b}, y_t) \ge R(a,b,y_t)$. If $\hat{b} \ne b$, $R(\hat{a}, \hat{b}, y_t) = p_t \tau C_T \ge R(a,b,y_t)$. Thus, $v_t^*(x_{ut}, x_{lt} + \alpha, y_t) = R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, x_{lt} + \alpha + a, y_{t+1}\Big)\Big] \le R(\hat{a}, \hat{b}, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, 0, y_{t+1}\Big)\Big] \le v_t^*(x_{ut}, x_{lt}, y_t)$ from the induction assumption. □

Proof of Lemma 3. Let $a = a_t^*(x_{ut}, x_{lt}, y_t)$ and $b = b_t^*(x_{ut}, x_{lt}, y_t)$. Pick an arbitrary y_t. Assume to the contrary that $a < x_{ut} - C_U \le 0$. Let $\hat{b} = \min\{b, C_T - (x_{ut} - C_U)/\theta\}$. We show that $(x_{ut} - C_U, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$: Since $(a,b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $-\min\{x_{lt}, C_P\} \le a < x_{ut} - C_U \le 0 \le \min\{x_{ut}, C_R\}$. Also, note that $0 \le \hat{b} \le b \le g(w_t)$. If $\hat{b} = b$, we have $(x_{ut} - C_U)/\theta + \hat{b} \le (x_{ut} - C_U)/\theta + C_T - (x_{ut} - C_U)/\theta = C_T$ and $-\tau C_T \le a/\theta + b < (x_{ut} - C_U)/\theta + \hat{b}$. If $\hat{b} = C_T - (x_{ut} - C_U)/\theta$, we have $(x_{ut} - C_U)/\theta + \hat{b} = C_T$. Hence $(x_{ut} - C_U, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. If $\hat{b} = b$, since $R(\cdot, b, y_t)$ is a non-decreasing function, $R(x_{ut} - C_U, \hat{b}, y_t) > R(a,b,y_t)$. If $\hat{b} \ne b$, $R(x_{ut} - C_U, \hat{b}, y_t) = p_t \tau C_T \ge R(a,b,y_t)$. The first inequality in Lemma 2 implies that $v_t^*(x_{ut}, x_{lt}, y_t) = R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, x_{lt} + a, y_{t+1}\Big)\Big] < R(x_{ut} - C_U, \hat{b}, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(C_U, x_{lt} + x_{ut} - C_U, y_{t+1}\Big)\Big]$. Since the action pair $(x_{ut} - C_U, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$ leads to a larger profit function, we have a contradiction. Thus $a = a_t^*(x_{ut}, x_{lt}, y_t) \ge x_{ut} - C_U$. We denote the maximum amount of wind energy that one can generate for a given water flow action a by

$$\bar{b} := \begin{cases} \min\{g(w_t), C_T - a/\theta\} & \text{if } a < 0 \\ \min\{g(w_t), C_T - \theta a\} & \text{if } a \ge 0. \end{cases}$$

Note that $b \le \bar{b}$ and $(a, \bar{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Since $R(a, \cdot, y_t)$ is an increasing function, the action pair $(a, \bar{b})$ is more profitable than the pair (a,b). Hence $b = \bar{b}$. □

Proof of Proposition 1. Note that $v_T^*(.)$ satisfies properties (a)–(c). Pick an arbitrary $t<T$. Assuming $v_{t+1}^*(.)$ satisfies properties (a)–(c), we will prove that $v_t^*(.)$ satisfies properties (a)–(c).

(a) First we prove that $a_{lhs} := v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) \le v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t) =: a_{rhs}$. Let $a = a_t^*(x_{ut}+\alpha, x_{lt}, y_t)$ and $c = a_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$. Also, let $b = b_t^*(x_{ut}+\alpha, x_{lt}, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}+\alpha+\beta, y_t)$. Lemma 3 implies that $C_U \ge x_{ut}+\alpha-a$ and $C_U \ge x_{ut}-c$. Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note $-x_{lt}-\beta < -x_{lt} \le a$. Hence $(a,b)\in\mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. We consider the following five scenarios to prove the statement:

(a1) Suppose that $\alpha+c>a$ and $C_L \ge x_{lt}+\alpha+\beta+c$: Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note $-x_{lt}-\alpha \le a-\alpha < c$. Hence, and since $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, we have $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$. Thus:

$$\begin{aligned}
a_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}\Big)\Big] \\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+\beta+a, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\alpha+\beta+c, y_{t+1}\Big)\Big] \le a_{rhs}.
\end{aligned}$$

The second inequality above holds in each of the following three cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha + a < C_U - x_{ut} + c$, as we assume $v_{t+1}^*(.)$ satisfies property (a), $v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1}) \le v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+\beta+a, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+\beta+c, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$, as we assume $v_{t+1}^*(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v_{t+1}^*(C_U,\cdot,y_{t+1})$), $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\beta+a, y_{t+1}) \le v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+a+C_U-x_{ut}+c-r_{t+1}, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\beta+a+C_U-x_{ut}+c-r_{t+1}, y_{t+1}) \le v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+\beta+c, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < C_U - x_{ut} + c < r_{t+1}$, as we assume $v_{t+1}^*(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v_{t+1}^*(C_U,\cdot,y_{t+1})$), $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\alpha+c, y_{t+1}) \le v_{t+1}^*(C_U, x_{lt}+\beta+a, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\alpha+\beta+c, y_{t+1})$.

(a2) Suppose that $\alpha+c>a$, $x_{lt}+\alpha+\beta+c > C_L \ge x_{lt}+a$, and $x_{lt}+\alpha+c \le C_L$: Recall from scenario (a1) that $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$ when $\alpha+c>a$. Thus:

$$\begin{aligned}
a_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}\Big)\Big] \\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L-\alpha-c+a, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \le a_{rhs}.
\end{aligned}$$

Our steps to prove the second inequality are similar to those in scenario (a1). By Lemma 2, since $C_L-\alpha-c+a < C_L$ and $C_L-\alpha-c < x_{lt}+\beta$, the third inequality holds.

(a3) Suppose that $\alpha+c>a$, $x_{lt}+\alpha+\beta+c > C_L \ge x_{lt}+a$, and $x_{lt}+\alpha+c > C_L$: Recall from scenario (a1) that $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$ when $\alpha+c>a$. Thus:

$$\begin{aligned}
a_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]
\end{aligned}$$

$$\leq R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big]$$
$$- R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \leq a_{rhs}.$$

By Lemma 2, since $C_L \geq x_{lt}+a$, the second inequality holds.

(a4) Suppose that $\alpha+c>a$ and $x_{lt}+\alpha+\beta+c>x_{lt}+a>C_L$: Recall from scenario (a1) that $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$ when $\alpha+c>a$. Thus: $a_{lhs} \leq R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] - R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \leq a_{rhs}$.

(a5) Suppose that $\alpha+c\leq a$: Let $\hat{b}=\min\{g(w_t), \max\{C_T-(a-\alpha)/\theta, C_T-\theta(a-\alpha)\}\}$ and $\hat{d}=\min\{g(w_t), \max\{C_T-(c+\alpha)/\theta, C_T-\theta(c+\alpha)\}\}$. Note that $c\leq a-\alpha<a$ and $c<c+\alpha\leq a$. Thus, $d\geq\hat{b}\geq b$ and $d\geq\hat{d}\geq b$. We show that $(a-\alpha,\hat{b})\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$: Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note that $-x_{lt}-\alpha\leq a-\alpha\leq x_{ut}$ and $a-\alpha<a\leq C_R$. Since $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, note that $-C_P\leq c\leq a-\alpha$. Hence, $-\min\{x_{lt}+\alpha, C_P\}\leq a-\alpha\leq\min\{x_{ut}, C_R\}$. Then, by the construction of $\hat{b}$, $(a-\alpha,\hat{b})\in\mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$. We also show that $(c+\alpha,\hat{d})\in\mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$: Since $(c,d)\in\mathbb{U}(x_{ut}, x_{lt}+\alpha+\beta, y_t)$, note that $-x_{lt}-\beta\leq c+\alpha\leq x_{ut}+\alpha$ and $-C_P\leq c\leq c+\alpha$. Since $(a,b)\in\mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$, note that $c+\alpha\leq a\leq C_R$. Hence, $-\min\{x_{lt}+\beta, C_P\}\leq c+\alpha\leq\min\{x_{ut}+\alpha, C_R\}$. Then, by the construction of $\hat{d}$, $(c+\alpha,\hat{d})\in\mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. Thus:

$$\begin{aligned}
a_{lhs} &\leq R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(a-\alpha,\hat{b},y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&= R(a,b,y_t) - R(a-\alpha,\hat{b},y_t)\\
&\leq R(c+\alpha,\hat{d},y_t) - R(c,d,y_t)\\
&= R(c+\alpha,\hat{d},y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\alpha+\beta+c, C_L\}, y_{t+1}\Big)\Big]\\
&\quad - R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\alpha+\beta+c, C_L\}, y_{t+1}\Big)\Big] \leq a_{rhs}.
\end{aligned}$$

Recall that $c\leq a-\alpha<a$, $c<c+\alpha\leq a$, $d\geq\hat{b}\geq b$, and $d\geq\hat{d}\geq b$. The second inequality above holds in each of the following six cases:

(1) Suppose that $d=\hat{d}=\hat{b}=b=g(w_t)$. Since $R(\cdot, g(w_t), y_t)$ is concave, $R(a, g(w_t), y_t) - R(a-\alpha, g(w_t), y_t) \leq R(c+\alpha, g(w_t), y_t) - R(c, g(w_t), y_t)$.

(2) Suppose that $d=\hat{d}=\hat{b}=g(w_t)$ and $b<g(w_t)$. Since $R(\cdot, g(w_t), y_t)$ is concave and $R(a,\cdot,y_t)$ is an increasing function, $R(a,b,y_t)-R(a-\alpha,\hat{b},y_t) = R(a,b,y_t)-R(a-\alpha,g(w_t),y_t) < R(a,g(w_t),y_t)-R(a-\alpha,g(w_t),y_t) \leq R(c+\alpha,g(w_t),y_t)-R(c,g(w_t),y_t) = R(c+\alpha,\hat{d},y_t)-R(c,d,y_t)$.

(3) Suppose that $d=\hat{d}=g(w_t)$, $\hat{b}<g(w_t)$, and $b<g(w_t)$. Note that $R(a,b,y_t)=R(a-\alpha,\hat{b},y_t)=p_t\tau C_T$. Since $R(\cdot,g(w_t),y_t)$ is an increasing function, $R(a,b,y_t)-R(a-\alpha,\hat{b},y_t)=0\leq R(c+\alpha,g(w_t),y_t) - R(c,g(w_t),y_t) = R(c+\alpha,\hat{d},y_t)-R(c,d,y_t)$.

(4) Suppose that $d=\hat{b}=g(w_t)$, $\hat{d}<g(w_t)$, and $b<g(w_t)$. Note that $R(a,b,y_t)=R(c+\alpha,\hat{d},y_t)=p_t\tau C_T$. Since $R(\cdot,g(w_t),y_t)$ is an increasing function, $R(a,b,y_t)-R(a-\alpha,\hat{b},y_t)=p_t\tau C_T - R(a-\alpha,g(w_t),y_t)\leq p_t\tau C_T - R(c,g(w_t),y_t) = R(c+\alpha,\hat{d},y_t)-R(c,d,y_t)$.

(5) Suppose that $d=g(w_t)$, $\hat{b}<g(w_t)$, $\hat{d}<g(w_t)$, and $b<g(w_t)$. Note that $R(a,b,y_t)=R(a-\alpha,\hat{b},y_t)=R(c+\alpha,\hat{d},y_t)=p_t\tau C_T$. Since $R(\cdot,\cdot,y_t)\leq p_t\tau C_T$, $R(a,b,y_t)-R(a-\alpha,\hat{b},y_t)=0\leq p_t\tau C_T - R(c,g(w_t),y_t) = R(c+\alpha,\hat{d},y_t)-R(c,d,y_t)$.

(6) Suppose that $d<g(w_t)$, $\hat{b}<g(w_t)$, $\hat{d}<g(w_t)$, and $b<g(w_t)$. Note that $R(a,b,y_t)=R(a-\alpha,\hat{b},y_t)=R(c+\alpha,\hat{d},y_t)=R(c,d,y_t)=p_t\tau C_T$. Hence $R(a,b,y_t)-R(a-\alpha,\hat{b},y_t)=R(c+\alpha,\hat{d},y_t)-R(c,d,y_t)$.

(b) Next we prove that $b_{lhs} := v_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t) - v_t^*(x_{ut}+\beta, x_{lt}+\alpha, y_t) \le v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}+\alpha, y_t) =: b_{rhs}$. Let $a = a_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$ and $c = a_t^*(x_{ut}, x_{lt}+\alpha, y_t)$. Also, let $b = b_t^*(x_{ut}+\alpha+\beta, x_{lt}, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}+\alpha, y_t)$. Recall from Lemma 3 that $C_U \ge x_{ut}+\alpha+\beta-a$ and $C_U \ge x_{ut}-c$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note $c \le x_{ut} < x_{ut}+\beta$. Hence $(c,d) \in \mathbb{U}(x_{ut}+\beta, x_{lt}+\alpha, y_t)$. We consider the following four scenarios to prove the statement:

(b1) Suppose that $\alpha+c > a$ and $C_L \ge x_{lt}+\alpha+c > x_{lt}+a$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\alpha, y_t)$, note $a < c+\alpha \le x_{ut}+\alpha$. Hence, and since $(a,b) \in \mathbb{U}(x_{ut}+\alpha+\beta, x_{lt}, y_t)$, we have $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned} b_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha+\beta-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\beta-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}\Big)\Big] \\ &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}\Big)\Big] \le b_{rhs}. \end{aligned}$$

The second inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha - \beta + a$, as we assume $v_{t+1}^*(.)$ satisfies property (b), $v_{t+1}^*(x_{ut}+\alpha+\beta-a+r_{t+1}, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a, y_{t+1}) \le v_{t+1}^*(x_{ut}+\beta-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v_{t+1}^*(.)$ satisfies property (b) and by Lemma 2, $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a, y_{t+1}) \le v_{t+1}^*(C_U+a-\alpha-c, x_{lt}+\alpha+c, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1}) \le v_{t+1}^*(\min\{x_{ut}+\beta-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} - \alpha + a$, by Lemma 2, $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\beta-c+r_{t+1}, C_U\}, x_{lt}+\alpha+c, y_{t+1}) \le v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\alpha+c, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\alpha+c, y_{t+1})$.

(b2) Suppose that $\alpha+c > a$ and $x_{lt}+\alpha+c > C_L \ge x_{lt}+a$: Recall from scenario (b1) that $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$ when $\alpha+c > a$. Thus:

$$\begin{aligned} b_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha+\beta-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\beta-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\ &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \le b_{rhs}. \end{aligned}$$

The second inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha - \beta + a$, as we assume $v_{t+1}^*(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v_{t+1}^*(\cdot, x_{lt}, y_{t+1})$), $v_{t+1}^*(x_{ut}+\alpha+\beta-a+r_{t+1}, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a, y_{t+1}) \le v_{t+1}^*(x_{ut}+x_{lt}+\alpha+\beta-C_L+r_{t+1}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+x_{lt}+\alpha-C_L+r_{t+1}, C_L, y_{t+1}) \le v_{t+1}^*(x_{ut}+\beta-c+r_{t+1}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, C_L, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v_{t+1}^*(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v_{t+1}^*(\cdot, x_{lt}, y_{t+1})$) and by Lemma 2, $v_{t+1}^*(C_U, x_{lt}+a, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a, y_{t+1}) \le v_{t+1}^*(C_U+x_{lt}+a-C_L, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}+x_{lt}+\alpha-C_L+r_{t+1}, C_L, y_{t+1}) \le v_{t+1}^*(C_U-\alpha+a-c, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, C_L, y_{t+1}) \le v_{t+1}^*(\min\{x_{ut}+\beta-c+r_{t+1}, C_U\}, C_L, y_{t+1}) - v_{t+1}^*(x_{ut}-c+r_{t+1}, C_L, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} - \alpha + a$, by Lemma 2, $v^*_{t+1}(C_U, x_{lt} + a, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) \le v^*_{t+1}(C_U, x_{lt} + a, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, x_{lt} + a, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(b3) Suppose that $\alpha + c > a$ and $x_{lt} + \alpha + c > x_{lt} + a > C_L$: Recall from scenario (b1) that $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$ when $\alpha + c > a$. Thus:

$$\begin{aligned} b_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha + \beta - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\ &\le R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \le b_{rhs}. \end{aligned}$$

The second inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha - \beta + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(x_{ut} + \alpha + \beta - a + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} + \beta - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$) and by Lemma 2, $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U - \alpha + a - c, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut} + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha - \beta + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} - \alpha + a$, by Lemma 2, $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(b4) Suppose that $\alpha + c \le a$: Let $\hat{b} = \min\{g(w_t), \max\{C_T - (a - \alpha)/\theta, C_T - \theta(a - \alpha)\}\}$ and $\hat{d} = \min\{g(w_t), \max\{C_T - (c + \alpha)/\theta, C_T - \theta(c + \alpha)\}\}$. Note that $c \le a - \alpha < a$ and $c < c + \alpha \le a$. Thus, $d \ge \hat{b} \ge b$ and $d \ge \hat{d} \ge b$. We show that $(a - \alpha, \hat{b}) \in \mathbb{U}(x_{ut} + \beta, x_{lt} + \alpha, y_t)$: Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha + \beta, x_{lt}, y_t)$, note that $-x_{lt} - \alpha \le a - \alpha \le x_{ut} + \beta$ and $a - \alpha < a \le C_R$. Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$, note that $-C_P \le c \le a - \alpha$. Hence, $-\min\{x_{lt} + \alpha, C_P\} \le a - \alpha \le \min\{x_{ut} + \beta, C_R\}$. Then, by the construction of $\hat{b}$, $(a - \alpha, \hat{b}) \in \mathbb{U}(x_{ut} + \beta, x_{lt} + \alpha, y_t)$. We also show that $(c + \alpha, \hat{d}) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$: Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt} + \alpha, y_t)$, note that $-x_{lt} \le c + \alpha \le x_{ut} + \alpha$ and $-C_P \le c \le c + \alpha$. Since $(a, b) \in \mathbb{U}(x_{ut} + \alpha + \beta, x_{lt}, y_t)$, note that $c + \alpha \le a \le C_R$. Hence, $-\min\{x_{lt}, C_P\} \le c + \alpha \le \min\{x_{ut} + \alpha, C_R\}$. Then, by the construction of $\hat{d}$, $(c + \alpha, \hat{d}) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned} b_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha + \beta - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \\ &\quad - R(a - \alpha, \hat{b}, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha + \beta - a + r_{t+1}, C_U\}, \min\{x_{lt} + a, C_L\}, y_{t+1}\Big)\Big] \\ &= R(a,b,y_t) - R(a - \alpha, \hat{b}, y_t) \\ &\le R(c + \alpha, \hat{d}, y_t) - R(c,d,y_t) \\ &= R(c + \alpha, \hat{d}, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + \alpha + c, C_L\}, y_{t+1}\Big)\Big] \\ &\quad - R(c,d,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + \alpha + c, C_L\}, y_{t+1}\Big)\Big] short \le b_{rhs}. \end{aligned}$$

Recall that $c \le a - \alpha < a$, $c < c + \alpha \le a$, $d \ge \hat{b} \ge b$, and $d \ge \hat{d} \ge b$. Our steps to prove the second inequality are similar to those in scenario (a5).

(c) Last we prove that $c_{lhs} := v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}+\beta, y_t) \le v_t^*(x_{ut}+\alpha, x_{lt}, y_t) - v_t^*(x_{ut}, x_{lt}, y_t) =: c_{rhs}$. Let $a = a_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $c = a_t^*(x_{ut}, x_{lt}, y_t)$. Also, let $b = b_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}, y_t)$. Lemma 3 implies $C_U \ge x_{ut}+\alpha-a$ and $C_U \ge x_{ut}-c$. We consider the following nine scenarios to prove the statement:

(c1) Suppose that $a > c$, $\alpha + c \ge a$, and $C_L \ge x_{lt}+\beta+c$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note $-x_{lt} \le c < a$. Hence, and since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, we have $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$. Also, since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\beta, y_t)$. Thus:

$$\begin{aligned}
&v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) - v_t^*(x_{ut}+\alpha, x_{lt}, y_t)\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big]\\
&- R(a,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&\le \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big]\\
&- \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L-\beta\}, y_{t+1}\Big)\Big]\\
&\le \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big]\\
&- \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L-\beta\}, y_{t+1}\Big)\Big]\\
&\le R(c,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\beta+c, y_{t+1}\Big)\Big]\\
&- R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}\Big)\Big]\\
&\le v_t^*(x_{ut}, x_{lt}+\beta, y_t) - v_t^*(x_{ut}, x_{lt}, y_t).
\end{aligned}$$

By Lemma 2, the second inequality holds. The third inequality holds as we assume $v_{t+1}^*(.)$ satisfies property (c). The fourth inequality holds as we assume $v_{t+1}^*(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v_{t+1}^*(x_{ut}, \cdot, y_{t+1})$).

(c2) Suppose that $a > c$, $\alpha + c \ge a$, and $x_{lt}+\beta+c > C_L$: Recall from scenario (c1) that $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}, y_t)$ and $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}+\beta, y_t)$ when $a > c$ and $\alpha + c \ge a$. Thus:

$$\begin{aligned}
c_{lhs} &\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&- R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&- R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&\le R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a, C_L\}, y_{t+1}\Big)\Big]\\
&- R(c,d,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big] \le c_{rhs}.
\end{aligned}$$

The second inequality holds as we assume $v_{t+1}^*(.)$ satisfies property (c). By Lemma 2, the third inequality holds.

(c3) Suppose that $a > c$, $\alpha + c < a$, and $C_L \ge x_{lt}+\beta+a$: Let $\hat{b} = \min\{g(w_t), \max\{C_T - (a-\alpha)/\theta, C_T - \theta(a-\alpha)\}\}$ and $\hat{d} = \min\{g(w_t), \max\{C_T - (c+\alpha)/\theta, C_T - \theta(c+\alpha)\}\}$. We show that $(a-\alpha, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $a - \alpha \le x_{ut}$ and $a - \alpha < a \le C_R$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $-\min\{x_{lt}, C_P\} \le c \le a - \alpha$. Hence, $-\min\{x_{lt}, C_P\} \le a - \alpha \le \min\{x_{ut}, C_R\}$. Then, by the construction of $\hat{b}$, $(a-\alpha, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus, and since $c = a_t^*(x_{ut}, x_{lt}, y_t)$ and $d = b_t^*(x_{ut}, x_{lt}, y_t)$, we have

$$\begin{aligned}
v_t^*(x_{ut}, x_{lt}, y_t) &= R(c,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big]\\
&\ge R(a-\alpha, \hat{b}, y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+a-\alpha, C_L\}, y_{t+1}\Big)\Big]. \qquad \text{(EC.1)}
\end{aligned}$$

We also show that $(c+\alpha, \hat{d}) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $c + \alpha \le x_{ut} + \alpha$, $-x_{lt} - \beta < -x_{lt} \le c < c + \alpha$, and $-C_P \le c \le c + \alpha$. Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $c + \alpha \le$

$a \le C_R$. Hence, $-\min\{x_{lt}+\beta, C_P\} \le c+\alpha \le \min\{x_{ut}+\alpha, C_R\}$. Then, by the construction of $\hat{d}$, $(c+\alpha,\hat{d}) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. Thus, and since $a = a_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $b = b_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, we have

$$\begin{aligned} v_t^*(x_{ut}+\alpha, x_{lt}+\beta, y_t) &= R(a,b,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big] \\ &\ge R(c+\alpha,\hat{d},y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+\alpha+c, C_L\}, y_{t+1}\Big)\Big]. \end{aligned} \tag{EC.2}$$

The inequalities (EC.1) and (EC.2) imply that

$$\begin{aligned} &R(c,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}\Big)\Big] \\ &- R(a-\alpha,\hat{b},y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a-\alpha, y_{t+1}\Big)\Big] \\ &\ge R(c+\alpha,\hat{d},y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\beta+\alpha+c, y_{t+1}\Big)\Big] \\ &- R(a,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+\beta+a, y_{t+1}\Big)\Big]. \end{aligned}$$

But this leads to a contradiction: We must have

$$\begin{aligned} &v_{t+1}^*(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a-\alpha, y_{t+1}) \\ &\le v_{t+1}^*(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+\beta+\alpha+c, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+\beta+a, y_{t+1}). \end{aligned}$$

The inequality above holds in each of the following three cases:

(1) If $r_{t+1} \le C_U - x_{ut} + c$, as we assume $v_{t+1}^*(.)$ satisfies property (a), $v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+c, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a-\alpha, y_{t+1}) \le v_{t+1}^*(x_{ut}-c+r_{t+1}, x_{lt}+\beta+\alpha+c, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+\beta+a, y_{t+1})$.

(2) If $C_U - x_{ut} + c < r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v_{t+1}^*(.)$ satisfies properties (a) and (c), $v_{t+1}^*(C_U, x_{lt}+c, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\beta+\alpha+c, y_{t+1}) \le v_{t+1}^*(C_U+\alpha-a+c, x_{lt}+a-\alpha, y_{t+1}) - v_{t+1}^*(C_U+\alpha-a+c, x_{lt}+\beta+a, y_{t+1}) \le v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+a-\alpha, y_{t+1}) - v_{t+1}^*(x_{ut}+\alpha-a+r_{t+1}, x_{lt}+\beta+a, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < r_{t+1}$, as we assume $v_{t+1}^*(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v_{t+1}^*(x_{ut}, \cdot, y_{t+1})$), $v_{t+1}^*(C_U, x_{lt}+c, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+a-\alpha, y_{t+1}) \le v_{t+1}^*(C_U, x_{lt}+\beta+\alpha+c, y_{t+1}) - v_{t+1}^*(C_U, x_{lt}+\beta+a, y_{t+1})$.

Also, recall from scenario (a5) that $R(c,d,y_t) - R(a-\alpha,\hat{b},y_t) \le R(c+\alpha,\hat{d},y_t) - R(a,b,y_t)$. Hence this scenario is not possible.

(c4) Suppose that $a > c$, $\alpha + c < a$, and $x_{lt}+\beta+a > C_L \ge x_{lt}+a-\alpha$: The inequalities (EC.1) and (EC.2) imply that

$$\begin{aligned} &R(c,d,y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}\Big)\Big] \\ &- R(a-\alpha,\hat{b},y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a-\alpha, y_{t+1}\Big)\Big] \\ &\ge R(c+\alpha,\hat{d},y_t) + \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+\alpha+c, C_L\}, y_{t+1}\Big)\Big] \\ &- R(a,b,y_t) - \mathbb{E}\Big[v_{t+1}^*\Big(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]. \end{aligned}$$

But this leads to a contradiction: We must have

$$\begin{aligned} &v_{t+1}^*(\min\{x_{ut}-c+r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, x_{lt}+a-\alpha, y_{t+1}) \\ &\le v_{t+1}^*(\min\{x_{ut}-c+r_{t+1}, C_U\}, C_L+\alpha-a+c, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L, y_{t+1}) \\ &\le v_{t+1}^*(\min\{x_{ut}-c+r_{t+1}, C_U\}, \min\{x_{lt}+\beta+\alpha+c, C_L\}, y_{t+1}) - v_{t+1}^*(\min\{x_{ut}+\alpha-a+r_{t+1}, C_U\}, C_L, y_{t+1}). \end{aligned}$$

By Lemma 2, the second inequality holds. The first inequality above holds in each of the following three cases:

(1) If $r_{t+1} \leq C_U - x_{ut} + c$, as we assume $v^*_{t+1}(.)$ satisfies property (a), $v^*_{t+1}(x_{ut} - c + r_{t+1}, x_{lt} + c, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, x_{lt} + a - \alpha, y_{t+1}) \leq v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L + \alpha - a + c, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, C_L, y_{t+1})$.

(2) If $C_U - x_{ut} + c < r_{t+1} \leq C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c), $v^*_{t+1}(C_U, x_{lt} + c, y_{t+1}) - v^*_{t+1}(C_U, C_L + \alpha - a + c, y_{t+1}) \leq v^*_{t+1}(C_U + \alpha - a + c, x_{lt} + a - \alpha, y_{t+1}) - v^*_{t+1}(C_U + \alpha - a + c, C_L, y_{t+1}) \leq v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, x_{lt} + a - \alpha, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, C_L, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < r_{t+1}$, as we assume $v^*_{t+1}(.)$ satisfies properties (a) and (c) (which together imply the concavity of $v^*_{t+1}(x_{ut}, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, x_{lt} + c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + a - \alpha, y_{t+1}) \leq v^*_{t+1}(C_U, C_L + \alpha - a + c, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Also, recall that $R(c, d, y_t) - R(a - \alpha, \hat{b}, y_t) \leq R(c + \alpha, \hat{d}, y_t) - R(a, b, y_t)$. Hence this scenario is not possible.

(c5) Suppose that $a > c$, $\alpha + c < a$, and $x_{lt} + a - \alpha > C_L$: The inequalities (EC.1) and (EC.2) imply that

$$\begin{aligned}
&R(c, d, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}\Big)\Big] \\
&- R(a - \alpha, \hat{b}, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big] \\
&\geq R(c + \alpha, \hat{d}, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + \beta + \alpha + c, C_L\}, y_{t+1}\Big)\Big] \\
&- R(a, b, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big].
\end{aligned}$$

But this leads to a contradiction: Lemma 2 implies that

$$\begin{aligned}
&v^*_{t+1}(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}) \\
&\leq v^*_{t+1}(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + \beta + \alpha + c, C_L\}, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}).
\end{aligned}$$

Also, recall that $R(c, d, y_t) - R(a - \alpha, \hat{b}, y_t) \leq R(c + \alpha, \hat{d}, y_t) - R(a, b, y_t)$. Hence this scenario is not possible.

(c6) Suppose that $c \geq a$ and $\beta + a \geq c$: Since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $a \leq c \leq x_{ut}$. Hence, and since $(a, b) \in \mathbb{U}(x_{ut} + \alpha, x_{lt} + \beta, y_t)$, we have $(a, b) \in \mathbb{U}(x_{ut}, x_{lt} + \beta, y_t)$. Also, since $(c, d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $(c, d) \in \mathbb{U}(x_{ut} + \alpha, x_{lt}, y_t)$. Thus:

$$\begin{aligned}
c_{lhs} &\leq R(a, b, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, \min\{x_{lt} + \beta + a, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(a, b, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + \beta + a, C_L\}, y_{t+1}\Big)\Big] \\
&\leq \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}\Big)\Big] \\
&\leq R(c, d, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - c + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}\Big)\Big] \\
&\quad - R(c, d, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}\Big)\Big] \leq c_{rhs}.
\end{aligned}$$

The second inequality holds as we assume $v^*_{t+1}(.)$ satisfies property (c). The third inequality above holds in each of the following four cases:

(1) If $r_{t+1} \leq C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(x_{ut} - a + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1}) \leq v^*_{t+1}(x_{ut} + \alpha - c + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha + a < r_{t+1} \leq C_U - x_{ut} - \alpha + c$, by Lemma 2 and as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(C_U, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt} + c, C_L\}, y_{t+1}) \leq v^*_{t+1}(C_U, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(C_U - \alpha, \min\{x_{lt} + c, C_L\}, y_{t+1}) \leq v^*_{t+1}(x_{ut} + \alpha - c + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + c < r_{t+1} \le C_U - x_{ut} + c$, by Lemma 2, $v^*_{t+1}(C_U, \min\{x_{lt}+c, C_L\}, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} - a + r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, \min\{x_{lt}+c, C_L\}, y_{t+1}) - v^*_{t+1}(x_{ut} - c + r_{t+1}, \min\{x_{lt} + c, C_L\}, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, \min\{x_{lt}+c, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{lt}+c, C_L\}, y_{t+1})$.

(c7) Suppose that $c \ge a$, $\beta + a < c$, and $C_L \ge x_{lt} + c$: Let $\hat{b} = \min\{g(w_t), \max\{C_T - (a+\beta)/\theta, C_T - \theta(a+\beta)\}\}$ and $\hat{d} = \min\{g(w_t), \max\{C_T - (c-\beta)/\theta, C_T - \theta(c-\beta)\}\}$. We show that $(a+\beta, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$: Since $(a,b) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, note that $-x_{lt} \le a+\beta$ and $-C_P \le a \le a+\beta$. Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $a + \beta < c \le \min\{x_{ut}, C_R\}$. Hence, $-\min\{x_{lt}, C_P\} \le a+\beta \le \min\{x_{ut}, C_R\}$. Then, by the construction of $\hat{b}$, $(a+\beta, \hat{b}) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$. Thus, and since $c = a^*_t(x_{ut}, x_{lt}, y_t)$ and $d = b^*_t(x_{ut}, x_{lt}, y_t)$, we have

$$\begin{aligned} v_t(x_{ut}, x_{lt}, y_t) &= R(c, d, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big] \\ &\ge R(a+\beta, \hat{b}, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a - \beta + r_{t+1}, C_U\}, \min\{x_{lt} + a + \beta, C_L\}, y_{t+1}\Big)\Big]. \end{aligned} \quad \text{(EC.3)}$$

We also show that $(c-\beta, \hat{d}) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$: Since $(c,d) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $-x_{lt} - \beta \le c - \beta$, $c - \beta < c \le x_{ut} < x_{ut} + \alpha$, and $c - \beta < c \le C_R$. Since $(a,b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, note that $-C_P \le a < c - \beta$. Hence, $-\min\{x_{lt}+\beta, C_P\} \le c-\beta \le \min\{x_{ut}+\alpha, C_R\}$. Then, by the construction of $\hat{d}$, $(c-\beta, \hat{d}) \in \mathbb{U}(x_{ut}+\alpha, x_{lt}+\beta, y_t)$. Thus, and since $a = a^*_t(x_{ut}+\alpha, x_{lt}+\beta, y_t)$ and $b = b^*_t(x_{ut}+\alpha, x_{lt}+\beta, y_t)$, we have

$$\begin{aligned} & v_t(x_{ut}+\alpha, x_{lt}+\beta, y_t) = R(a, b, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha - a + r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big] \\ & \ge R(c-\beta, \hat{d}, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha+\beta - c + r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big]. \end{aligned} \quad \text{(EC.4)}$$

The inequalities (EC.3) and (EC.4) imply that

$$\begin{aligned} & R(c, d, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big] \\ & - R(a+\beta, \hat{b}, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - a - \beta + r_{t+1}, C_U\}, \min\{x_{lt}+a+\beta, C_L\}, y_{t+1}\Big)\Big] \\ & \ge R(c-\beta, \hat{d}, y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha+\beta - c + r_{t+1}, C_U\}, \min\{x_{lt}+c, C_L\}, y_{t+1}\Big)\Big] \\ & - R(a, b, y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut}+\alpha - a + r_{t+1}, C_U\}, \min\{x_{lt}+\beta+a, C_L\}, y_{t+1}\Big)\Big]. \end{aligned}$$

But this leads to a contradiction: We must have

$$\begin{aligned} & v^*_{t+1}(\min\{x_{ut} - c + r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} - \beta - a + r_{t+1}, C_U\}, x_{lt}+\beta+a, y_{t+1}) \\ & \le v^*_{t+1}(\min\{x_{ut}+\alpha+\beta - c + r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(\min\{x_{ut}+\alpha - a + r_{t+1}, C_U\}, x_{lt}+\beta+a, y_{t+1}). \end{aligned}$$

The inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies property (b), $v^*_{t+1}(x_{ut} - c + r_{t+1}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, x_{lt}+\beta+a, y_{t+1}) \le v^*_{t+1}(x_{ut}+\alpha+\beta - c + r_{t+1}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(x_{ut}+\alpha - a + r_{t+1}, x_{lt}+\beta+a, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} + \beta + a$, as we assume $v^*_{t+1}(.)$ satisfies property (b) and by Lemma 2, $v^*_{t+1}(x_{ut} - c + r_{t+1}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, x_{lt}+\beta+a, y_{t+1}) \le v^*_{t+1}(C_U + \beta + a - c, x_{lt}+c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\beta+a, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut}+\alpha+\beta - c + r_{t+1}, C_U\}, x_{lt} + c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\beta+a, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} + \beta + a$, by Lemma 2, $v^*_{t+1}(x_{ut} - c + r_{t+1}, x_{lt} + c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\beta+a, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut}+\alpha+\beta - c + r_{t+1}, C_U\}, x_{lt}+c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \beta + a, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, x_{lt}+c, y_{t+1}) - v^*_{t+1}(C_U, x_{lt}+\beta+a, y_{t+1})$.

Also, we must have $R(c,d,y_t) - R(a+\beta,\hat{b},y_t) \le R(c-\beta,\hat{d},y_t) - R(a,b,y_t)$. Our steps to prove this inequality are similar to those in scenario (a5). Hence this scenario is not possible.

(c8) Suppose that $c \ge a$, $\beta + a < c$, and $x_{lt} + c > C_L \ge x_{lt} + \beta + a$: The inequalities (EC.3) and (EC.4) imply that

$$\begin{aligned}
&R(c,d,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&- R(a+\beta,\hat{b},y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - \beta - a + r_{t+1}, C_U\}, x_{lt} + \beta + a, y_{t+1}\Big)\Big]\\
&\ge R(c-\beta,\hat{d},y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&- R(a,b,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, x_{lt} + \beta + a, y_{t+1}\Big)\Big].
\end{aligned}$$

But this leads to a contradiction: We must have

$$\begin{aligned}
&v^*_{t+1}(\min\{x_{ut} - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} - \beta - a + r_{t+1}, C_U\}, x_{lt} + \beta + a, y_{t+1})\\
&\le v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, x_{lt} + \beta + a, y_{t+1}).
\end{aligned}$$

The inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha + \beta - c + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} + x_{lt} - C_L + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + x_{lt} - C_L + \alpha + \beta + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, x_{lt} + \beta + a, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, x_{lt} + \beta + a, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} + \beta + a$, by Lemma 2 and as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U + \beta + a - c, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} + x_{lt} - C_L + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U + x_{lt} - C_L + \beta + a, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, x_{lt} + \beta + a, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \beta + a, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} + \beta + a$, by Lemma 2, $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \beta + a, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \beta + a, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{lt} + \beta + a, y_{t+1})$.

Also, recall from scenario (c7) that $R(c,d,y_t) - R(a+\beta,\hat{b},y_t) \le R(c-\beta,\hat{d},y_t) - R(a,b,y_t)$. Hence this scenario is not possible.

(c9) Suppose that $c \ge a$, $\beta + a < c$, and $x_{lt} + \beta + a > C_L$: The inequalities (EC.3) and (EC.4) imply that

$$\begin{aligned}
&R(c,d,y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&- R(a+\beta,\hat{b},y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} - \beta - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&\ge R(c-\beta,\hat{d},y_t) + \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big]\\
&- R(a,b,y_t) - \mathbb{E}\Big[v^*_{t+1}\Big(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}\Big)\Big].
\end{aligned}$$

But this leads to a contradiction: We must have

$$\begin{aligned}
&v^*_{t+1}(\min\{x_{ut} - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} - \beta - a + r_{t+1}, C_U\}, C_L, y_{t+1})\\
&\le v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(\min\{x_{ut} + \alpha - a + r_{t+1}, C_U\}, C_L, y_{t+1}).
\end{aligned}$$

The inequality above holds in each of the following four cases:

(1) If $r_{t+1} \le C_U - x_{ut} - \alpha + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$), $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(x_{ut} + \alpha + \beta - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} + \alpha - a + r_{t+1}, C_L, y_{t+1})$.

(2) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} \le C_U - x_{ut} + \beta + a$, as we assume $v^*_{t+1}(.)$ satisfies properties (b) and (c) (which together imply the concavity of $v^*_{t+1}(\cdot, x_{lt}, y_{t+1})$) and by Lemma 2, $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(x_{ut} - \beta - a + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U + \beta + a - c, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(3) If $C_U - x_{ut} - \alpha + a < r_{t+1} \le C_U - x_{ut} + c$ and $r_{t+1} > C_U - x_{ut} + \beta + a$, by Lemma 2, $v^*_{t+1}(x_{ut} - c + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(\min\{x_{ut} + \alpha + \beta - c + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(4) If $C_U - x_{ut} + c < r_{t+1}$, both sides become $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Also, recall that $R(c, d, y_t) - R(a + \beta, \hat{b}, y_t) \le R(c - \beta, \hat{d}, y_t) - R(a, b, y_t)$. Hence this scenario is not possible.

Hence $v^*_t(.)$ satisfies properties (a), (b), and (c). □

Proof of Lemma 4. Suppose that $\alpha > 0$ and $\beta > 0$. Fix x_{ut}, x_{lt}, and y_t. Without loss of generality, we assume that $\alpha \ge \beta$. We consider the following sixteen cases to show that $V_t(z_{ut}, x_{ut}, x_{lt}, y_t)$ is concave in z_{ut}, that is,

$$
\begin{aligned}
& V_t(z_{ut}, x_{ut}, x_{lt}, y_t) - V_t(z_{ut} + \alpha, x_{ut}, x_{lt}, y_t) \\
&= \mathbb{E}\Big[v^*_{t+1}\Big(\min\{z_{ut} + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut}, C_L\}, y_{t+1}\Big)\Big] \\
&- \mathbb{E}\Big[v^*_{t+1}\Big(\min\{z_{ut} + \alpha + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}\Big)\Big] \\
&\le \mathbb{E}\Big[v^*_{t+1}\Big(\min\{z_{ut} + \beta + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut} - \beta, C_L\}, y_{t+1}\Big)\Big] \\
&- \mathbb{E}\Big[v^*_{t+1}\Big(\min\{z_{ut} + \alpha + \beta + r_{t+1}, C_U\}, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(z_{ut} + \beta, x_{ut}, x_{lt}, y_t) - V_t(z_{ut} + \alpha + \beta, x_{ut}, x_{lt}, y_t).
\end{aligned}
$$

(1) If $r_{t+1} \le C_U - z_{ut} - \alpha - \beta$ and $x_{ut} + x_{lt} - z_{ut} \le C_L$, by properties (a) and (b) of Proposition 1, $v^*_{t+1}(z_{ut} + r_{t+1}, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) \le v^*_{t+1}(z_{ut} + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(2) If $r_{t+1} \le C_U - z_{ut} - \alpha - \beta$ and $x_{ut} + x_{lt} - z_{ut} - \beta \le C_L < x_{ut} + x_{lt} - z_{ut}$, by Lemma 2 and by properties (b) and (a) of Proposition 1, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}) \le v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(z_{ut} + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(3) If $r_{t+1} \le C_U - z_{ut} - \alpha - \beta$ and $x_{ut} + x_{lt} - z_{ut} - \alpha - \beta \le C_L < x_{ut} + x_{lt} - z_{ut} - \beta$, by properties (b) and (c) of Proposition 1 and by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, C_L - \alpha, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1}) \le v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(4) If $r_{t+1} \le C_U - z_{ut} - \alpha - \beta$ and $C_L < x_{ut} + x_{lt} - z_{ut} - \alpha - \beta$, by properties (b) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$), $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(z_{ut} + \alpha + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \alpha + \beta + r_{t+1}, C_L, y_{t+1})$.

(5) If $C_U - z_{ut} - \alpha - \beta < r_{t+1} \le C_U - z_{ut} - \beta$ and $x_{ut} + x_{lt} - z_{ut} \le C_L$, by properties (b) and (a) of Proposition 1 and by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) \le v^*_{t+1}(C_U - \beta, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) \le v^*_{t+1}(C_U - \beta, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1}) \le v^*_{t+1}(\min\{z_{ut} + \alpha + r_{t+1}, C_U\}, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(6) If $C_U - z_{ut} - \alpha - \beta < r_{t+1} \le C_U - z_{ut} - \beta$ and $x_{ut} + x_{lt} - z_{ut} - \alpha \le C_L < x_{ut} + x_{lt} - z_{ut}$, by Lemma 2 and by properties (b) and (a) of Proposition 1, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, \min\{x_{ut} + x_{lt} - z_{ut} - \beta, C_L\}, y_{t+1}) \le v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L - \beta, y_{t+1}) \le v^*_{t+1}(C_U - \beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L - \beta, y_{t+1}) \le v^*_{t+1}(C_U - \beta, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1}) \le v^*_{t+1}(\min\{z_{ut} + \alpha + r_{t+1}, C_U\}, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(7) If $C_U - z_{ut} - \alpha - \beta < r_{t+1} \le C_U - z_{ut} - \beta$ and $x_{ut} + x_{lt} - z_{ut} - \alpha - \beta \le C_L < x_{ut} + x_{lt} - z_{ut} - \alpha$, since $\alpha \ge \beta$, by properties (b) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$) and by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U - \beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(\min\{z_{ut} + \alpha + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(8) If $C_U - z_{ut} - \alpha - \beta < r_{t+1} \le C_U - z_{ut} - \beta$ and $C_L < x_{ut} + x_{lt} - z_{ut} - \alpha - \beta$, by properties (b) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(\cdot, C_L, y_{t+1})$) and by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(z_{ut} + \beta + r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U - \beta, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(\min\{z_{ut} + \alpha + r_{t+1}, C_U\}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(9) If $C_U - z_{ut} - \beta < r_{t+1} \le C_U - z_{ut}$ and $x_{ut} + x_{lt} - z_{ut} \le C_L$, since $\alpha \ge \beta$, by Lemma 2 and by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(z_{ut} + r_{t+1}, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(10) If $C_U - z_{ut} - \beta < r_{t+1} \le C_U - z_{ut}$ and $x_{ut} + x_{lt} - z_{ut} - \beta \le C_L < x_{ut} + x_{lt} - z_{ut}$, since $\alpha \ge \beta$, by Lemma 2 and by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(11) If $C_U - z_{ut} - \beta < r_{t+1} \le C_U - z_{ut}$ and $x_{ut} + x_{lt} - z_{ut} - \alpha - \beta \le C_L < x_{ut} + x_{lt} - z_{ut} - \beta$, since $\alpha \ge \beta$, by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le 0 \le v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(12) If $C_U - z_{ut} - \beta < r_{t+1} \le C_U - z_{ut}$ and $C_L < x_{ut} + x_{lt} - z_{ut} - \alpha - \beta$, since $\alpha \ge \beta$, by Lemma 2, $v^*_{t+1}(z_{ut} + r_{t+1}, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(13) If $C_U - z_{ut} < r_{t+1}$ and $x_{ut} + x_{lt} - z_{ut} \le C_L$, by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(14) If $C_U - z_{ut} < r_{t+1}$ and $x_{ut} + x_{lt} - z_{ut} - \beta \le C_L < x_{ut} + x_{lt} - z_{ut}$, by Lemma 2 and by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L - \alpha, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \beta, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(15) If $C_U - z_{ut} < r_{t+1}$ and $x_{ut} + x_{lt} - z_{ut} - \alpha - \beta \le C_L < x_{ut} + x_{lt} - z_{ut} - \beta$, by Lemma 2, $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{ut} + x_{lt} - z_{ut} - \alpha, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, x_{ut} + x_{lt} - z_{ut} - \alpha - \beta, y_{t+1})$.

(16) If $C_U - z_{ut} < r_{t+1}$ and $C_L < x_{ut} + x_{lt} - z_{ut} - \alpha - \beta$, both sides become $v^*_{t+1}(C_U, C_L, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Hence $V_t(z_{ut}, x_{ut}, x_{lt}, y_t)$ is concave in z_{ut}. □

Proof of Theorem 1. Let $a = a^*_t(x_{ut}, x_{lt}, y_t)$ and $b = b^*_t(x_{ut}, x_{lt}, y_t)$. Fix x_{ut}, x_{lt}, and y_t.

(i) Suppose that $(x_{ut}, x_{lt}, w_t) \in \Psi^0$. Thus $g(w_t) > C_T + \min\{x_{lt}, C_P, C_U - x_{ut}\}/\theta$. Since $(a, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $a \ge -\min\{x_{lt}, C_P, C_U - x_{ut}\}$. Thus $g(w_t) > C_T - a/\theta$. Lemma 3 implies that $b = C_T - a/\theta$ if $a \le 0$ and $b = C_T - \theta a$ if $a > 0$. Hence $R(a, b, y_t) = p_t \tau C_T$. We consider the following problem: $\max_{z_{ut} \in [0, C_U]} \left\{ V_t(z_{ut}, x_{ut}, x_{lt}, y_t) \right\}$. Note that $S^{(\mathsf{CS})}_t$ yields the maximum value in this problem. Taking into account the capacity constraints, by Lemma 4, we obtain

$$a = \begin{cases} -\min\{S^{(\mathsf{CS})}_t - x_{ut}, x_{lt}, C_P\} & \text{if } x_{ut} \le S^{(\mathsf{CS})}_t, \\ \min\{x_{ut} - S^{(\mathsf{CS})}_t, C_R\} & \text{if } S^{(\mathsf{CS})}_t < x_{ut}. \end{cases}$$

By Lemma 3, we also obtain

$$b = \begin{cases} C_T - a/\theta & \text{if } x_{ut} \le S^{(\mathsf{CS})}_t, \\ C_T - \theta a & \text{if } S^{(\mathsf{CS})}_t < x_{ut}. \end{cases}$$

(ii) Suppose that $(x_{ut}, x_{lt}, w_t) \in \Psi^1$. Thus $C_T < g(w_t) \le C_T + \min\{x_{lt}, C_P, C_U - x_{ut}\}/\theta$. In order to characterize the optimal water flow policy, we consider the following three cases:

- Suppose that $b = C_T - \theta a$. By Lemma 3, $a \ge 0$. We consider the following problem: $\max_{z_{ut}\in[0,x_{ut}]}\left\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t)\right\}$. By Lemma 4, $\min\{S_t^{(\mathsf{CS})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraints, by Lemma 4, we obtain $a = \min\{x_{ut} - S_t^{(\mathsf{CS})}, C_R\}$ if $x_{ut} > S_t^{(\mathsf{CS})}$.
- Suppose that $b = C_T - a/\theta$. By Lemma 3, $g(w_t) \ge C_T - a/\theta$ and $a \le 0$. Note that $C_T < g(w_t)$. Hence $0 \ge a \ge -\theta(g(w_t) - C_T)$. We consider the following problem: $\max_{z_{ut}\in[x_{ut},C_U]}\left\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t)\right\}$. By Lemma 4, $\max\{S_t^{(\mathsf{CS})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = -\min\{S_t^{(\mathsf{CS})} - x_{ut}, \theta(g(w_t) - C_T), x_{lt}, C_P\}$ if $x_{ut} \le S_t^{(\mathsf{CS})}$.
- Suppose that $b = g(w_t)$. By Lemma 3, $g(w_t) \le C_T - a/\theta$ if $a \le 0$ and $g(w_t) \le C_T - \theta a$ if $a > 0$. Since $g(w_t) > C_T$, it follows that $C_T < C_T - a/\theta$ if $a \le 0$ and $C_T < C_T - \theta a$ if $a > 0$. Since the case with $a > 0$ leads to a contradiction, we must have $a \le 0$. We consider the following two cases that correspond to PS and PP decisions, respectively:
  — Suppose that $0 \ge a/\theta > -g(w_t)$: Recall that $g(w_t) \le C_T - a/\theta$. Hence $0 > -\theta(g(w_t) - C_T) > a > -\theta g(w_t)$. We consider the following problem: $\max_{z_{ut}\in[x_{ut}+\theta(g(w_t)-C_T),C_U]}\left\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\mathsf{PS})}(x_{ut} - z_{ut}, y_t)\right\}$. By Lemma 4, $\max\{S_t^{(\mathsf{PS})}, x_{ut} + \theta(g(w_t) - C_T)\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = -\min\{S_t^{(\mathsf{PS})} - x_{ut}, \theta g(w_t), x_{lt}, C_P\}$ if $x_{ut} \le S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T)$.
  — Suppose that $0 \ge -g(w_t) \ge a/\theta$: Recall that $b = g(w_t)$. Since $(a, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $a/\theta + g(w_t) \ge -\tau C_T$. Hence $-\theta g(w_t) \ge a \ge -\theta(\tau C_T + g(w_t))$. We consider the following problem: $\max_{z_{ut}\in[x_{ut}+\theta g(w_t),C_U]}\left\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\mathsf{PP})}(x_{ut} - z_{ut}, y_t)\right\}$. By Lemma 4, $\max\{S_t^{(\mathsf{PP})}, x_{ut} + \theta g(w_t)\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = -\min\{S_t^{(\mathsf{PP})} - x_{ut}, \theta(\tau C_T + g(w_t)), x_{lt}, C_P\}$ if $x_{ut} \le S_t^{(\mathsf{PP})} - \theta g(w_t)$.

Combining all of the above observations, we obtain

$$a = \begin{cases} -\min\{S_t^{(\mathsf{PP})} - x_{ut}, \theta(\tau C_T + g(w_t)), x_{lt}, C_P\} & \text{if } x_{ut} \le S_t^{(\mathsf{PP})} - \theta g(w_t), \\ -\min\{S_t^{(\mathsf{PS})} - x_{ut}, \theta g(w_t), x_{lt}, C_P\} & \text{if } S_t^{(\mathsf{PP})} - \theta g(w_t) < x_{ut} \le S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T), \\ -\min\{S_t^{(\mathsf{CS})} - x_{ut}, \theta(g(w_t) - C_T), x_{lt}, C_P\} & \text{if } S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T) < x_{ut} \le S_t^{(\mathsf{CS})}, \\ \min\{x_{ut} - S_t^{(\mathsf{CS})}, C_R\} & \text{if } S_t^{(\mathsf{CS})} < x_{ut}. \end{cases}$$

By Lemma 3, we also obtain

$$b = \begin{cases} g(w_t) & \text{if } x_{ut} \le S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T), \\ C_T - a/\theta & \text{if } S_t^{(\mathsf{PS})} - \theta(g(w_t) - C_T) < x_{ut} \le S_t^{(\mathsf{CS})}, \\ C_T - \theta a & \text{if } S_t^{(\mathsf{CS})} < x_{ut}. \end{cases}$$

(iii) Suppose that $(x_{ut}, x_{lt}, w_t) \in \Psi^2$. Thus $g(w_t) \le C_T$. In order to characterize the optimal water flow policy, we consider the following three cases:

- Suppose that $b = C_T - \theta a$. By Lemma 3, $a \ge 0$ and $g(w_t) \ge C_T - \theta a$. Hence $a \ge (C_T - g(w_t))/\theta$. We consider the following problem: $\max_{z_{ut}\in[0,x_{ut}-(C_T-g(w_t))/\theta]}\left\{V_t(z_{ut}, x_{ut}, x_{lt}, y_t)\right\}$. By Lemma 4, $\min\{S_t^{(\mathsf{CS})}, x_{ut} - (C_T - g(w_t))/\theta\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = \min\{x_{ut} - S_t^{(\mathsf{CS})}, C_R\}$ if $x_{ut} > S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta$.
- Suppose that $b = C_T - a/\theta$. By Lemma 3, $g(w_t) \ge C_T - a/\theta$ and $a < 0$. These two inequalities together imply that $(C_T - g(w_t))\theta < 0$. But this leads to a contradiction since $g(w_t) \le C_T$. Thus, this case is not possible.
- Suppose that $b = g(w_t)$. We consider the following three cases that correspond to RS, PS, and PP decisions, respectively:

— Suppose that $a > 0$: By Lemma 3, $g(w_t) \le C_T - \theta a$. Hence $0 < a \le (C_T - g(w_t))/\theta$. We consider the following problem: $\max_{z_{ut} \in [0, x_{ut}]} \left\{ V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\mathsf{RS})}(x_{ut} - z_{ut}, y_t) \right\}$. By Lemma 4, $\min\{S_t^{(\mathsf{RS})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = \min\{x_{ut} - S_t^{(\mathsf{RS})}, (C_T - g(w_t))/\theta, C_R\}$ if $x_{ut} > S_t^{(\mathsf{RS})}$ and $a = 0$ if $x_{ut} \le S_t^{(\mathsf{RS})}$.

— Suppose that $0 \ge a/\theta > -g(w_t)$: Thus $0 \ge a > -\theta g(w_t)$. We consider the following problem: $\max_{z_{ut} \in [x_{ut}, C_U]} \left\{ V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\mathsf{PS})}(x_{ut} - z_{ut}, y_t) \right\}$. By Lemma 4, $\max\{S_t^{(\mathsf{PS})}, x_{ut}\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = -\min\{S_t^{(\mathsf{PS})} - x_{ut}, \theta g(w_t), x_{lt}, C_P\}$ if $x_{ut} \le S_t^{(\mathsf{PS})}$.

— Suppose that $0 \ge -g(w_t) \ge a/\theta$: Recall that $b = g(w_t)$. Since $(a, b) \in \mathbb{U}(x_{ut}, x_{lt}, y_t)$, we have $a/\theta + g(w_t) \ge -\tau C_T$. Hence $-\theta g(w_t) \ge a \ge -\theta(\tau C_T + g(w_t))$. We consider the following problem: $\max_{z_{ut} \in [x_{ut} + \theta g(w_t), C_U]} \left\{ V_t(z_{ut}, x_{ut}, x_{lt}, y_t) + R_t^{(\mathsf{PP})}(x_{ut} - z_{ut}, y_t) \right\}$. By Lemma 4, $\max\{S_t^{(\mathsf{PP})}, x_{ut} + \theta g(w_t)\}$ yields the maximum value in this problem. Taking into account the capacity constraints, again by Lemma 4, we obtain $a = -\min\{S_t^{(\mathsf{PP})} - x_{ut}, \theta(\tau C_T + g(w_t)), x_{lt}, C_P\}$ if $x_{ut} \le S_t^{(\mathsf{PP})} - \theta g(w_t)$.

Combining all of the above observations, we obtain

$$a = \begin{cases} -\min\{S_t^{(\mathsf{PP})} - x_{ut}, \theta(\tau C_T + g(w_t)), x_{lt}, C_P\} & \text{if } x_{ut} \le S_t^{(\mathsf{PP})} - \theta g(w_t), \\ -\min\{S_t^{(\mathsf{PS})} - x_{ut}, \theta g(w_t), x_{lt}, C_P\} & \text{if } S_t^{(\mathsf{PP})} - \theta g(w_t) < x_{ut} \le S_t^{(\mathsf{PS})}, \\ 0 & \text{if } S_t^{(\mathsf{PS})} < x_{ut} \le S_t^{(\mathsf{RS})}, \\ \min\{x_{ut} - S_t^{(\mathsf{RS})}, (C_T - g(w_t))/\theta, C_R\} & \text{if } S_t^{(\mathsf{RS})} < x_{ut} \le S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta, \\ \min\{x_{ut} - S_t^{(\mathsf{CS})}, C_R\} & \text{if } S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta < x_{ut}. \end{cases}$$

By Lemma 3, we also obtain

$$b = \begin{cases} g(w_t) & \text{if } x_{ut} \le S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta, \\ C_T - \theta a & \text{if } S_t^{(\mathsf{CS})} + (C_T - g(w_t))/\theta < x_{ut}. \end{cases}$$

We next show that $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{PS})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{CS})}(x_{ut}, x_{lt}, y_t)$: Fix x_{ut}, x_{lt}, and y_t. For each $\nu \in \{\mathsf{PP}, \mathsf{PS}, \mathsf{RS}, \mathsf{CS}\}$, let $S^{(\nu)} = S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$. By definition of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$, the following inequalities hold.

$$V_t(S^{(\mathsf{PP})}, x_{ut}, x_{lt}, y_t) - p_t S^{(\mathsf{PP})}/(\theta\tau) \ge V_t(S^{(\mathsf{PS})}, x_{ut}, x_{lt}, y_t) - p_t S^{(\mathsf{PS})}/(\theta\tau),$$
$$V_t(S^{(\mathsf{PS})}, x_{ut}, x_{lt}, y_t) - p_t \tau S^{(\mathsf{PS})}/\theta \ge V_t(S^{(\mathsf{PP})}, x_{ut}, x_{lt}, y_t) - p_t \tau S^{(\mathsf{PP})}/\theta.$$

The summation of the above inequalities implies that $p_t(\tau - 1/\tau)S^{(\mathsf{PP})}/\theta \ge p_t(\tau - 1/\tau)S^{(\mathsf{PS})}/\theta$. Since $0 < \theta$ and $0 < \tau \le 1$, $S_t^{(\mathsf{PP})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{PS})}(x_{ut}, x_{lt}, y_t)$. Again, by definition of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$, the following inequalities hold.

$$V_t(S^{(\mathsf{PS})}, x_{ut}, x_{lt}, y_t) - p_t \tau S^{(\mathsf{PS})}/\theta \ge V_t(S^{(\mathsf{RS})}, x_{ut}, x_{lt}, y_t) - p_t \tau S^{(\mathsf{RS})}/\theta,$$
$$V_t(S^{(\mathsf{RS})}, x_{ut}, x_{lt}, y_t) - p_t \tau\theta S^{(\mathsf{RS})} \ge V_t(S^{(\mathsf{PS})}, x_{ut}, x_{lt}, y_t) - p_t \tau\theta S^{(\mathsf{PS})}.$$

The summation of the above inequalities implies that $p_t\tau(\theta - 1/\theta)S^{(\mathsf{PS})} \ge p_t\tau(\theta - 1/\theta)S^{(\mathsf{RS})}$. Since $0 < \theta \le 1$ and $0 < \tau$, $S_t^{(\mathsf{PS})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t)$. Once again, by definition of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$, the following inequalities hold.

$$V_t(S^{(\mathsf{RS})}, x_{ut}, x_{lt}, y_t) - p_t \tau\theta S^{(\mathsf{RS})} \ge V_t(S^{(\mathsf{CS})}, x_{ut}, x_{lt}, y_t) - p_t \tau\theta S^{(\mathsf{CS})},$$
$$V_t(S^{(\mathsf{CS})}, x_{ut}, x_{lt}, y_t) + p_t \tau C_T \ge V_t(S^{(\mathsf{RS})}, x_{ut}, x_{lt}, y_t) + p_t \tau C_T.$$

The summation of the above inequalities implies that $-p_t\theta\tau S^{(\mathsf{RS})} \ge -p_t\theta\tau S^{(\mathsf{CS})}$. Since $0 < \theta$ and $0 < \tau$, $S_t^{(\mathsf{RS})}(x_{ut}, x_{lt}, y_t) \le S_t^{(\mathsf{CS})}(x_{ut}, x_{lt}, y_t)$.

We now show that $S_t^{(\nu)}(x_{ut},x_{lt},y_t)=S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t)$ if $x_{ut}+x_{lt}\ge C_U$ for each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$ and $\alpha>0$: By Lemma 2, for any $z_{ut}\in[0,C_U]$ note that

$$\begin{aligned}V_t(z_{ut},x_{ut},x_{lt},y_t)&=\mathbb{E}\left[v_{t+1}^*\Big(\min\{z_{ut}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}-z_{ut},C_L\},y_{t+1}\Big)\right]\\&=\mathbb{E}\left[v_{t+1}^*\Big(\min\{z_{ut}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-z_{ut},C_L\},y_{t+1}\Big)\right]=V_t(z_{ut},x_{ut},x_{lt}+\alpha,y_t).\end{aligned}$$

Thus, for each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$,

$$\begin{aligned}S_t^{(\nu)}(x_{ut},x_{lt},y_t)&=\underset{z_{ut}\in[0,C_U]}{\arg\max}\,\{V_t(z_{ut},x_{ut},x_{lt},y_t)+R_t^{(\nu)}(x_{ut}-z_{ut},y_t)\}\\&=\underset{z_{ut}\in[0,C_U]}{\arg\max}\,\{V_t(z_{ut},x_{ut},x_{lt}+\alpha,y_t)+R_t^{(\nu)}(x_{ut}-z_{ut},y_t)\}=S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t).\end{aligned}$$

We also show that $S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t)=S_t^{(\nu)}(x_{ut}+\alpha,x_{lt},y_t)$ for each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$ and $\alpha>0$: By definition of $V_t(\cdot)$, for any $z_{ut}\in[0,C_U]$ note that

$$V_t(z_{ut},x_{ut},x_{lt}+\alpha,y_t)=\mathbb{E}\left[v_{t+1}^*\Big(\min\{z_{ut}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-z_{ut},C_L\},y_{t+1}\Big)\right]=V_t(z_{ut},x_{ut}+\alpha,x_{lt},y_t).$$

Thus, for each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$,

$$\begin{aligned}S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t)&=\underset{z_{ut}\in[0,C_U]}{\arg\max}\,\{V_t(z_{ut},x_{ut},x_{lt}+\alpha,y_t)+R_t^{(\nu)}(x_{ut}-z_{ut},y_t)\}\\&=\underset{z_{ut}\in[0,C_U]}{\arg\max}\,\{V_t(z_{ut},x_{ut}+\alpha,x_{lt},y_t)+R_t^{(\nu)}(x_{ut}-z_{ut},y_t)\}=S_t^{(\nu)}(x_{ut}+\alpha,x_{lt},y_t).\end{aligned}$$

Lastly, we show that $S_t^{(\nu)}(x_{ut},x_{lt},y_t)\le S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t)$ for each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$ and $\alpha>0$: For each $\nu\in\{\mathsf{PP},\mathsf{PS},\mathsf{RS},\mathsf{CS}\}$, let $S_t^{(\nu)}(x_{ut},x_{lt},y_t)=S_1^{(\nu)}$ and $S_t^{(\nu)}(x_{ut},x_{lt}+\alpha,y_t)=S_2^{(\nu)}$. Assume to the contrary that $S_1^{(\nu)}>S_2^{(\nu)}$. We consider the following nine cases to show that

$$\begin{aligned}&\mathbb{E}\left[v_{t+1}^*\Big(\min\{S_1^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}-S_1^{(\nu)},C_L\},y_{t+1}\Big)\right]\\&-\mathbb{E}\left[v_{t+1}^*\Big(\min\{S_2^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}-S_2^{(\nu)},C_L\},y_{t+1}\Big)\right]\\&\le\mathbb{E}\left[v_{t+1}^*\Big(\min\{S_1^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},C_L\},y_{t+1}\Big)\right]\\&-\mathbb{E}\left[v_{t+1}^*\Big(\min\{S_2^{(\nu)}+r_{t+1},C_U\},\min\{x_{ut}+x_{lt}+\alpha-S_2^{(\nu)},C_L\},y_{t+1}\Big)\right].\end{aligned}$$

(1) If $r_{t+1}\le C_U-S_1^{(\nu)}<C_U-S_2^{(\nu)}$ and $x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}\le C_L$, by property (a) of Proposition 1, $v_{t+1}^*(S_1^{(\nu)}+r_{t+1},x_{ut}+x_{lt}-S_1^{(\nu)},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},x_{ut}+x_{lt}-S_2^{(\nu)},y_{t+1})\le v_{t+1}^*(S_1^{(\nu)}+r_{t+1},x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},x_{ut}+x_{lt}+\alpha-S_2^{(\nu)},y_{t+1})$.

(2) If $r_{t+1}\le C_U-S_1^{(\nu)}<C_U-S_2^{(\nu)}$ and $x_{ut}+x_{lt}-S_2^{(\nu)}\le C_L<x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}$, by property (a) of Proposition 1 and by Lemma 2, $v_{t+1}^*(S_1^{(\nu)}+r_{t+1},x_{ut}+x_{lt}-S_1^{(\nu)},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},x_{ut}+x_{lt}-S_2^{(\nu)},y_{t+1})\le v_{t+1}^*(S_1^{(\nu)}+r_{t+1},C_L+S_2^{(\nu)}-S_1^{(\nu)},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},C_L,y_{t+1})\le v_{t+1}^*(S_1^{(\nu)}+r_{t+1},\min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},C_L\},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},C_L,y_{t+1})$.

(3) If $r_{t+1}\le C_U-S_1^{(\nu)}<C_U-S_2^{(\nu)}$ and $C_L<x_{ut}+x_{lt}-S_2^{(\nu)}$, by Lemma 2, $v_{t+1}^*(S_1^{(\nu)}+r_{t+1},\min\{x_{ut}+x_{lt}-S_1^{(\nu)},C_L\},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},C_L,y_{t+1})\le v_{t+1}^*(S_1^{(\nu)}+r_{t+1},\min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},C_L\},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},C_L,y_{t+1})$.

(4) If $C_U-S_1^{(\nu)}<r_{t+1}\le C_U-S_2^{(\nu)}$ and $x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}\le C_L$, by properties (a) and (c) of Proposition 1, $v_{t+1}^*(C_U,x_{ut}+x_{lt}-S_1^{(\nu)},y_{t+1})-v_{t+1}^*(C_U,x_{ut}+x_{lt}+\alpha-S_1^{(\nu)},y_{t+1})\le v_{t+1}^*(C_U+S_2^{(\nu)}-S_1^{(\nu)},x_{ut}+x_{lt}-S_2^{(\nu)},y_{t+1})-v_{t+1}^*(C_U+S_2^{(\nu)}-S_1^{(\nu)},x_{ut}+x_{lt}+\alpha-S_2^{(\nu)},y_{t+1})\le v_{t+1}^*(S_2^{(\nu)}+r_{t+1},x_{ut}+x_{lt}-S_2^{(\nu)},y_{t+1})-v_{t+1}^*(S_2^{(\nu)}+r_{t+1},x_{ut}+x_{lt}+\alpha-S_2^{(\nu)},y_{t+1})$.

(5) If $C_U - S_1^{(\nu)} < r_{t+1} \le C_U - S_2^{(\nu)}$ and $x_{ut}+x_{lt}-S_2^{(\nu)} \le C_L < x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}$, by Lemma 2 and by properties (a) and (c) of Proposition 1, $v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, C_L+S_2^{(\nu)}-S_1^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U+S_2^{(\nu)}-S_1^{(\nu)}, x_{ut}+x_{lt}-S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U+S_2^{(\nu)}-S_1^{(\nu)}, C_L, y_{t+1}) \le v^*_{t+1}(S_2^{(\nu)}+r_{t+1}, x_{ut}+x_{lt}-S_2^{(\nu)}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)}+r_{t+1}, C_L, y_{t+1})$.

(6) If $C_U - S_1^{(\nu)} < r_{t+1} \le C_U - S_2^{(\nu)}$ and $C_L < x_{ut}+x_{lt}-S_2^{(\nu)}$, by Lemma 2, $v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)}+r_{t+1}, C_L, y_{t+1}) \le v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(S_2^{(\nu)}+r_{t+1}, C_L, y_{t+1})$.

(7) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $x_{ut}+x_{lt}+\alpha-S_2^{(\nu)} \le C_L$, by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$), $v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U, x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}, y_{t+1})$.

(8) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $x_{ut}+x_{lt}-S_2^{(\nu)} \le C_L < x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}$, by properties (a) and (c) of Proposition 1 (which together imply the concavity of $v^*_{t+1}(C_U, \cdot, y_{t+1})$) and by Lemma 2, $v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, x_{ut}+x_{lt}-S_2^{(\nu)}, y_{t+1}) \le v^*_{t+1}(C_U, C_L+S_2^{(\nu)}-S_1^{(\nu)}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

(9) If $C_U - S_1^{(\nu)} < C_U - S_2^{(\nu)} < r_{t+1}$ and $C_L < x_{ut}+x_{lt}-S_2^{(\nu)}$, by Lemma 2, $v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1}) \le v^*_{t+1}(C_U, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}) - v^*_{t+1}(C_U, C_L, y_{t+1})$.

Hence:

$$\begin{aligned}
&\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}-S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}-S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(S_1^{(\nu)}, x_{ut}, x_{lt}, y_t) - V_t(S_2^{(\nu)}, x_{ut}, x_{lt}, y_t) \\
&\le V_t(S_1^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) - V_t(S_2^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) \qquad \text{(EC.5)} \\
&= \mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

By definitions of $S_t^{(\nu)}(x_{ut}, x_{lt}, y_t)$ and $S_t^{(\nu)}(x_{ut}, x_{lt}+\alpha, y_t)$, the following inequalities hold.

$$V_t(S_1^{(\nu)}, x_{ut}, x_{lt}, y_t) + R_t^{(\nu)}(x_{ut}-S_1^{(\nu)}, y_t) \ge V_t(S_2^{(\nu)}, x_{ut}, x_{lt}, y_t) + R_t^{(\nu)}(x_{ut}-S_2^{(\nu)}, y_t),$$
$$V_t(S_2^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) + R_t^{(\nu)}(x_{ut}-S_2^{(\nu)}, y_t) \ge V_t(S_1^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) + R_t^{(\nu)}(x_{ut}-S_1^{(\nu)}, y_t).$$

The summation of the above inequalities implies that

$$\begin{aligned}
&\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}-S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}-S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&= V_t(S_2^{(\nu)}, x_{ut}, x_{lt}, y_t) - V_t(S_1^{(\nu)}, x_{ut}, x_{lt}, y_t) \\
&\le V_t(S_2^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) - V_t(S_1^{(\nu)}, x_{ut}, x_{lt}+\alpha, y_t) \\
&= \mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_2^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}+\alpha-S_2^{(\nu)}, C_L\}, y_{t+1}\Big)\Big] \\
&-\mathbb{E}\Big[v^*_{t+1}\Big(\min\{S_1^{(\nu)}+r_{t+1}, C_U\}, \min\{x_{ut}+x_{lt}+\alpha-S_1^{(\nu)}, C_L\}, y_{t+1}\Big)\Big].
\end{aligned}$$

This leads to a contradiction with the inequality in EC.5. Thus $S_1^{(\nu)} \le S_2^{(\nu)}$. □

The proof of Lemma 5 (in Section 6) is available upon request from the authors.