Rewards Must Be Proportional in the Core of Large Claim Games

Esat Doruk Cetemen
University of Rochester

Kevin Edward Hasker*
Bilkent University

Emin Karagözoğlu
Bilkent University

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Abstract

We model the bankruptcy problem as a cooperative game played by a continuum of claimants and allow the claimants to split their claims or merge them with others' claims. Our approach (in line with *partition function form games*) differs from the standard one (à la O'Neill, 1982) used in modeling cooperative claim (or bankruptcy) games in that the stand-alone value of a coalition depends on the partition in our model. We show, under mild assumptions, that the rewards must be *proportional* in the core of our game. Our result complements axiomatic results in the literature on the immunity of the proportional rule to manipulation via merging and splitting. Moreover, it can be considered as a policy-ineffectiveness result for bankruptcy problems.

Keywords: Claim Games, Coalition Formation, Core, Merging, Partition Function Form Games, Proportionality, Splitting.

JEL Codes: C71, C79, D63, D74.

* *Corresponding Author:* Bilkent University, Department of Economics, 06800 Bilkent, Ankara-Turkey. E-mail: hasker@bilkent.edu.tr. Tel.: +90-312-2901875, Fax:+90-312-2665140.

1 Introduction

The bankruptcy problem, introduced formally by O'Neill (1982), describes a situation in which there is a perfectly divisible estate to be allocated to a finite number of agents, whose claims over the estate add up to an amount larger than the estate. Many real life problems (e.g., distributing a will to inheritants, liquidating the assets of a bankrupt company, rationing problems, taxation, and cost sharing) can be studied using bankruptcy models. There is a vast literature analyzing this problem with an axiomatic approach, characterizing allocation rules with normatively desirable properties (see Moulin, 2002; Thomson, 2003 for comprehensive reviews).

The cooperative game theoretical approach has been proven to be another fruitful approach to study bankruptcy problems. Under this approach, the bankruptcy problem is transformed into a transferable utility game (or a coalitional bargaining game); and cooperative solution concepts such as core, kernel, nucleolus etc are studied.[1]

In the current paper, we follow the cooperative game theoretical approach. In particular, we define a natural claim game among a continuum of claimants and study the properties of the core of this game. In our claim game, given the allocation rule to be used in resolving the bankruptcy problem, claimants are allowed to split their (exogenously given) claims or merge their claims with others', before submitting them. Agents make these decisions simultaneously. The resulting claims are then submitted to the authority who will apply a bankruptcy (or reward) rule to distribute the estate on the basis of these claims. In defining the game, we are inspired by the presence of merging/splitting behavior in some real life bankruptcy problems (e.g., spouses can act like a single claimant or the partners of a single firm can present themselves as different claimants (see de Frutos, 1999)) and the incentives some bankruptcy rules give to claimants (e.g., the constrained equal awards rule encourages splitting whereas the constrained equal losses rule encourages merging).

We consider reward functions that satisfy two very basic properties: *budget-balancedness*, stipulating that the resource is completely distributed and *anonymity*, stipulating that the identity of the claimants does not matter but only the size of their claims. Both of these assumptions on the reward

[1]For the cooperative game theoretical approach to claims problems, the reader is referred to Aumann and Maschler (1985), Young (1985), Curiel, Maschler, and Tijs (1987), and Dagan and Volij (1993) among others.

function are primitive. In fact, the former is almost always embedded in the definition of a bankruptcy rule and the latter assumption is a standard fairness assumption used in the literature.[2] Our main result is that the rewards are always proportional in the core of our game. We should emphasize that our result only concerns the behavior of reward functions in the core. Outside the core a great deal of freedom can be attained.

The game we analyze is essential and primitive. A legal system must have a clear and understandable method for determining how rewards will be determined if the case goes to court. Once the bankruptcy is officially filed and the reward rule is in place, it is natural to consider how agents react to this environment, i.e., how the court rule influences pre-court negotiations where agents may trade claims. We analyze these negotiations as a coalitional game.

Our analysis differs from standard cooperative games used to model bankruptcy problems in that the reward rule in our model is exogenously given. As we mentioned above, it can be interpreted as determined *a-priori* by the court. This implies that the stand-alone value of a coalition is not well defined, which is in contrast to the standard approach à la O'Neill (1982). No matter which coalition forms (naturally, except the grand coalition), its reward may depend on which other coalitions are formed. Most other analyses of bankruptcy with a cooperative game begins by characterizing the stand-alone value of a coalition, which is rule-independent and from that they derive the core or another solution concept. This literature is primarily interested in the relationship between various reward rules and solution concepts (e.g., Aumann and Maschler, 1985; Curiel et al., 1987; Serrano, 1995; Benoît, 1997 among others).

Our main result is related to some other results on the proportional rule in the axiomatic literature. One of these studies is Chambers and Thomson (2002). These authors show in a continuum of agents setup that the proportional rule is the only rule that satisfies equal treatment of equal groups. We do not assume this property in our model. Nevertheless, it holds in the core of our game. Moreover, Chambers and Thomson (2002) provide an axiomatic result, whereas ours comes from the analysis of a cooperative game. Finally, proportionality is a *core-property* in our model, whereas it is universally char-

[2]We should note that the sequential priority rules that are widely used in many countries in case of corporate bankruptcy violate this property. Anonymity is still satisfied within each priority class though.

acterized in Chambers and Thomson (2002). Several other papers (O'Neill, 1982; Moulin, 1987; Chun, 1988; de Frutos, 1999; Ching and Kakkar, 2001; Ju, 2003; Ju, Miyagawa, and Sakai, 2007) show, in different contexts, that the proportional rule can be characterized with (i) *no advantageous merging* and *no advantageous splitting* (or equivalently, it is the only rule that cannot be manipulated via merging or splitting of claims), (ii) no-arbitrage condition, or (iii) no advantageous reallocation.[3] Our result differs from the ones in these studies since there can be many rules that behave proportionally in the core but almost arbitrarily elsewhere and core can be non-empty even under reward rules that do not satisfy no-arbitrage type conditions.

Finally, we believe that our main result has an important implication: the support for proportionality is stronger than what the literature on bankruptcy problems indicates.[4] The reason is that almost all studies in the literature focused on particular rules; but our main result shows that even if a rule other than the proportional rule is used, the equilibrium allocation can still be proportional. Hence, from another perspective our result can be interpreted as a *policy-ineffectiveness* result: a policymaker who wants to implement an allocation other than the proportional one may get nothing but proportionality since agents adjust their behavior according to the policy.

2 The Model

We have an uncountable, compact set of agents, denoted by $N \subseteq \mathbb{R}$, which is equipped with the Euclidean metric. Let $\mathcal{B}$ be the $\sigma-$algebra generated by Borel measurable subsets of N. We use C to denote the (Borel) measure of claims with associated density denoted by c. There is a minimal size that a claim can be submitted in, κ (e.g., one cent or one stock) and all claims are multiples of this amount. Thus, for $\chi_i \in \mathbb{N} \setminus \{0\}$, $c(i) = \chi_i \kappa > 0$ is the claim of $i \in N$. We assume that C is non-atomic and we normalize $C(N) = 1$. A *coalition* denoted by S_k is a measureble subset of N. A *partition* S is a set of disjoint coalitions $(S_1, S_2, ..., S_K)$ such that their union is N. Let $C(S)$

[3] More recently, Ju (2013) offers a full characterization of non-manipulable rules in allocation problems over networks, where exogenously given network stucture contrains the set of feasible coalitions.

[4] Reader is refered to Chun (1988), Bergantiños and Sanchez (2002), Chun and Thomson (2005), Ju, Miyagawa, and Sakai (2007), and Karagözoğlu (2014) among others for studies providing support for proportionality in bankruptcy problems

denote the vector of claims so that $C_k \equiv C(S_k)$ and $C(S) \equiv [C(S_k)]_{S_k \in S}$. The resource available for distribution is E, where $\kappa \leq E \leq 1-\kappa$. Since E will be fixed in our analysis, we suppress it in our notation. E will be distributed according to a reward function, R. Formally, $R : \mathcal{S} \to \mathbb{R}^{\infty}$ with the restriction that $R(S) \in \mathbb{R}^{|C(S)|}$, where $\mathcal{S}$ denotes the space of all possible partitions and $|C(S)|$ is the cardinality of the claims vector. Hence, $R(S_k|S) \equiv R_k(S)$ stands for the reward of (potentially a subset of) agents in S_k given partition S. Let $v(i,S)$ be the reward of agent i from $R(S)$. As in almost all models of bankruptcy, we assume (for R) that rewards are bounded below by zero and above by claims. In addition to these, we impose the following two restrictions.

Definition 1 *$R(\cdot|\cdot)$ is* budget balanced *if $\int_{i\in S} v(i,S)d\mu = E$.*

Assumption 1 *$R(S)$ is budget balanced.*

Definition 2 *$R(\cdot|\cdot)$ is* anonymous *if for every $S \in \mathcal{S}$ and every $S_i \in S$ and every perturbation Π of identifiers, $R(S_k|S) = R\left(S_{\Pi(k)}|\tilde{S}\right)$ where $\tilde{S}$ is the perturbed partition.*

Assumption 2 *$R(S)$ is anonymous.*

The first one is an implication of efficiency.[5] The second one is a primitive fairness axiom, which is satisfied by almost all well-known bankruptcy rules except the sequential priority rules.[6]

In our game, any claimant can decide to split his claim into multiple smaller claims (down to a minimal size of κ) or merge his claim with others' claims and submit them as one larger claim.[7] This creates a transferable

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[5] We do not label the property as *efficiency* since this term has welfare connotations. We think it is more appropriate to perceive it as a simple accounting property here.

[6] As mentioned in the introduction, Chambers and Thomson (2002) characterizes the proportional rule with equal treatment of equal groups in a continuum of agents setup. It is worthwhile saying here that anonymity and equal treatment of equal groups are logically not related. Furthermore, in our model reward assignments within coalitions are determined in the core whereas in their model reward functions also deal with within coalition reward assignments.

[7] Suppose that some claimants merged their claims and submitted the sum as one large claim, then the reward function, R, will assign a reward to this single, large coalition. Obviously, this reward later needs to be distributed among the members of the coalition. It is worthwhile saying here that we do not impose any division rule for the intra-coalition division problem. Nevertheless, we show in Remark 2 that the intra-coalitional division is also proportional for any coalition in the core.

utility (TU) game (N, R), where R is the reward (or characteristic) function for this game. Our model is similar to *partition function form games*, where coalitional rewards depend on the partition structure.[8] We follow Aumann (1964) in that we require a coalition of agents to be a subset of σ-algebra $S_k \subseteq \mathcal{B}$. Finally, we work with positive measure (non-null) coalitions throughout the paper.

The following lemma shows that the average reward (later utilized in the proof of our main result) is well-defined.

Lemma 1 *If the reward rule is anonymous then, for all $S_k \in S$, $\frac{R(C_k|S)}{C_k}$ is bounded.*

Proof. We prove this result in two steps: for S_k closed and S_k not closed. First assume that S_k is closed in S. Notice that it is bounded since it is a subset of a compact space. Then,

$$\frac{R\left(C_k|S\right)}{C_k} = \frac{\int_{i\in S_k} v\left(i,S\right)}{C_k} \leq \frac{\max_{i\in S_k} \frac{v(i,S)}{c(i)} \int_{i\in S_k} c\left(i\right)}{C_k} = \max_{i\in S_k} \frac{v\left(i,S\right)}{c\left(i\right)}$$

Now, assume that S_k is not closed in S; and let S' be the partition where $S'_k = \bar{S}_k$ and it is otherwise unchanged. Since C is non-atomic, $C\left(S_k\right) = C\left(S'_k\right)$ follows. By anonymity, $R\left(S_k|S\right) = R\left(S'_k|S'\right)$. Thus $\frac{R(S_k|S)}{C_k} = \frac{R\left(S'_k|S'\right)}{C_k} \leq \max_{i\in S_k} \frac{v(i,S')}{c(i)} \leq 1$. ■

In the definition of core in a TU-game, a partition S (and the corresponding allocation) is *blocked* by a non-null coalition if there exists $X \subseteq \mathcal{B}$ such that $R\left(X|S'\right) > \int_{i\in X} v(i,S)$ where S' is the resulting partition after deviations. A coalition is non-null if $C\left(\cdot\right) > 0$.[9] Note that, by assumption, S' is *feasible* (i.e., $\int_{i\in N} v\left(i,S'\right) d\mu \leq E$). If S is not blocked by any feasible coalition, then we say it belongs to *core*.

[8] Partition function form games are introduced by Thrall and Lucas (1963). For more recent research on the topic, see Funaki and Yamato (1999), Koczy (2007), Grabisch (2010), Grabisch and Funaki (2012), Bloch and van den Nouweland (2014).

[9] We assume that if some agents leave a coalition, the other members of the coalition still stay together. This notion is known as $\delta-$*stability* in the literature (see d'Aspremont et al. 1983; Chander and Tulkens, 1997; Chander 2007). An alternative assumption is $\gamma-$*stability* of Hart and Kurz (1983) where all members of the coalition will break up to singletons if and when some agents leave a coalition. In this paper, we take the behaviour of the non-deviating agents exogoenous. The non-cooperative coalition formation literature presents some attempts for endogenizing it. Reader is referred to the extensive survey of Ray and Vohra (2015) for such studies.

3 Rewards in the core

Lemma 2 clarifies the importance of the average reward of an agent in a coalition, in claim games with continuum of agents investigated here.

Lemma 2 *For every* $\alpha \in (0,1]$ *and* $\hat{S}_k \subseteq S_k$ *such that* $C\left(\hat{S}_k\right) = \alpha C\left(S_k\right)$ [10]

$$\frac{R\left(S_k|S\right)}{C_k} \geq \min_{\hat{S}_k \subseteq S_k} \int_{i \in \hat{S}_k} \frac{v\left(i,S\right)}{\alpha C_k} d\mu.$$

Proof. If $\alpha = 1$, the statement holds by definition. Assume that $\alpha < 1$ and let $\hat{S}_k$ be any subset that attains the minimum. Note that by the definition of $\hat{S}_k$, for all $S'_k \subseteq S_k$ (disjoint from $\hat{S}_k$) such that $C\left(S'_k\right) = \beta C\left(S_k\right)$ and $\beta < \alpha$, $\int_{i \in \hat{S}_k} \frac{v(i,S)}{\alpha C_k} d\mu \leq \int_{i \in S'_k} \frac{v(i,S)}{\beta C_k} d\mu$. Thus, let $\{\beta_j\}_{j=1}^{J}$ be such that for all j, $\beta_j < \alpha$ and $\Sigma_{j=1}^{J} \beta_j = 1 - \alpha$. Now, suppose $\alpha R\left(S_k|S\right) < \min_{\hat{S}_k \subseteq S_k} \int_{i \in \hat{S}_k} v\left(i,S\right) d\mu$. Replacing $v\left(i,S\right)$ with its lower bound, we have $\int_{i \in S_k} v\left(i,S\right) d\mu > \alpha R\left(S_k|S\right) + \left(1-\alpha\right) R\left(S_k|S\right) = R\left(S_k|S\right)$, a contradiction. ■

Note that $\hat{S}_k \subseteq S_k$ in this lemma is the *cheapest* group to get to agree to block the current partition. Thus, a goal within a coalition should be to maximize the reward of this group. Across coalitions this would imply maximizing the average reward; and those with the highest would be in the best position to block current coalitions.

The result in the following lemma is superseded by our main proposition. It is related to the property known as *regressivity* in the literature on bankruptcy problems. Regressivity stipulates that if $C_k \geq C_l$, then S_k should receive proportionally at most as much as S_l does. Here, we find this property to be a mere by-product of strategic repositioning.

Lemma 3 *If* S *is in the core and* $C_k \geq C_l > 0$*, then* $\frac{R(S_k|S)}{C_k} \leq \frac{R(S_l|S)}{C_l}$.

Proof. Suppose by contradiction that there exists S_l and S_k such that $C_k \geq C_l > 0$ but $\frac{R(S_k|S)}{C_k} > \frac{R(S_l|S)}{C_l}$ in the core.

Now, we show that there exists a blocking coalition. To do that, create a new partition denoted by S' by choosing an $\hat{S}_k \subseteq S_k$ and $C\left(\hat{S}_k\right) +$

[10]The existence of such sets is guaranteed by *Lyapunov Theorem* (see Aliprantis and Border, 2006 for further details).

$C\left(S_k \setminus \hat{S}_k\right) = C(S_k)$. Notice that this implies there exists $\beta \in (0,1]$ such that $\beta C(S_k) = C\left(\hat{S}_k\right)$ and $(1-\beta)C(S_k) = C(S_l) = C\left(S_k \setminus \hat{S}_k\right)$. Then, have the agents in $\hat{S}_k$ propose to unite with the agents in S_l and offer them $R(S_l|S) + \varepsilon$. This can be done in a way that everyone in S_l strictly prefers this change for some $\epsilon > 0$.

Furthermore, $C(S_k|S) = C\left(\hat{S}_k \cup S_l|S\right)$. Now, what is left for agents in $\hat{S}_k$ is $R(S_k|S) - R(S_l|S) - \varepsilon$. Hence, we need to show that $R(S_k|S) - R(S_l|S) - \varepsilon > \int_{i \in \hat{S}_k} v(i,S)\,d\mu$. From Lemma 2, this reduces to showing that $R(S_k|S) - R(S_l|S) - \varepsilon > \beta R(S_k|S)$ holds. Since $R(S_l|S) < (1-\beta) R(S_k|S)$, this inequality is satisfied for some $\varepsilon > 0$. ■

Remark 1 *The result in Lemma 3 rules out, for example, the constrained equal losses rule:*[11] *under the constrained equal losses rule, the game will have an empty core. To see this, recognize that there is always a $C < 1$ such that for all S, $R(C|S) = E$. But who should be in this coalition? There will be never-ending negotiations to be in this coalition implying that everyone will compromise more and more in order not to get zero. This implies that each member of this coalition must be getting zero : a contradiction. More generally , the game will have an empty core under any strictly progressive reward rule.*

Next, we show that the rewards are proportional in the core. We are concerned with the cases where core is non-empty.

Proposition 1 *Under Assumption 1 and 2, if S is in the core, then for all $S_k \in S$, we have $R(S_k|S) = C_k E$.*

Proof. Consider creating an S' by splitting S_k into two coalitions $\hat{S}_k$ and $S_k \backslash \hat{S}_k$ such that $C\left(\hat{S}_k\right) = \alpha C_k$ for $0 < \alpha < 1$ and $C\left(S_k \backslash \hat{S}_k\right) = (1-\alpha) C_k$.

Now if $R(\alpha C_k|S') > \alpha R(C_k|S) \geq \min_{\hat{S}_k \subset S_k} \phi\left(\hat{S}_k|S\right)$ (where $\phi(\cdot,\cdot)$ is the amount the subcoalition $\hat{S}_k$ receives from $R(C_k|S)$) then $\hat{S}_k$ gains from blocking. Likewise if $R((1-\alpha)C_k|S') > (1-\alpha) R(C_k|S)$. This implies that

$$\max\left(\frac{R(\alpha C_k|S')}{\alpha C_k}, \frac{R((1-\alpha)C_k|S')}{(1-\alpha)C_k}\right) \leq \frac{R(C_k|S)}{C_k}$$

[11] In our continuum of agents setup, the constrained equal losses rule can be defined as follows: $R(S_k|S) = \max\{C(S_k) - \lambda, 0\}$, where λ is chosen to satisy efficiency.

or the average payoff of one of the groups is strictly larger; and this would imply that they could profitably split.

Now, by budget balancedness, the other coalitions have in total $E - R(C_k|S)$ in S, whereas they have in total $E - R(\alpha C_k|S') - R((1-\alpha)C_k|S')$ in S'.

If $E - R(C_k|S) < E - R(\alpha C_k|S') - R((1-\alpha)C_k|S')$, then other coalitions can afford to pay for the split.[12] Thus, we must have $R(C_k|S) = R(\alpha C_k|S') + R((1-\alpha)C_k|S')$ or $\frac{R(C_k|S)}{C_k} = \alpha\frac{R(\alpha C_k|S')}{\alpha C_k} + (1-\alpha)\frac{R((1-\alpha)C_k|S')}{(1-\alpha)C_k}$. Combining this with the weak inequality above, we have $\frac{R(\alpha C_k|S')}{\alpha C_k} = \frac{R(C_k|S)}{C_k}$.

Finally, choose S_j so that $S_j \in \arg\max_{S_i \in S} C(S_i)$. Then, there exists an $\alpha_k \leq 1$ such that $C_j = \alpha_k C_k$ for all k. Therefore, $R(\alpha_k C_k|S') = R(C_j|S') = R(S_j|S)$ or $R(S_j|S) = \frac{C_j}{C_k}R(C_k|S)$ (by anonymity and the definition of core) and $\int_{S_j \in S}\frac{C_j}{C_k}R(C_k|S)\,ds = \frac{R(C_k|S)}{C_k}\int_{S_j \in S} C_j ds = \frac{R(C_k|S)}{C_k} = E$ (by budget balancedness). Thus, $R(C_k|S) = C_k E$ and the result follows. ■

Remark 2 *Our result implies that for all* $\hat{S}_k \subseteq S_k$*, if* $C\left(\hat{S}_k\right) = \alpha C(S_k)$ *then* $\phi\left(\hat{S}_k|S\right) = \alpha C_k E$*. This shows that the proportionality of rewards is not only valid across coalitions in the core, but also valid within coalitions in the core: a major strengthening of our main result.*

Remark 3 *What would happen under the constrained equal awards rule, which gives every (positive measure) coalition an incentive to split?*[13] *Assuming that a minimum size bound (similar to* κ*) also for coalitions, we can see that any positive-measure claimant with a claim larger than the minimum claim-size will have an incentive to split his claim into multiple, smaller claims. This will lead all claimants to split their claims to the smallest claim possible and submit these smaller claims. This suggests that in the core, agents' total rewards will be proportional (to the total number of units of shares they have).*

Finally, Corollary 1 below shows that we can extend the result in Proposition 1 in an interesting manner. In particular, we can say that the reward

[12] We implicitly assume that transfers are possible across coalitions. Reader is referred to Gomes and Jehiel (2005), Ray (2007), and Ray and Vohra (2015) for studies using a similar assumption).

[13] In our continuum of agents setup, the constrained equal awards rule can be defined as follows: $R(S_k|S) = \min\{C(S_k), \lambda\}$, where λ is chosen to satisy efficiency.

rule must be *downward proportional*. Or equivalently, if S' is derived from S by splitting coalitions in S up, then the rewards must still be proportional.

Corollary 1 *If S is in the core and S' can be derived from S by finitely many splits of coalitions in S, then S' must also be in the core.*

Proof. The proof of Proposition 1 shows that any S' derived from S is also locally proportional; and therefore in the core. Thus, by induction, the statement follows. ■

4 Concluding Remarks

In this paper, we model a bankruptcy problem as a cooperative game with continuum of agents. Under mild assumptions, we show that the rewards in the core of the game are proportional, independent of the reward rule used. Our game allows claimants to split their claims or merge them with other claimants' and submit it as a single claim. The way we model bankruptcy problem is different than the standard approach (see O'Neill, 1982) in that we consider circumstances where a reward rule is exogenously given by the court (or the authority) and the value of a coalition depends on the partition. We think that these are reasonable deviations from the standard model since the rule to be used if the case goes to court is pre-determined in many real life bankruptcy problems; and given this assumption the value of a coalition will depend on other coalitions. The second deviation has already a long history in coalitional games literature (e.g., partition function form games).

Our main result provides another possible explanation as to why proportional allocation of resources in many real-life problems is a very popular practice. Our work can be interpreted as an attempt to investigate bankruptcy problems where the final (submitted) claims are endogenously/strategically determined. These problems are of great relevance since merging/splitting is allowed in many real-life bankruptcy problems. Under this interpretation, our main result implies that if there are *large* number of claimants and merging/splitting is not explicitly prohibited, then for a great variety of reward rules, agents adjust their claims to such extent that the reward rule behaves proportionally in equilibrium. In other words, if we predict that the core outcome will be observed in the real life, then a policy that aims to implement an allocation other than the proportional one will be ineffective. In the light of this, we argue that the practice of allocating a resource proportionally would be observed even more frequently than the use of the proportional rule in real-life claim problems. Hence, if one focuses on allocation vectors rather than particular rules, the support for proportionality is even stronger.[14] One possible caveat for our results is that they would not be valid in a model with finite number of claimants. Investigating the limit properties (i.e., by replication or increasing the number of agents) in such a model is left for future research.

Acknowledgments: Will be added later.

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[14] A similar result in a different framework is due to Chun and Thomson (2005) who showed that the *random arrival rule* (O'Neill, 1982) behaves like the proportional rule in a k-times replicated bankruptcy problem, where k goes to infinity.

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