The Emergent Seed: A Representation Theorem for Models of Stochastic Evolution.

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Abstract

In a model of stochastic evolution there is an underlying structure that helps determine the long run viability of limit sets. This structure is called the *emergent seed*. Relative to this structure, long run viability is the additive component of the *radius* and the *core attraction rate*. The radius is a local characteristic of a limit set; the core attraction rate is the solution to a shortest path problem; thus this results in a local/linear representation of viability.

The usefulness of this methodology is shown in several applications. Using this technique I find the stochastically stable limit set in three dimensional lattices and the speed of evolution. I also am able to completely characterize the stochastically stable limit set in contract games and gift-giving games

Key words: Stochastic Evolution, Emergent Seed, Radius, Coradius, Directed Graphs, Matching Games.

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1 Introduction

Since the seminal papers of Kandori, Mailath and Rob [12] and Young [17] stochastic evolution has become a viable alternative to assuming equilibrium behavior. However our general understanding of what will survive the evolutionary process is still rather weak.

Young [17] shows that the analyst can focus on stable sets of states or *limit sets.* This simplifies analysis but still one must find the *stochastic potential* of every limit set. This difficult task requires that one finds a particular minimal cost connected graph—a *tree*. A limit set will be evolutionarily successful (*stochastically stable*) if it has the minimum stochastic potential. Thus all we can say about general problems is that a limit set with the least costly tree will be evolutionarily successful. To this day this stands as the only general representation. I show that if one first finds an intermediate structure, called the *emergent seed(s)*, then stochastic stability is determined by a local characteristic of a limit set (its *radius*) and the solution to a shortest path problem (its *core attraction rate*).

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The emergent seed is a minimal cost graph that connects all limit sets. It differs from trees because every limit set has a successor and in principle it must only be found once per analysis. The radius is defined in Ellison [6] as the minimal cost way to exit a limit set. The *core* is the subset of limit sets in the emergent seed that can be reached from every other limit set, and the core attraction rate is the cost of getting from that substructure to the given limit set. To have a low stochastic potential a limit set should have a high radius and a low core attraction rate, a local/linear representation of stochastic potential.

Only a few other papers focus on simplifying the general problem. Generally these papers develop various *tree surgery* techniques, a term first used by Young [17]. Essentially the methodology is to find sufficient conditions for a given limit set such that if you changed a tree to terminate at that limit set it would have lower stochastic potential. Clearly the most significant and general of these techniques was developed by Ellison [6]. This paper was motivated by two concerns, first to bound the speed of evolution and second to find a local characterization of when a limit set is stochastically stable. The local characterization is the radius/(modified) coradius, if the radius is higher than the (modified) coradius this is sufficient for a limit set to be stochastically stable, and furthermore the (modified) coradius is an upper bound on the speed of evolution. The coradius is the maximum minimal cost of getting to a given limit set from any other limit set, and the modified coradius is the same using a modified cost function. Thus that characterization is local/linear like the characterization in this paper, and furthermore the modified coradius is closely related to the core attraction rate in many problems. A contribution of my paper is to show that a modification of the radius/modified coradius method is not only sufficient but necessary for a limit set to be stochastically stable.

One paper in this literature does focus on simplifying the general problem, but the simplification is one of methodology, the general representation is the same as in Young [17]. In an early paper Kandori and Rob [13] derive the optimized cost function (the minimum cost of transitioning between any two states) and use this to find a simpler proof of the representation in Young [17]. Several papers have shown that if a limit set is *half-dominant* (Ellison [6]) or *globally pairwise risk dominant* (Kandori and Rob [14]) then it is stochastically stable. A limit set is half-dominant if some distribution in it is a best response to any distribution that places at least probability one half on any element(s) of the limit set. This is a strong but simple result that implies the radius is greater than the coradius. Binmore, Samuelson, and Young [2] provide the latest contributions. They independently state a radius/coradius condition and actually propose a test that checks to see if the emergent seed has a particularly simple form. Like in the bargaining game (Young [18]) sometimes a limit set in the core has maximal radius, in which case it is stochastically stable. They call this the *naive minimization test*, in this paper I refer to this as *core dominance*.

However the naive minimization test highlights the fundamental problem with sufficient methods. If this test does not find the stochastically stable limit set what are you to do? The methodology here tells you the next step, and can lead to a simple solution. Likewise the radius/modified coradius method relies on an overly strong characterization; thus there can be many problems where this method has nothing to say.

This brings us to a second benefit of the emergent seed method. Not only does it increase our understanding but it also simplifies analysis in applications, as will be shown below. In contrast Ellison [6] states that the radius/(modified) coradius methodology may not actually simplify analysis in applications, though clearly it is more likely to as the number of limit sets increases. I will provide complete characterizations in several examples where the sufficient methods above fail.

The first example is to extend the analysis in Ellison [6] from matching on two dimensional lattices to matching on three dimensional lattices. Matching on lattices responds to the concern about how slow stochastic evolution can be. In a standard model the speed of evolution decreases with the population size and thus it can take an essentially infinite amount of time to reach the stochastically stable limit set(s). Ellison [5] is a first response to this concern by showing that if a person is matched only with his two nearest neighbors (a one dimensional lattice) then in a standard symmetric two action coordination game it takes one mutation to get to the stochastically stable limit set. However this raises a secondary concern, does this reduction also hold in higher dimensional lattices? If the number of mutations was to explode then this result has little practical value. Ellison [6] responds to this question by analyzing two dimensional lattices, that paper finds that the number of mutations needed is at most three; below I improve this bound to two. Unfortunately two observations in the sequence doe not answer the general concern, which is about matching on an L dimensional lattice for all L. Perhaps the rate of increase is linear in L or perhaps it is exponential. I am able to extend his analysis to three dimensions, and find that it takes seven mutations to reach the half dominant equilibrium. Given three points in the series I conjecture that for general L the number of mutations needed will be on the order of 2^{L-1}.

Another application where all other methods fail is Young [19]. This paper extends the analysis of bargaining games in [18] to contract games. In a contract game players propose a payoff for both parties (a contract) and the proposals must agree to be accepted. Despite there being a potentially large number of Nash equilibria which are limit sets in this game, Young [19] shows that as the number of contracts becomes dense in a well-behaved feasible set the unique stochastically stable contract is the Kalai-Smordinsky solution. In this application the radius/(modified) coradius methodology fails because there could be no equilibria passing this test. The naive minimization test fails because the limit sets in the core have very low radii. In contrast I provide a simple analytic formula to find which equilibrium is stochastically stable.

Another application that shows the benefit of a necessary methodology is Johnson, Levine, and Pesendorfer [11]. That paper analyzes the evolution of cooperative behavior in a gift giving game. Since there are many ways to support cooperation this is a very complex interaction. Johnson et al. [11] use the radius/coradius methodology and show that if the expected benefit of giving a gift is less than two there will be no cooperation and if it is greater than four there will be. Below it is shown that one can use core dominance (or the naive minimization test.) With this technique we realize that if the expected benefit of giving the gift is below three there is no cooperation, if it is greater than three there is.

While these illustrations of how the emergent seed can simplify analysis will be of interest to most readers, it should be clear that this is not the primary contribution of this paper. The new representation theorem allows us to understand the fundamentals of what makes a limit set stochastically stable in a new light. We now realize it is a combination of being hard to leave that limit set (a high radius) and having a low cost of getting from a certain key set of limit sets to the given limit set. It should be clear that this method will not simplify every analysis. For example if you increase the payoff from a given pair of actions enough it will be half-dominant, while the emergent seed can remain arbitrarily complex. Another example is the analysis of super-modular games in Kandori and Rob [13]. That paper develops game specific methods that provide a solution when none of the methods discussed above will work. However I have yet to find a challenging example where one of the other general methods is superior to the emergent seed.

Recognize that the method here can be applied to all models of stochastic evolution and even a wide class of problems outside of this literature. The fundamentals in this analysis are a set of states and a cost of transitioning between these states. The method here is recommended as long as the costs of transitions are not symmetric by construction. Thus this methodology can be applied to questions like the optimal location of a water treatment plant or conference. One needs to be careful, as well, that one's cost function is not graph dependent. Consider, for example, the literature on network games (see Jackson and Wolinsky [9] or Bala and Goyal [1].) In that literature the benefit of establishing a connection depends on the current graph—i.e. who a person is connected to—and therefore the emergent seed methodology would not seem to be appropriate. However Jackson and Watts [8] shows that in this case one only has to look at the state of the system as including the network, and then they are able to derive a cost function over these states. At this point all general methodologies in the literature can be applied directly, including the emergent seed.

In the next section I show how using the emergent seed can extend our understanding of matching on a lattice. This section makes reference to results found in the full text but a reader should be able to get a general understanding of the methodology from this section alone. In this section we also explain how the cost function can be derived, something that is taken as a given in the general text. In the general model (section 2) the fundamentals are taken as a set of states over which evolution will occur and a cost function for transitioning between these states. In that section I also briefly summarize some key results from the literature. After this, I turn to finding the emergent seed in section 3. The key insights are in section 3.1, section 3.2 shows how to iterate the process to find the general representation. In section 4 I show how to use this methodology in analysis. In section 4.1, I show how it can simplify and extend our understanding of stochastic stability in contract games, and in section 4.2, I show how it can significantly increase our understanding of when cooperation will evolve. In section 5 I provide some guidelines on cases where the emergent seed methodology may not be the most efficient way to proceed, and section 6 concludes.

1.1 Extending our Understanding of Evolution on a Lattice.

Analyzing evolution on a lattice is motivated by the very long time evolution can take in a standard model. In a standard model of stochastic evolution one begins with a large population playing a normal form game and all players are matched by equal likelihood. Usually the people in the population either do not change their strategy or they switch to a myopic best response to the current distribution, but with probability ε they play a different action at random—or "mutate." The conclusion of the analysis is generally that if the population is large enough the most likely long run state is that the entire population will be playing one particular Nash equilibrium (that equilibrium is *stochastically stable*). Unfortunately the long run can be very long, unreasonably long, especially since our analysis assumes that the probability of a mutation is nearly zero. While the force of stochastic evolution may drive society to one particular choice it may take so long to get there that this is only of academic interest. As a concrete example assume that the population is playing a symmetric coordination game.

	a	b
a	0	u_{ab}
b	u_{ba}	1

where $0 > u_{ba}$, $1 > u_{ab}$ and $1 - u_{ab} < -u_{ba}$ so that the two Nash equilibria are (b, b) and (a, a) and (a, a) is half dominant. In this game the half dominant equilibrium is stochastically stable, and if initially the entire population is using the strategy b and there are $H * L$ people in the population then it will take $\left\lceil H * L * \frac{-u_{ba}}{1-u_{ba}-u_{ab}} \right\rceil$ mutations to reach the stochastically stable equilibrium.[1] Thus for reasonable population sizes society could play the (b, b) equilibrium for any foreseeable time horizon.

Ellison has addressed this problem by considering matching on a lattice. In an L dimensional lattice players are arranged into an L dimensional hyper-cube, with H players in each dimension. Thus the *location* of a player can be denoted as $h = (h_1, h_2, h_3, ..., h_L)$. A player in location $h_l \in \{0, 1, 2, ..., H - 1\}$ of dimension $l \in \{1, 2, 3, ..., L\}$ will only interact with his two *neighbors* in that dimension—the players at $(h_l + 1) \bmod H$ and $(h_l - 1) \bmod H$ in that dimension and the same location in all other dimensions. In order for the two neighbors of a player to be different we will assume throughout that $H \geq 3$.

A *state* in this model is the action of every player at every location, denoted z with the set of states being Z. We will assume that with probability $(1 - \chi)$ a player will use the action he used in the last period, with probability $\chi (1 - \varepsilon)$ the player will switch to a myopic best response to the distribution of actions in his neighborhood, with probability $\chi\varepsilon$ he will switch to the other action—this is called a *mutation*. This motivates the cost function $C(z|z')$ which is the number of players who must mutate in order to switch from z' to z in one period. To understand this notation I would like to explicitly calculate the probability of a transition. Let $P_{t+1}(z|z')$ be the probability of being in state z in period $t+1$ given that the state in period

[1] $\lceil x \rceil$ is the smallest integer greater than or equal to $x \in \mathbb{R}$.

t is z'. To do this we need to divide the population into four groups, based on whether they are currently taking the right action and whether they have to switch to a best response. We denote each group by two letters: XY; $X = C$ if the person is currently using the correct strategy, N otherwise; $Y = N$ if the person must switch to a best response, B otherwise. Then the four groups are NN, NB, CN, and CB; note that $NN + NB + CN + CB = H * L$ and $C(z|z') = NN$. Then

$$P_{t+1}(z|z') = (\varepsilon\chi)^{C(z|z')} (\chi(1-\varepsilon))^{NB} \sum_{k=0}^{CN} \binom{CN}{k} (1-\chi)^k (\chi\varepsilon)^{CN-k} \sum_{k=0}^{CB} \binom{CB}{k} (1-\chi)^k (\chi(1-\varepsilon))^{CB-k} . \tag{1}$$

As the probability of mutations (ε) is driven to zero this probability is on the order of $\varepsilon^{C(z|z')}$ and the more mutations that are required the less likely the transition. Or in other words if $C(z'|z)$ decreases a transition is more likely, and since we are interested in the most likely long run state one can see that the most likely long run state will be reached with minimal cost.

Notice that since we are looking for long run results we do not actually care if it takes one period or (finitely) many to go from z' to z, and that taking multiple steps may reduce the cost of the transition. For example we could go from z' to some intermediate state $\tilde{z}$ and then just have everyone best respond to get to z. This motivates the optimized cost function $\overleftarrow{C}(z|z')$ which is the least number of players who must mutate in order to switch from z' to z in a finite number of periods. The final element we need for our analysis is the *limit sets*, in this model $\Omega \in Z$ is a limit set if $\overleftarrow{C}(z|\Omega) > 0$ for every $z \in Z\backslash\Omega$, denote the set $\mathcal{W}$. In general these limit sets may be sets of states, but that is not the case here.

Ellison [5] considered matching on a one dimensional lattice. In this case there are only two limit sets, the one where everyone plays b and the one where everyone plays a, denote the former state Ω^b and the latter Ω^a—we will use this notation throughout the analysis. Now $\overleftarrow{C}\left(\Omega^a|\Omega^b\right) = 1$, because if everyone is playing b and one person switches to a then the next period that persons two neighbors will switch to a by using a myopic best response. The following period two more can, until in the end all people have switched to a. By the same analysis $\overleftarrow{C}\left(\Omega^b|\Omega^a\right) = H$, and since it takes longer to switch from Ω^b to Ω^a than the reverse Ω^a is stochastically stable and furthermore the expected waiting time is ε^{-1}. Thus there is hope that when interaction is "local" that evolution might be fast, fast enough to occur in a reasonable amount of time.

1.1.1 Matching on a two dimensional lattice

However the one dimensional lattice is clearly a special case, does this intuition generalize to higher dimensional lattices? In Ellison [6] he analyzes two dimensional lattices and finds a similar result, but the analysis is much more difficult. I should note that in this section I assume that $u_{ab} < \frac{1}{3}u_{ba} + 1$, otherwise the only limit sets are Ω^a and Ω^b and the analysis is the same as above. Assuming this there are now many more limit sets. In a limit set everyone must have two or more neighbors that are playing a or one or fewer. So for example a square of people playing a is a limit set, and so is a line of people playing a, and with these two basic examples one can construct many more.[2]

To find the stochastically stable limit set he develops the radius/coradius and the radius/modified coradius theorems. To do this it is convenient to extend the $\overleftarrow{C}(z|z')$ terminology to sets, so for $Z' \subseteq Z$ and $Z'' \subseteq Z$ let $\overleftarrow{C}(Z'|Z'') = \min_{z' \in Z', z'' \in Z''} \overleftarrow{C}(z'|z'')$. Then the *radius* of a limit set is

$$R(\Omega) = \min_{z \in Z\backslash\Omega} \overleftarrow{C}(z|\Omega) = \min_{\Omega' \in \mathcal{W}\backslash\Omega} \overleftarrow{C}(\Omega'|\Omega) \tag{2}$$

and the *coradius* is

$$CR(\Omega) = \max_{\Omega' \in \mathcal{W}\backslash\Omega} \overleftarrow{C}(\Omega'|\Omega) \tag{3}$$

[2] To be precise, a *square* is a set of a players of whom four have two neighbors playing b. A *line* is a set of players who have all locations but one in common and every player in that line has two neighbors playing a.

and he shows that if $R(\Omega) > CR(\Omega)$ then Ω is stochastically stable. Notice that in practice finding the coradius could be very difficult, however in this example we expect that $CR(\Omega^a) = \overleftarrow{C}\left(\Omega^b|\Omega^a\right)$, but we have a still further difficulty. To get from Ω^b to Ω^a at the least cost we will want to transition between a series of intermediate limit sets, but the number of limit sets increases in the number of players so the coradius of any limit set probably increases without bound. Thus this theorem could have limited usefulness. Furthermore he shows that the maximal expected waiting time to transition to Ω is $\varepsilon^{-CR(\Omega)}$, so does this mean that in a two dimensional grid the time to reach the stochastically stable limit set increases without bound?

The answer is no, and to show this he has to find a more complicated cost, the modified coradius. To define this I find it most convenient to define the first difference cost function:

$$\Delta C(z|z') = \begin{cases} \overleftarrow{C}(z|z') - R(\Omega) & \text{if } z' \in \Omega \text{ and } z \in \mathcal{W}\backslash\Omega \\ \overleftarrow{C}(z|z') & \text{else} \end{cases} \tag{4}$$

and we define $\Delta\overleftarrow{C}(z|z')$ and $\Delta\overleftarrow{C}(Z'|Z'')$ in the same way as we did for $C(\cdot|\cdot)$. The *modified coradius* is then

$$CR^*(\Omega) = \max_{\Omega' \in \mathcal{W}\backslash\Omega} \left\{ \Delta\overleftarrow{C}(\Omega|\Omega') + R(\Omega') \right\} . \tag{5}$$

He shows that Ω is stochastically stable if $R(\Omega) > CR^*(\Omega)$, and that the expected waiting time until Ω is reached is on the order time of $\varepsilon^{-CR^*(\Omega)}$. To understand this expression assume that in the critical path we go from Ω' to some $\tilde{\Omega}$ and from $\tilde{\Omega}$ to Ω, then

$$CR^*(\Omega) = \overleftarrow{C}\left(\Omega|\tilde{\Omega}\right) - R\left(\tilde{\Omega}\right) + \overleftarrow{C}\left(\tilde{\Omega}|\Omega'\right) . \tag{6}$$

and the expected length of the transition is $\varepsilon^{-\overleftarrow{C}(\tilde{\Omega}|\Omega')}\frac{\varepsilon^{-\overleftarrow{C}(\Omega|\tilde{\Omega})}}{\varepsilon^{-R(\tilde{\Omega})}} = \varepsilon^{-CR^*(\Omega)}$. The rational for $\varepsilon^{-\overleftarrow{C}(\tilde{\Omega}|\Omega')}$ is transparent because that is the expected waiting time necessary to start the transition, but why do we condition $\varepsilon^{-\overleftarrow{C}(\Omega|\tilde{\Omega})}$ on $\varepsilon^{-R(\tilde{\Omega})}$? This is because we are looking for the length of time it will take to transition from $\tilde{\Omega}$ to Ω *conditional on a transition occurring.* The most likely transition is $R\left(\tilde{\Omega}\right)$, so the conditional expected waiting time is $\frac{\varepsilon^{-\overleftarrow{C}(\Omega|\tilde{\Omega})}}{\varepsilon^{-R(\tilde{\Omega})}}$. Using this tighter bound on the waiting time he is able to show that Ω^a is always stochastically stable and that $CR^*(\Omega^a) \leq 3$, thus the hope provided by the one dimensional lattice does extend to the two dimensional lattice.

At this point I would like to replicate his analysis using the *emergent seed*. To do this we have to go back to Young [17] and discuss the general conditions for Ω to be stochastically stable. To do this we will construct a graph over the limit sets, call a *direct successor* of a limit set the limit set that comes immediately after it in this graph, and a *successor* of a limit set a limit set that is arrived at after a finite number of iterations of the direct successor relationship. Then a *tree with base* Ω is a directed graph over the set of limit sets, such that Ω is the successor of all other limit sets but has no successors itself. As explained above transitioning to Ω is less likely in the long run if the cost of reaching Ω is higher, thus the likelihood of Ω is determined by the cost of it's least cost tree, this is called the *stochastic potential* of a limit set. A limit set is *stochastically stable* if it has the lowest stochastic potential.

Before I present the emergent seed I would like to explain how we can look at the radius/modified coradius theorem as a type of *tree surgery.* Imagine a tree with arbitrary base Ω', now we want to construct a new tree with base Ω. Clearly this will at least decrease the cost of the tree by the radius of Ω, and at most increase the cost by the coradius of Ω. But consider the $\tilde{\Omega}$ in the path between Ω' and Ω, clearly these $\tilde{\Omega}$ must have had a successor in the tree with base Ω'? Thus we can subtract from the cost of this path the radius of every such $\tilde{\Omega}$ and this results in the modified coradius, and clearly if $R(\Omega) > CR^*(\Omega)$ then $-R(\Omega) + CR^*(\Omega) < 0$ so Ω has a lower stochastic potential than every such Ω' and it is stochastically stable.

Returning to the general argument, the methodology is to find the minimal cost tree for each limit set and then compare these costs. But if one considers this problem when there are a reasonably large number of limit sets one realizes that one is probably doing the same thing over and over again. Surely, unless there is good reason not to, for each Ω there is some Ω' that will always be its direct successor? The answer to this question is yes, and this fundamental underlying structure I call the *emergent seed.* But how do you find this structure? Take any tree, and give the base of that tree a direct successor. Then minimize the cost of this structure and you have the emergent seed(s).

Definition 1 *An* emergent seed, *E, is a minimal cost graph such that:*

1. *Every $\Omega \in \mathcal{W}$ has a successor.*

2. *There exists some $\Omega^* \subseteq \mathcal{W}$ that are the successors of all other $\Omega \in \mathcal{W}$.*

For a precise definition of a minimal cost graph please see section 2. Note that there is an apparent abuse of a notational convention here, in general one would assume Ω^* is a special limit set, instead it is set of limit sets. The rational for this is that is the unique limit set with regards to a new cost function, as will be explained after the following proposition. For now notice that all the limits sets in Ω^* form a cycle, each is the successor of the others, call this cycle the *core*. In matching on a two dimensional grid this structure is easy to find, and it makes the conclusions in Ellison [6] simple to derive.

Define $\Omega^a(\Omega)$ as the set of limit sets where if $\Omega' \in \Omega^a(\Omega)$ then the only difference between Ω and Ω' is that some players have switched from the b action to the a action. We continue to denote Ω^a as the limit set were everyone plays a and Ω^b as the limit set where everyone plays b.

Proposition 1 *When matching is on a two dimensional lattice:*

1. *There is an emergent seed where the direct successor of Ω is in $\Omega^a(\Omega)$ for $\Omega \neq \Omega^a$, and therefore Ω^a is in the core of this emergent seed.*

2. *$R(\Omega^a) > R(\Omega')$ for all $\Omega' \in \mathcal{W} \backslash \Omega^a$ thus Ω^a is stochastically stable.*

3. *Furthermore $CR^*(\Omega^a) = R(\Omega^b) = 2$.*

Proof. Let j and j' be two players who have two neighbors in common but are not neighbors themselves.

Assume that $\Omega \notin \{\Omega^a, \Omega^b\}$, and choose a j who is playing b and for whom there is a j' who is using the strategy a. Have j mutate to playing the strategy a. Now the best response of both of the common neighbors of j and j' will be a. Since Ω is a limit set at least one of these two neighbors must have been playing b, thus we have exited Ω. There may be a chain of other people who switch to a because j and j''s common neighbors have, but after a finite number of players change their strategy one will be in a new limit set where more players are using strategy a. Note that it only took one mutation to do this, thus I have shown that there is an emergent seed where the direct successor of every Ω is in $\Omega^a(\Omega)$ for $\Omega \notin \{\Omega^a, \Omega^b\}$.

Clearly the direct successor of Ω^b is in $\Omega^a(\Omega^b)$, and note that if j and j' both mutate to a then their two common neighbors will switch to a and this is a limit set, thus $R(\Omega^b) = 2$. Furthermore $R(\Omega^a) > 2$, because unless three people mutate to b no player's best response is b, thus $R(\Omega^a) \geq 3$.

To finish the proof we will refer to general results derived in this paper. First by Corollary 1 since $R(\Omega^a) > R(\Omega')$ for all $\Omega' \in \mathcal{W} \backslash \Omega^a$ Ω^a is stochastically stable, second by Corollary 2 $CR^*(\Omega^a) = \max_{\Omega' \in \mathcal{W} \backslash \Omega^a} R(\Omega') = R(\Omega^b) = 2$. ∎

Thus I have improved on the bound Ellison [6] found for the modified coradius. At this point I can explain why the core of the emergent seed uses the same notation as a limit set. In every respect the first difference cost function is just a standard cost function for our analysis, and Ω^* is the only limit set with regards to this cost function. Thus Ω^* is a limit set, but for a different cost function.

1.1.2 Matching on a three dimensional lattice

At this point I have strengthened the conclusion in Ellison [5] and Ellison [6] to show that the waiting time in an L dimensional lattice is on the order of e^{-L} if $L \in \{1, 2\}$. Does this insight extend or are the first two elements of the sequence special cases? One would guess that these are special cases, but to see this for sure we must study at least one more dimension. Here I will extend his analysis to three dimensional lattices. In the three dimensional lattice the analysis is more difficult because the first structure we find (the *base* of the emergent seed) will not have a core. I will solve the problem by using the same methodology I used to find the base with a new cost function—the first difference cost function defined above. Like above I will assume that the payoffs are such that we can not simplify the problem to one of the two above, in this case we need $u_{ab} < \frac{1}{2}u_{ba} + 1$ so that a is not a best reply when two of a player's six neighbors are playing a. In this model a cube of players playing a is a limit set, so is a plane of player playing a, and many other limit sets can be found by the combination of these two limit sets.[3]

Defining $\Omega^a(\Omega)$ as we did in a two dimensional lattice, the following results is immediate.

Lemma 1 *If* $\Omega \neq \Omega^a$, $\overleftarrow{C}(\Omega^a(\Omega)|\Omega) \leq 4$

Proof. Clearly the most difficult case is when we want to switch players from playing b to a in an area where there are currently no players playing a. In such an area choose four players who have two common neighbors with each of the other three but are not neighbors themselves. Have these players mutate to playing a. Then the best response of all the remaining players is to play a, and we are at an element of $\Omega^a(\Omega)$. ∎

Now a limit set can be described as a set of a *components.* An a component is a connected set of players who are all playing a. An a component is *convex* if there is no j and j' in the component that have a common neighbor playing b. We can go from any limit set with non-convex a components to one where all a components are convex in a series of single mutations, or:

Lemma 2 *Assume* Ω *has non-convex* a *components. Then there is a finite sequence of limit sets* $\{\Omega(t)\}_{t=1}^{T}$ *such that* $\Omega(1) = \Omega$*, for* $t < T$, $\Omega(t+1) \in \Omega^a(\Omega(t))$, $\overleftarrow{C}(\Omega(t+1)|\Omega(t)) = 1$ *and* $\Omega(T)$ *has only convex* a *components.*

Proof. If Ω has a non-convex a component then there is a line of players who have exactly two neighbors playing a. If one player in this line mutates to a then every other player in this line will switch by best response, resulting in a new limit set. If this process is repeated in the end there will be no non-convex components in the resulting limit set. ∎

Given this corollary I will simply assume from now on that every a component is convex. For limit sets with a convex a component the maximum cost of reaching $\Omega^a(\Omega)$ is lower.

Lemma 3 *Assume that* Ω *has a convex* a *component and* $\Omega \neq \Omega^a$*, then* $\overleftarrow{C}(\Omega^a(\Omega)|\Omega) \leq 2$.

Proof. Given Ω has an a component there are at least four players playing b who have exactly one neighbor playing a. Choose two of these players who each have a common neighbor playing b and are not neighbors themselves and have them mutate to playing a. Then the other two players in this square will switch to playing a and we are at an element of $\Omega^a(\Omega)$. ∎

Up to now we have focused on "good news," mutations that lead us to Ω^a. Now we switch focus to mutations that lead to Ω^b. Define $\Omega^b(\Omega)$ as the set of limit sets where if $\Omega' \in \Omega^b(\Omega)$ then the only difference between Ω and Ω' is that some players have switched from the a action to the b action. Notice that we are only interested in elements of $\Omega^b(\Omega)$ that have convex a components due to Lemma 2.

[3] To be precise a *cube of a players* is a set of a players of whom eight have three neighbors who are playing b. A *plane of a players* is a set of players who share all but two locations in common, and all players have four neighbors who are playing a.

Now since I assume that all a components are convex, an a component can be completely described by its three dimensions: length, width, and height. I say that the *size* of an a component is the size of its smallest dimension unless the component is a plane in which case it's size is H. Another characteristic of an a component is its *faces* formally this is a set of players for whom $\exists l \in \{1,2,3\}$ such that h_l is the same for all of them, and their neighbors at location $(h_l + 1) \bmod H$ or $(h_l - 1) \bmod H$ are all playing b.

Lemma 4 *If Ω has an a component with a size of two then $\overleftarrow{C}\left(\Omega^b(\Omega)\,|\Omega\right) = 1$, otherwise $\overleftarrow{C}\left(\Omega^b(\Omega)\,|\Omega\right) \geq 2$.*

Proof. First if Ω has an a component with a size of two then it has a subset of players who have exactly three neighbors playing a. If one of these players mutates to b then each of his neighbors in the subset will switch to b, and after a finite number of periods this will lead to an element of $\Omega^b(\Omega)$ which has convex a components.

On the other hand if Ω does not have an a component with a size of two then clearly the worst case will be when there is an a component that is $3 \times 3 \times 3$. I need to show that no single mutation can lead to $\Omega^b(\Omega)$. In such a cube only the players on the corners will have only three neighbors playing a. Thus if a single mutation is going to lead to a new limit set *with convex a components* then one of the neighbors of a player on a corner should mutate to b. If this player is a neighbor of two corner players than a line of players will switch to b. However the resulting limit set will have a non-convex a component and we can convexify it with single mutations. Thus we are not in the set $\Omega^b(\Omega)$ and the proof is done. ■

Finally we will find the radii of the two important limit sets, Ω^a and Ω^b

Lemma 5 *$R\left(\Omega^b\right) = 4$ and $R(\Omega^a) \geq 5$.*

Proof. If we start at the state Ω^b then the only way to get a limit set with an a component is to have four people who all have two neighbors in common to mutate to the strategy a, thus $R\left(\Omega^b\right) = 4$.

If we start at the state Ω^a then one limit set with a b component is a plane of players who are all playing b. If $H = 3$ then to get the rest of the plane to switch to b by best response you must have five players who are not neighbors all switch to b.

I will be done when I show that there is no limit set that takes fewer mutations to reach, I will do this by showing that every limit set with a b component must have at least H^2 players playing b. This is obvious because each player must have four neighbors playing b, thus $R(\Omega^a) \geq 5$. ■

I am now ready to find the base of the emergent seed by connecting limit sets to the limit set that determines their radii.

Proposition 2 *In a base of an emergent seed:*

1. *If $\Omega \in \mathcal{W}\backslash\Omega^a$ has an a component with a size of 3 or more then its direct successor is in $\Omega^a(\Omega)$.*
2. *Else if $\Omega \in \mathcal{W}\backslash\Omega^b$ then its direct successor is in $\Omega^b(\Omega)$.*
3. *Finally if the direct successor of Ω is in $\Omega^a(\Omega)$, $R(\Omega^a) > R(\Omega)$, and if it is in $\Omega^b(\Omega)$ $R\left(\Omega^b\right) > R(\Omega)$.*

Proof. In Lemma 3 I showed that $\overleftarrow{C}(\Omega^a(\Omega)\,|\Omega) \leq 2$ for $\Omega \neq \Omega^a$. Thus if $\overleftarrow{C}\left(\Omega^b(\Omega)\,|\Omega\right) \geq 2$ I can construct a base of the emergent seed where the direct successor of Ω is in $\Omega^a(\Omega)$. In Lemma 4 I showed that if Ω has an a component of size 2 or smaller then $\overleftarrow{C}\left(\Omega^b(\Omega)\,|\Omega\right) = 1$, this in this case I may have to have the direct successor of Ω be $\Omega^b(\Omega)$.

Potentially it may be that for each of these states $\overleftarrow{C}(\Omega^a(\Omega)\,|\Omega) = 1$, in which case the base of the emergent seed does have a core and I am done. To show that this is not true consider an Ω where there are exactly 8 players playing a. These players must be be arranged in a cube, and to get anyone else to switch to a there must be exactly two mutations. ■

Notice that there are clearly other bases. We can have a components which are $3 \times 3 \times k$ (for $k \geq 3$) have successors in $\Omega^b(\Omega)$. On the other hand if two a components have only one b player between them then they can have a successor in $\Omega^a(\Omega)$.

Now what do we do? Notice that the first difference cost function—$\Delta\overleftarrow{C}$—is essentially exactly the same as a normal cost function except that the only limit sets with regards to this cost function are Ω_1^a and Ω_1^b. These are both sets of limit sets, Ω_1^a is the limit set Ω^a and a limit set where there is only one plane of players playing b. Ω_1^b is Ω^b and only one cube of size 2 of players playing a.

Thus now all we have to do is find out which of these two sets is stochastically stable with regards to the new cost function, and then we will show that one of the limit sets in those sets is stochastically stable with regards to the original cost function.

Lemma 6 *$\Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) = 3$ and $\Delta\overleftarrow{C}\left(\Omega_1^b|\Omega_1^a\right) \geq 3$.*

Proof. In Ω_1^b there is Ω^b and one of the limit sets that has one a component that is $2 \times 2 \times 2$. Choose three faces of this cube and have two players that are not neighbors on each face mutate to a. When we convexify the resulting a component we have a limit set with an a component that is $3 \times 3 \times 3$. Each time two of the players on a face mutate we have a new limit set, and the radius of all limit sets in Ω_1^b is one, thus $\Delta C\left(\Omega_1^a|\Omega_1^b\right) = 3$.

In the core of Ω_1^a there is Ω^a and one of the limit sets that has a plane of players playing b. (Note there are H^2 of these players). Notice that if there is a plane of players playing b with two planes of players playing a on either side then with one mutation everyone in the plane will switch to a since everyone in that plane only has four neighbors playing b. Now in order to get to a convex element of $\Omega^b(\Omega)$ we must at the very least have H players who have a neighbor playing b, share two common neighbors, but are not neighbors themselves mutate to b, thus $\Delta C\left(\Omega_1^b|\Omega_1^a\right) \geq 3$. ■

We will conclude by proving that Ω^a is the stochastically stable limit set and finding its modified coradius.

Proposition 3 *In a three dimensional lattice Ω^a is stochastically stable and $CR^*(\Omega^a) = 7$.*

Proof. First of all notice that by Corollary 5 and Proposition 2 we know that Ω^b and Ω^a are the only possible stochastically stable states.

In the emergent seed without loss of generality we can assume that the successor of the state where there is only one cube of a players is an Ω such that its direct successor is in $\Omega^a(\Omega)$, thus by Theorem 1 we can see that the stochastic potential of Ω^a is:

$$C(E) - R(\Omega^a)$$

the stochastic potential of Ω^b is:

$$C(E) - R\left(\Omega^b\right) - \Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) + \Delta\overleftarrow{C}\left(\Omega_1^b|\Omega_1^a\right)$$

To understand this expression notice that (like above) if we are looking for a tree that terminates in Ω^b we obviously do not need to connect Ω^b to its direct successor, saving $R\left(\Omega^b\right)$, we also do not need Ω_1^a to be the successor of Ω_1^b, so we save $\Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right)$, but on the other hand we now need Ω_1^b to be the successor of Ω_1^a, costing us $\Delta\overleftarrow{C}\left(\Omega_1^b|\Omega_1^a\right)$. Note that:

$$C(E) - R\left(\Omega^b\right) - \Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) + \Delta\overleftarrow{C}\left(\Omega_1^b|\Omega_1^a\right) \geq C(E) - R\left(\Omega^b\right) > C(E) - R(\Omega^a)$$

since $\Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) \leq \Delta\overleftarrow{C}\left(\Omega_1^b|\Omega_1^a\right)$ and $R(\Omega^a) > R\left(\Omega^b\right)$. Thus Ω^a is stochastically stable.

Since Ω^a is in the core and the only cost that is not zero for the first difference cost function is $\Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right)$ the modified coradius is $CR^*(\Omega^a) = \Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) + \max_{\Omega \in \Omega_1^b} R(\Omega) = \Delta\overleftarrow{C}\left(\Omega_1^a|\Omega_1^b\right) + R\left(\Omega^b\right) = 7$. ■

Based on these three observations we can begin to develop some understanding of how long evolution will take on an L dimensional lattice. This understanding is based on the fact that $CR^*(\Omega^a) \geq R\left(\Omega^b\right)$, and that $R\left(\Omega^b\right)$ will be determined by a limit set with the smallest possible a components. In general this will be a hyper-cube with 2 people on each face, and like above the cost of this will be 2^{L-1}. To be certain that the speed of evolution is on the order of 2^{L-1} I need to further argue that $\Delta C\left(\Omega_1^a|\Omega_1^b\right)$ will increase at a slower rate. but since 2^{L-1} is an exponential rate this is unlikely. However this argument can not be based on the current analysis.

I speculate that for any L most of the argument above will directly generalize. In the base of every emergent seed either limit sets will go towards Ω^b or Ω^a, thus the emergent seed can be found using only the first difference cost function. There will be a critical size for an a component, if the size of the largest a component is larger than this the successor of the limit set can be Ω^a, smaller then it must be Ω^b. It may be that this size is increasing with the number of dimensions, but I suspect it is not and if so one may find that $\Delta C\left(\Omega_1^a|\Omega_1^b\right) = L$.

However at this point all I can safely conjecture is that the speed of evolution is on the order of 2^{L-1}. While this is a significant decrease in the speed of evolution notice that it is still independent of population size. If this conjecture is true then evolution depends only on the number of people someone is matched with, which would greatly increase the probability of being in the long run limit set.

One natural extension of this analysis would be to consider what happens if some people are only matched with two neighbors, some with three, and so on. A model that has this characteristic is Galeotti et al. [7], however a caution must be provided to interested analysts. In that model actions are determined prior to the realization of the network, and then the network is determined by global matching. In such a setting evolution will almost surely occur at a speed on the order of the population size, unlike here. Indeed one feature of that model is that the incomplete information limits the number of equilibria and thus potentially the number of limit sets. That simplifies the analysis in that model but in this analysis the number of intermediate limit sets speeds up evolution. On the other hand the analysis of diffuse networks in Jackson and Yariv [10] suggests that as the global matching becomes more diffuse the transition to the stochastically stable limit set will be quicker. The results above could be considered as what happens when the matching becomes maximally diffused.

2 The Model and Preliminary Analysis

In this section I generalize and formalize the terms introduced in the example above. For simplicity of presentation I will dispense with showing how the model used in the analysis is generated from the underlying evolutionary process. In this paper an *evolutionary system* is a pair $\Upsilon = \{Z, C\}$ where Z is a finite set of states and C is a cost function $C : \{Z, \emptyset\} \times Z \to \mathbb{R}_+$, satisfying:

1. $\forall \{z, z'\} \in Z \times Z$, $C\left(z'|z\right) \geq 0$, and
2. $C\left(\emptyset|z\right) = 0$.

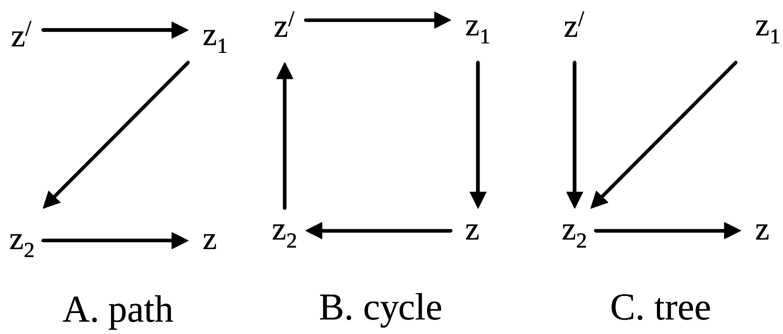
Note that in general $C\left(z|z'\right) \neq C\left(z'|z\right)$ or this is a *directed cost function.* For a general derivation of such a cost function see Ellison [6], for specific examples see above and section 4.

For clarity it is best to formally define a *directed graph* over Z. This graph can be represented as a function from Z to Z and the empty set, $s : Z \to Z \cup \emptyset$. [4] If $z' = s\left(z\right)$ then z' is z's *direct successor* in s and z' is z's *direct predecessor.* This means that in this graph in some period t we are at state z and then in $t+1$ we are at z'. If there is a finite number of iterations of s such that $z' = s\left(s\left(s\left(...s\left(z\right)\right)\right)\right)$ then z' is z's *successor* and z is z''s *predecessor.* I have included the empty set in the range of z to indicate that a given state has no successors in the given graph, and indeed if z has no direct predecessors or successors then we say z is *disconnected.* We say that a state is *connected* if $s\left(z\right) \neq \emptyset$, and a graph s is *more connected* than a

[4] Note that $s\left(z\right)$ is a function. For our analysis this restriction is non-binding.

graph s' if $s'(z) \neq \emptyset$ implies that $s(z) \neq \emptyset$. The cost of a graph is $C(s) = \Sigma_{z\in Z} C(s(z)|z)$, and I use the same notation for other cost functions. Given a set of graphs S that satisfy given characteristics, a *minimal cost graph* is $s \in \arg\min_{s\in S} C(s)$.

There are three types of graphs used in the analysis. The simplest is a *path*; $s(z|z')$ is a path if z' is the predecessor of every $\widetilde{z} \in \widetilde{Z}$ that is not disconnected and z is the successor of every such $\widetilde{z}$. A path is just a sequence of states, one occurring after the other as is illustrated in graph A in the figure below. Another simple example is a *cycle*; a graph is a cycle if every z that is not disconnected is a successor of every other z' that is not disconnected. This is graph B in the figure below. The most important type of graph in the analysis are *trees with base* z, denoted t_z^χ where χ indicates which of the subsets of Z the graph is over, let t_z be a graph where every $z \in Z$ is connected. In a tree the only restriction is that z has no successors and is the successor of every other $\widetilde{z} \in Z\backslash z$. An example is graph C in the figure below. A tree looks like an inverted extensive form game; the decision nodes are now states with z as the initial "decision node." The dynamics are reversed as well; instead of starting with z one starts from the ends of the branches and travel back to z.



An arrow from $\widetilde{z}$ to $\widetilde{\widetilde{z}}$ means that $\widetilde{\widetilde{z}}$ is the direct successor of $\widetilde{z}$ in the illustrated graph.

For simplicity of exposition I will use the *optimized cost function* which was first introduced in Kandori and Rob [13] and I used in section 1.1. If $S(z'|z)$ is the set of paths from z to z' then $\overleftarrow{C}(z'|z) = \min_{s\in S(z'|z)} C(s)$, and I will follow Ellison [6] by extending the cost function to a set notation, for two non-empty subsets of Z—Z' and Z''—define $\overleftarrow{C}(Z''|Z') = \min_{z''\in Z'', z'\in Z', s\in S(z''|z')} C(s)$. I assume that $\max_{\{z',z\}\subseteq Z} \overleftarrow{C}(z'|z) < \infty$, if this is not true either one can ignore some states *a priori* or there is no solution to the problem in this paper.

The goal is to find the states with the minimal cost trees. If the set of trees with base z is T_z then the *stochastic potential* of z is $C_z = \min_{t_z\in T_z} C(t_z)$, and z is *stochastically stable* if and only if $z \in \arg\min_{z\in Z} C_z$.

Notice that the cardinality of Z might be quite large. In a simple example Z is the set of feasible distributions over some strategy space A given the population size is J, where J is large. However Young [17] shows that the analysis is much simpler than it seems. Young proves that all we have to consider are trees over *limit sets*.

Definition 2 *A* limit set *is an* $\Omega \subseteq Z$ *such that:*

1. *For every* $z \in Z\backslash\Omega$ $\overleftarrow{C}(z|\Omega) > 0$.
2. *Given a graph* s *where all* $z \in Z\backslash\Omega$ *are disconnected,* $\overleftarrow{C}(s) = 0$.

Let the set of limit sets be $\mathcal{W}$. In principle there could be a large number of limit sets, but in general it is a much smaller set than Z and does not increase with the population size.[5] Note that if $z \notin \mathcal{W}$ then there is some $\Omega \in \mathcal{W}$ such that $\overleftarrow{C}(\Omega|z) = 0$. If this were not so, then z itself would be part of a limit set. In other words there are two types of states in our analysis: those in limit sets, and those that can reach limit sets at zero cost. The second class of states is clearly less important, and indeed one of the key results in Young [17] is that one can essentially ignore such states. Given Ω, call the states z that can reach Ω with zero cost Ω's *basin of attraction*, denoted $B(\Omega) = \left\{ z \in Z | \overleftarrow{C}(\Omega|z) = 0 \right\}$.

Now consider the trees t_Ω^0 over $\mathcal{W}$ and T_Ω^0 be the set of these trees (the 0 superscript indicates that it is a graph over $\mathcal{W}$, not Z).[6] Let

$$\overleftarrow{C}_\Omega = \min_{t_\Omega^0 \in T_\Omega^0} \overleftarrow{C}\left(t_\Omega^0\right) \tag{7}$$

then Young's primary results can be summarized in the following lemma.

Lemma 7 *Given* $\overleftarrow{C}$ *and* $\mathcal{W}$ *then for all* $z \in \Omega \in \mathcal{W}$*,*

$$C_z = \overleftarrow{C}_\Omega,$$

and for all $z \notin \mathcal{W}$*,*

$$C_z = \min_{\Omega \in \mathcal{W}} \left\{ \overleftarrow{C}(z|\Omega) + \overleftarrow{C}_\Omega \right\}.$$

The first claim states that for limit sets it is only necessary to consider trees over other limit sets. The second claim implies that you can ignore all other states. Thus the problem simplifies to one which depends on the cardinality of $\mathcal{W}$, not Z. There are currently two proofs of the first part of this claim in the literature. The first (in Young [17]) uses a tree surgery method, the second (in Kandori and Rob [13]) is based on reduced chains. The following proof is similar to Young's proof and is useful in the analysis that follows. It is also the first to formally establish the second part of the claim. This analysis is the first to make extensive use of this part of the statement.

Proof. Define $\hat{z}$ as a *juncture* if there are two limit sets who's first common successor is $\hat{z}$. Let Ω be such that $z \in \Omega$, $\hat{t}_\Omega^0$ be the tree that has cost $\overleftarrow{C}_\Omega$ and $\hat{t}_z$ be the tree that has cost C_z.

First for $z \in \Omega$, $\overleftarrow{C}_\Omega \geq C_z$ since a path can be added from Ω to z at zero cost and for every $\tilde{z} \in B\left(\tilde{\Omega}\right)$ a path can be added from $\tilde{z}$ to $\tilde{\Omega}$ at zero cost. Therefore $\hat{t}_\Omega^0$ can be represented as a graph over Z without increasing its cost.

To go in the other direction is somewhat more complicated, but we will show that $\overleftarrow{C}_\Omega \leq C_z$ by reducing the cost $\hat{t}_z$ until it can be represented as a graph over $\mathcal{W}$. First since there is a graph with zero cost that connects each limit set if $z' \in \Omega'$ occurs at some point in the tree we can say that Ω' occurs at that point. Second assume that Ω' and Ω'' are two limit sets for which $\hat{z} \in B\left(\hat{\Omega}\right)$ is the juncture. Both Ω' and Ω'' can not be the successor of $\hat{\Omega}$, thus add a path from the one that is not to $\hat{\Omega}$ and $\hat{z}$ is no longer a juncture. Finally if $\tilde{z} \in B\left(\tilde{\Omega}\right)$ is not a predecessor of Ω in the resulting graph add a path from $\tilde{z}$ to $\tilde{\Omega}$ at zero cost.

To prove the second claim notice that otherwise there would be two limit sets for which only z would be the first common successor. In this case we can make the successor of either one of them the Ω where $z \in B(\Omega)$ and the cost of the tree would not be increased, thus the representation is without loss of generality. ■

[5] As a counter example consider the model in section 1.1.

[6] To be precise this means that all $z \in Z \backslash \mathcal{W}$ are disconnected.

3 The Emergent Seed

In this section we will show how to find the emergent seed in a general problem. The structure we are seeking is:

Definition 3 *An* emergent seed*, E, is a minimal cost graph such that:*

1. *Every $\Omega \in \mathcal{W}$ has a successor.*
2. *There exists some $\Omega^* \subseteq \mathcal{W}$ that are the successors of all other $\Omega \in \mathcal{W}$.*

We will then find the stochastic potential of each limit set relative to the cost of this structure. How do we do this? In general we can always subtract the cost of connecting Ω to its direct successor, but then we may need to construct a path of Ω''s from Ω^* to Ω, but by the definition of the emergent seed we do not need to make any further changes. In other words we have changed a tree minimization problem into a shortest path problem.

Like in the example above (section 1.1) there will be multiple limit sets in Ω^*, and these limit sets are in a cycle which we call the *core*. The reason that the notation for Ω^* is the same as a limit set is because it is the unique limit set for a modified cost function we will use to find the stochastic potential.

Clearly there may be multiple emergent seeds, and section 4.1 shows this may even be true for a generic set of games. However for purposes of exposition it is simpler to treat the emergent seed as unique. We will find this structure iteratively, the base of the emergent seed is denoted E_0 and is derived from $\mathcal{W}$ and $\overleftarrow{C}$. Using E_0 we then define the first difference cost function $\Delta\overleftarrow{C}$and $\mathcal{W}_1$—which are the limit sets of $\Delta\overleftarrow{C}$and also the cycles in E_0. The first "level" of the emergent seed is then denoted E_1, which is derived from $\mathcal{W}_1$ and $\Delta^1\overleftarrow{C}$, after this we continue the iteration and E is then defined as the projection of $E_0, E_1, E_2 ... E_K$ onto $\mathcal{W}$.

3.1 The Base of the Emergent Seed

The essential strategy is to only connect limit sets via the least cost connections, those that define the radius of each limit set. The base of the emergent seed is then nothing more than connecting each element of $\mathcal{W}$ to one of the limit sets that determines its radius.

Definition 4 *The* base of the emergent seed*—E_0—is a graph where all $z \in Z \backslash \mathcal{W}$ are disconnected and*

$$s_0(\Omega) \in \arg\min_{\Omega' \in \mathcal{W} \backslash \Omega} \overleftarrow{C}(\Omega' | \Omega) = R(\Omega) \quad . \tag{8}$$

Clearly E_0 must contain at least one cycle. Each cycle in E_0 is an Ω_1, and the set of Ω_1's is called $\mathcal{W}_1$. If there is an E_0 for which $\mathcal{W}_1$ has one element, then this is an emergent seed since this implies every $\Omega \in \mathcal{W}_1$ is the successor of all other $\Omega' \in \mathcal{W}$.

If we want to take something out of this E_0 graph what is the relevant cost function? If we make the successor of Ω some Ω' then we are increasing the cost by $\overleftarrow{C}(\Omega'|\Omega)$ but we are also decreasing the cost by $R(\Omega)$. This motivates a new cost function, which I call the *first difference cost function*:

$$\Delta C(z|z') = \begin{cases} \overleftarrow{C}(z|z') - R(\Omega) & \text{if } z' \in \Omega \text{ and } z \in \mathcal{W} \backslash \Omega \\ \overleftarrow{C}(z|z') & \text{else} \end{cases}$$

and we can find $\Delta\overleftarrow{C}$ as before. Notice that while Ω_1 is a cycle in E_0 it is properly defined as a limit set with regards to the $\Delta\overleftarrow{C}$ cost function. In every respect $\Delta\overleftarrow{C}$ is a standard cost function, just like $\overleftarrow{C}$.

Lemma 8 *$\Delta C$is a standard cost function and furthermore if Ω_1 is a limit set with respect to ΔC then $z \in \Omega_1$ means there is a Ω such that $z \in \Omega$.*

Proof. The properties of a cost function are $\forall \{z, z'\} \in Z \times Z$, $\Delta C(z'|z) \geq 0$, and $\Delta C(\emptyset|z) = 0$. The latter property is adapted directly from $C(\cdot|\cdot)$, to see that the former property holds notice that by definition for $z' \in \Omega$ and $z \in \mathcal{W} \backslash \Omega$ $\overleftarrow{C}(z|z') \geq R(\Omega)$.

Now let Ω_1 be a limit set with regards to ΔC, then this means that for all $z \in Z \backslash \Omega_1$ $\Delta \overleftarrow{C}(z|\Omega_1) > 0$. Let $z' \in \Omega_1$ and assume that $z' \notin \mathcal{W}$, then by definition $\Delta C(z|z') = \overleftarrow{C}(z|z')$ thus if $\Delta C(z|z') > 0$ then $\overleftarrow{C}(z|z') > 0$ for all $z \in Z \backslash \Omega_1$ and $z' \in \Omega_1 \in \mathcal{W}$, a contradiction. ■

Notice what this means about the relationship between C and ΔC. If there was a way to exist a state at zero cost with C there still is with ΔC, and many states that were in limit sets with regards to C can now be exited at zero cost. Since in Lemma 7 we have shown that all we need to take care of is limit sets, this means finding the optimal tree with regard to ΔC will be much easier than finding it with regard to C. Indeed the strategy of this paper is to repeat the technique by which we found E_0 and ΔC until finding the optimal tree is trivial.

In fact in application I have often found that all ones need to do is find E_0. This is signified by their being a core for E_0, denoted Ω^*. Another way of putting it is that there is a unique limit set with respect to ΔC, denoted Ω_1. In this case the *core attraction rate* is $Ca(\Omega) = \Delta \overleftarrow{C}(\Omega|\Omega_1) = \Delta \overleftarrow{C}(\Omega|\Omega^*)$. I now show that when this is the case there is a simple linear representation of the stochastic potential of a limit set in terms of the radius and the core attraction rate.

Proposition 4 *If E_0 has a core then:*

$$\overleftarrow{C}_\Omega = \overleftarrow{C}(E_0) - R(\Omega) + Ca(\Omega) \tag{9}$$

and Ω is stochastically stable if

$$\Omega \in \arg\max_{\Omega \in \mathcal{W}} \{R(\Omega) - Ca(\Omega)\} \tag{10}$$

The general representation is the same as this one except that the core attraction rate are radius are modified. The proof of this proposition assumes that E_0 has a core. It is necessary that there are more than three limit sets for E_0 not to have a core. Furthermore in Ellison [6], Young [18], and Young [19] E_0 has a core even though there is potentially an infinite number of limit sets in each model. Notice that the point of representation 10 is that one does not actually need to calculate the cost of the emergent seed, once one has found it all that matters is the radii and core attraction rates of the various limit sets.

Proof. For a given t^0_Ω notice that by definition:

$$\overleftarrow{C}(t^0_\Omega) = \overleftarrow{C}(E_0) - R(\Omega) + \Delta \overleftarrow{C}(t^0_\Omega) \tag{11}$$

indeed the only part of this expression that needs explanation is to note that since in E_0 Ω has a successor and in t^0_Ω it does not we must subtract the cost of $\overleftarrow{C}(s_0(\Omega)|\Omega) = R(\Omega)$. Then by Lemma 7 we know that:

$$\overleftarrow{C}_\Omega = \overleftarrow{C}(E_0) - R(\Omega) + \min_{\Omega_1 \in \mathcal{W}_1} \left\{\Delta \overleftarrow{C}(\Omega|\Omega_1) + \Delta \overleftarrow{C}_{\Omega_1}\right\} \tag{12}$$

by assumption we know that there is a unique Ω_1 and that $\Delta \overleftarrow{C}_{\Omega_1} = 0$ thus we have:

$$\begin{aligned} \overleftarrow{C}_\Omega &= \overleftarrow{C}(E_0) - R(\Omega) + \Delta \overleftarrow{C}(\Omega|\Omega^*) \\ &= \overleftarrow{C}(E_0) - R(\Omega) + Ca(\Omega) \end{aligned} \tag{13}$$

which is representation 9. Assume that Ω has minimal stochastic potential, then this means that for every Ω'

$$\begin{aligned} \overleftarrow{C}(E_0) - R(\Omega) + Ca(\Omega) &\leq \overleftarrow{C}(E_0) - R(\Omega') + Ca(\Omega') \\ R(\Omega) - Ca(\Omega) &\geq R(\Omega') - Ca(\Omega') \end{aligned} \tag{14}$$

which gives us representation 10. ■

Using the emergent seed methodology one can immediately dismiss the possibility that some of the limit sets are stochastically stable.

Corollary 1 (Predecessor Dominance) *If Ω is a successor of Ω' in the emergent seed and $R(\Omega) > R(\Omega')$ then Ω' is not stochastically stable.*

Proof. Since Ω is a successor of Ω' it has zero first difference cost to go from Ω' to Ω; thus $Ca(\Omega') \geq Ca(\Omega)$. Since $R(\Omega) > R(\Omega')$, Ω' is not stochastically stable. ■

When this holds for some element of the core then I call this *core-dominance.* Core-dominance combined with the base of the emergent seed being the emergent seed is what Binmore, Samuelson, and Young [2] call the naive minimization.

The emergent seed methodology also allows us to bound the modified coradius in Ellison [6]. If one does not find the emergent seed then finding the modified coradius can be difficult since one has to solve $|\mathcal{W}|$ shortest path problems and Ω is stochastically stable only if $R(\Omega) \geq CR^*(\Omega)$. An advantage of the radius/modified coradius methodology is that it places an upper bound on the maximum waiting time until the system is in the stochastically stable limit set.

Corollary 2 *If E_0 has a core then for all $\Omega \in \mathcal{W}$*

$$Ca(\Omega) + \max_{\Omega' \in \mathcal{W} \backslash \Omega} \{R(\Omega') - Ca(\Omega')\} \leq CR^*(\Omega) \leq Ca(\Omega) + \max_{\Omega' \in \mathcal{W} \backslash \Omega} R(\Omega') \tag{15}$$

and if Ω is in the core then:

$$CR^*(\Omega) = \max_{\Omega' \in \mathcal{W} \backslash \Omega} R(\Omega'). \tag{16}$$

Proof. Notice that

$$\Delta \overleftarrow{C}(\Omega|\Omega') \leq \Delta \overleftarrow{C}(\Omega|\Omega_1) + \Delta \overleftarrow{C}(\Omega_1|\Omega') = Ca(\Omega) \tag{17}$$

because one path from Ω' to Ω is to first go to the core and then from the core to Ω. Using the same logic makes it clear that $Ca(\Omega) \leq \Delta \overleftarrow{C}(\Omega|\Omega') + Ca(\Omega')$ since in order to get to Ω one can always go from the core to Ω' and then from Ω' to Ω.

If Ω is in the core then both of these bounds are the same and I derive condition 16. ■

This corollary can potentially expand the application of the modified coradius. Especially if one finds the game is solvable by core dominance then the modified coradius is merely the second highest radius. Of course this can only be directly applied if E_0 has a core.

Notice how close the condition in equation 15 is to representation 10. The radius being greater than the lower bound is a restatement of representation 10. If the radius is greater than the upper bound then

$$R(\Omega) - Ca(\Omega) \geq \max_{\Omega' \in \mathcal{W} \backslash \Omega} R(\Omega')$$

which is equivalent to assuming that the core attraction rate of all other limit sets is zero. This insight helps explain the difference between the necessary representation in this paper and the sufficient representation in Ellison [6].

3.2 The Representation Theorem.

As stated above given the first difference cost function if Ω' is a successor of some Ω in E_0 then $\Delta\overleftarrow{C}(\Omega'|\Omega) = 0$, or every $\Omega \in \mathcal{W}$ is in the basin of attraction of some $\Omega_1 \in \mathcal{W}_1$. Using this I can immediately apply Lemma 7 to find the stochastic potential of all Ω.

Corollary 3 *If $\mathcal{W}_1$ has more than one element then:*

$$\overleftarrow{C}_\Omega = \overleftarrow{C}(E_0) - R(\Omega) + \min_{\Omega_1 \in \mathcal{W}_1} \left\{ \Delta\overleftarrow{C}(\Omega|\Omega_1) + \Delta\overleftarrow{C}_{\Omega_1} \right\}$$

Proof. This is equation 12 from Proposition 4. ■

Thus the new problem is to find graphs over $\mathcal{W}_1$ instead of graphs over $\mathcal{W}$. Instead of directly solving this problem I will do what I did in the last section and simplify the problem. In the last section we found the base of the emergent seed, we will now find it's higher "levels," defined as the least cost way to make the partial emergent seed more connected. Let $\Delta^0 C = C$, $\Delta^1 C = \Delta C$, and $\mathcal{W}_0 = \mathcal{W}$, then for $k \in \{1, 2, 3, ...\}$, the procedure we will follow is given $\Delta^{k-1}C$ find the limit sets of this cost function—$\mathcal{W}_{k-1}$; then find the next level of the emergent seed E_k; and using this define $\Delta^k C$. There is one difference between finding E_k and finding E_0. Now if there is a cycle in an $\tilde{E}_k$ we want to break the most costly link since we are interested in a graph over $\mathcal{W}$, not $\mathcal{W}_k$. Given $\Delta^k\overleftarrow{C}$:

$$\Delta^k R(\Omega_k) \equiv \min_{\Omega'_k \in \mathcal{W}_k \backslash \Omega_k} \Delta^k\overleftarrow{C}(\Omega'_k|\Omega_k).$$

Definition 5 *For $k \geq 1$ Given $\mathcal{W}_k$ and $\Delta^k C$ the* k'th level of the emergent seed*—E_k—is a graph where all $z \in Z\backslash\mathcal{W}_k$ are disconnected. In $\tilde{E}_k$:*

$$s_k(\Omega_k) \in \arg\min_{\Omega'_k \in \mathcal{W}_k \backslash \Omega_k} \Delta^k\overleftarrow{C}(\Omega'_k|\Omega_k) \ .$$

then E_k is found by finding all cycles in $\tilde{E}_k$ and letting $s_k(\Omega_k) = \emptyset$ for one Ω_k such that $\Delta^k R(\Omega_k) \geq \Delta^k R(\Omega'_k)$ for any other Ω'_k in the same cycle. Then $\mathcal{W}_{k+1}$ is the $\Omega_k \in \mathcal{W}_k$ such that $s(\Omega_k) = \emptyset$.

For $k \geq 1$ define

$$\Delta^{k+1}\overleftarrow{C}(z'|z) = \begin{cases} \Delta^k\overleftarrow{C}(z'|z) - \Delta^k R(\Omega_k) & \text{if } z \in \Omega_k \in \mathcal{W}_k\backslash\mathcal{W}_{k+1} \text{ and } z' \in \mathcal{W}_k\backslash\Omega_k \\ \Delta^k\overleftarrow{C}(z'|z) & \text{else} \end{cases}$$

and iterate. Notice an important difference between $\Delta^{k+1}\overleftarrow{C}(z'|z)$ and $\Delta\overleftarrow{C}(z'|z)$. If a state is in $\mathcal{W}_{k+1}$ then we do not subtract anything from its cost because it has no successor in the k'th level. Each iteration results in a cost function with fewer and fewer limit sets, until finally there is one limit set and we are done.

Definition 6 *The emergent seed—E—is found by projecting $\{E_k\}_{k=1}^K$ onto $\mathcal{W}$. If z has a direct successor in E_{k^*} but no $k > k^*$ then that direct successor is z's direct successor in E.*

The core attraction rate is then:

$$Ca(\Omega) \equiv \Delta^K\overleftarrow{C}(\Omega|\Omega^*) \ . \tag{18}$$

If we define $\Delta^0 R(\cdot) = R(\cdot)$ and let

$$\Omega_k(\Omega) = \begin{cases} \Omega'_k & \text{where} \quad \Omega \subseteq \Omega'_k \in \mathcal{W}_k \quad \text{if it exists} \\ \Omega & \text{else} \end{cases}$$

then the generalized radius is

$$R_E(\Omega) = \Sigma_{k=0}^{K} \Delta^k R(\Omega_k(\Omega)) \ .$$

To understand the difference between this and the standard radius first notice that there can be at most two k for which $\Delta^k R(\Omega_k(\Omega)) > 0$. We will always have $R(\Omega) > 0$ and if $\Omega \subseteq \Omega_1$ then there may be a higher k such that $\Delta^k R(\Omega_k(\Omega)) > 0$. After this point $\Omega \nsubseteq \mathcal{W}_{k+1}$ and $\Delta^{k+l} R(\Omega) = 0$ for $l \in \{k+1, k+2, ...K\}$. Now recognize that if we are looking for a tree with base $\Omega \subseteq \Omega_1$ then we do not have to have direct successor for Ω and for the Ω_1 that contains this Ω. For example in section 1.1.2 when we were calculating the stochastic potential of the limit set where everyone played b (Ω^b) the expression we found was:

$$\overleftarrow{C}_{\Omega^b} = \overleftarrow{C}(E) - \left(R\left(\Omega^b\right) + \Delta \overleftarrow{C}\left(\Omega_1^a | \Omega_1^b\right)\right) + \Delta \overleftarrow{C}\left(\Omega_1^b | \Omega_1^a\right)$$

the generalized radius is $R_E\left(\Omega^b\right) = R\left(\Omega^b\right) + \Delta \overleftarrow{C}\left(\Omega_1^a | \Omega_1^b\right)$. The general representation theorem can now be stated.

Theorem 1 *For* $\Omega \in \mathcal{W}$:

$$\overleftarrow{C}_{\Omega} = \overleftarrow{C}(E) - R_E(\Omega) + Ca(\Omega) \tag{19}$$

and Ω *is stochastically stable if*

$$\Omega \in \arg\max_{\Omega \in \mathcal{W}} \{R_E(\Omega) - Ca(\Omega)\} \tag{20}$$

Proof. Given E_0 and ΔC we notice that in Proposition 4 equation 12 holds regardless of the number of elements in $\mathcal{W}_1$. Given this we can define the new problem,

$$\min_{\Omega_1 \in \mathcal{W}_1} \left\{\Delta^1 \overleftarrow{C}(\Omega | \Omega_1) + \Delta^1 \overleftarrow{C}_{\Omega_1}\right\}$$

One method to solve this problem is to define E_1 and $\Delta^2 C$ as is done above. Notice that Lemma 8 generalizes so we can still use Lemma 7 to find that

$$\Delta \overleftarrow{C}(\Omega | \Omega_1) + \Delta \overleftarrow{C}_{\Omega_1} = \Delta \overleftarrow{C}(E_1) - \Delta R(\Omega) + \min_{\Omega_2 \in \mathcal{W}_2} \left\{\Delta^2 \overleftarrow{C}(\Omega | \Omega_2) + \Delta^2 \overleftarrow{C}_{\Omega_2}\right\}$$

and by iteration we can find that

$$\begin{aligned} \overleftarrow{C}_{\Omega} &= \Sigma_{k=0}^{K} \Delta^k \overleftarrow{C}(E_k) - \Sigma_{k=0}^{K} \Delta R(\Omega) + \Delta^K \overleftarrow{C}(\Omega | \Omega^*) \\ &= \Sigma_{k=0}^{K} \Delta^k \overleftarrow{C}(E_k) - R_E(\Omega) + Ca(\Omega) \ . \end{aligned}$$

The first part of the claim is finished when we show that $\Sigma_{k=0}^{K} \Delta^k \overleftarrow{C}(E_k) = \overleftarrow{C}(E)$. Notice that by definition if Ω has a successor in E_k then its radius is

$$\begin{aligned} \Delta^k R(\Omega) &\equiv \Delta^k \overleftarrow{C}(s_k(\Omega) | \Omega) \\ &= \Delta^{k-1} \overleftarrow{C}(s_k(\Omega) | \Omega) - \Delta^{k-1} R(\Omega) \\ &= \overleftarrow{C}(s_k(\Omega) | \Omega) - \Delta R(\Omega) \ . \end{aligned}$$

Where the last step is realized by noting that Ω will have a successor at most twice, once in the base level and once in a higher level. Thus the total cost associated with Ω is:

$$\Delta^k R(\Omega) + R(\Omega) = \overleftarrow{C}(s_k(\Omega) | \Omega) - \Delta R(\Omega) + R(\Omega) = \overleftarrow{C}(s_k(\Omega) | \Omega)$$

thus the claim is established. The second claim is established in the same manner as it was in Proposition 4. ■

A useful fact is:

Corollary 4 *For $k \geq 1$, $\mathcal{W}_{k+1} \subseteq \mathcal{W}_k$*

After the base of the emergent seed is found the analysis looks at smaller and smaller subsets of $\mathcal{W}_1$, and given a new cost metric finds minimal cost trees over them.

Corollary 1 can be generalized to take advantage of the additional information in each E_k. However in general we need to perform the test with regards to $R_k(\Omega) = \Sigma_{l=0}^{k} \Delta^l R(\Omega_k(\Omega))$. We write it with regards to this radius because it clarifies that you may elicit information about which states are not stochastically stable at every step in the process.

Corollary 5 (Predecessor Dominance) *If Ω is a successor of Ω' in E_k for $k \geq 0$ and $R_k(\Omega) > R_k(\Omega')$ then Ω' is not stochastically stable.*

4 Two Further Applications

To show how this methodology is potentially useful in applications I analyze two more papers in the literature where the results can be strengthened and the derivation simplified by using the emergent seed. In a review of the applications literature it is surprising how much of the literature relies directly on half dominance. Generally I have found that papers that do not rely on half dominance can have their analysis simplified and/or generalized by using the emergent seed methodology. There are exceptions to this rule, for example Kandori and Rob [13] which I will discuss further below, but in at least the two following cases and the analysis of matching on a lattice above I have found the emergent seed helpful.

In both of the following examples there is a 2 role game, $G = \{A, u\}$ where $A = A_1 \times A_2$ and $u : A \to \mathbb{Q}^2$, and for $i \in \{1, 2\}$, u_i is the payoff of role i. There will be J players of each role. These players will be matched using uniform random matching and the cost function will have the form of $C(\Omega'|\Omega) = \lceil c(\Omega'|\Omega) J \rceil$ where $c(\Omega'|\Omega) \in [0, 1]$. Thus I will normalize the costs by J and assume that J is large enough that the difference between $C(u'|u)$ and $c(u'|u) J$ is insignificant. I use the lower case for the radius and core attraction rate when it is generated from this cost function.

4.1 Evolution in the Contract Game

Young [19] analyzes the *contract game*. If two parties are simply dividing a surplus—like in standard bargaining (Young [18])—what happens when the parties disagree can be well specified, but in more complicated situations where they are deciding (for example) how to produce the surplus the outcome if they disagree is not transparent. In the contract game one assumes that since they have failed to agree no surplus is produced. If we ignore the production methodology then both parties must propose an outcome for both him and the person he is interacting with, and if the two proposals do not agree then both parties receive their outside option. The primary result in Young [19] is that the stochastically stable contract is always Pareto efficient and generally *egalitarian* which means that the surplus is split evenly between the parties. To be precise this means that when the number of contracts becomes dense the stochastically stable contract converges to the Kalai-Smordinsky solution. However that paper relies on tree surgery methods so when there are a small number of contracts it only finds sufficient conditions. The emergent seed in this game is simple and intuitive, thus I am able to provide a general characterization.

In the contract game it is easiest to characterize the strategies by the payoffs they give. Thus there are $M < \infty$ possible contracts $\{u^m\}_{m=1}^{M} \equiv U$, $u^m = (u_1^m, u_2^m) \in \mathbb{Q}^2$. The Player in role one offers a contract u^m and the player in role two offers a contract $u^{m'}$. If $u^m = u^{m'}$ then player one gets u_1^m and player two gets u_2^m, otherwise the two players get $u^d = (u_1^d, u_2^d)$. One can easily see that if $u \geq\geq u^d$ then u^m is a Nash equilibrium.[7] I refer the reader to Young [19] to see that if $u^m >> u^d$ then u^m is a limit set and there are no

[7] For vectors x and y, $x \geq\geq y$ if for every dimension i $x_i \geq y_i$, $x >> y$ means that for all i $x_i > y_i$.

other limit sets. Denote $\bar{u}^1$ as contract that gives player one his highest payoff, and likewise $\bar{u}^2$ and for now assume these contracts are also limit sets. Normalize payoffs so that $u_1^d = u_2^d = 0$ and for simplicity denote the limit set where everyone offers the contract u as u.

One can easily see that:

$$c(u'|u) = \min\left\{\frac{u_1}{u_1+u_1'}, \frac{u_2}{u_2+u_2'}\right\} \tag{21}$$

$$r(u) = \min\left\{\frac{u_1}{u_1+\bar{u}_1^1}, \frac{u_2}{u_2+\bar{u}_2^2}\right\} . \tag{22}$$

Thus in the emergent seed the successor of every contract is either $\bar{u}^1$ or $\bar{u}^2$, and these two contracts are the core. However these contracts will rarely be stochastically stable. Thus I must define the first difference cost function

$$\Delta c(u'|u) = \min\left\{\frac{u_1}{u_1+u_1'}, \frac{u_2}{u_2+u_2'}\right\} - r(u) . \tag{23}$$

and find the core attraction rate. For arbitrary u $ca(u) = \min_{\{u^t\}_{t=1}^{T-1}} \Sigma_{t=1}^T \Delta c\left(u^t|u^{t-1}\right)$ where $u^0 \in \left\{\bar{u}^1, \bar{u}^2\right\}$ and $u^T = u$. However the only limit sets that matter for the analysis are those that are strongly Pareto efficient. Say that u strongly Pareto dominates u' if $u >> u'$, and that u is strongly Pareto efficient if it is not strongly Pareto dominated in U. Finding the core attraction rate of contracts that are strongly Pareto efficient is not difficult.

Lemma 9 *If u' is not strongly Pareto efficient then it is not stochastically stable. If u is strongly Pareto efficient then the core attraction rate is*

$$ca(u) = \min\left\{\frac{\bar{u}_2^1}{\bar{u}_2^1+u_2} - r\left(\bar{u}^1\right), \frac{\bar{u}_1^2}{\bar{u}_1^2+u_1} - r\left(\bar{u}^2\right)\right\} . \tag{24}$$

Proof. Notice first of all that if $u >> u'$ then $r(u) > r(u')$. As well from equation 23 one can immediately see that for $t < T$ u^{t+1} is strongly Pareto efficient given u^t. Thus $u = u^T$ must be strongly Pareto efficient to be stochastically stable.

Now I will directly minimize $\Delta c(u'|u)$ over u, when I show that $u \in \left\{\bar{u}^1, \bar{u}^2\right\}$ minimizes this cost it is immediate that $ca(u')$ is given by equation 24. First of all notice that if $c(u'|u)$ is determined by u_1 and $r(u)$ is determined by u_2 then $\Delta c(u'|u)$ can easily be decreased by increasing u_2 and decreasing u_1. Thus the only case to consider is when $c(u'|u)$ and $r(u)$ are both a function of u_1 (or u_2). In this case one can show that

$$\frac{\partial \Delta c(u'|u)}{\partial u_1} = \left(\frac{\bar{u}_1^1}{\bar{u}_1^1+u_1} - \frac{u_1}{u_1+u_1'}\right)\frac{\bar{u}_1^1-u_1'}{u_1+u_1'}\frac{1}{u_1+\bar{u}_1^1} .$$

Note that $\frac{\bar{u}_1^1}{\bar{u}_1^1+u_1} > \frac{1}{2}$. Since by assumption $c(u'|u) = \frac{u_1}{u_1+u_1'}$, $\frac{u_1}{u_1+u_1'} \le \frac{1}{2}$ since both u and u' are strongly Pareto efficient or $u = \bar{u}^2$, if $u_2 > u_2'$ then $u_1 \le u_1'$ and $c(u'|u) \le \frac{1}{2}$. Therefore $\frac{\partial \Delta c\left(u'|u\right)}{\partial u_1} < 0$ and one should reduce u_1 as much as possible. Notice that if u is strongly Pareto efficient then since $\bar{u}_2^2 > u_2$, $u_1 \ge \bar{u}_1^2$, thus $u = u^0 = \bar{u}^2$. ■

Thus if u is strongly Pareto efficient the stochastic potential is

$$c_u^* = c(E) - \min\left\{\frac{u_1}{u_1+\bar{u}_1^1}, \frac{u_2}{u_2+\bar{u}_2^2}\right\} + \min\left\{\frac{\bar{u}_2^1}{\bar{u}_2^1+u_2} - r\left(\bar{u}^1\right), \frac{\bar{u}_1^2}{\bar{u}_1^2+u_1} - r\left(\bar{u}^2\right)\right\} \tag{25}$$

and using this function one can easily find the stochastically stable contract. However with a bit of analysis one can simplify this problem further. Assume that the radius is determined by u_2 and core attraction rate is determined by u_1. Then by decreasing u_1 and increasing u_2 you can increase the radius and decrease the core attraction rate, thus clearly decreasing the stochastic potential. In order to decrease the radius you need

$$u_2 < u_1 \frac{\bar{u}_2^2}{\bar{u}_1^1} = f_r(u_1) \ , \tag{26}$$

and to decrease the coradius

$$u_2 < \bar{u}_2^1 \frac{\left(r\left(\bar{u}^2\right) - r\left(\bar{u}^1\right)\right)\left(\bar{u}_1^2 + u_1\right) + u_1}{\bar{u}_1^2 - \left(r\left(\bar{u}^2\right) - r\left(\bar{u}^1\right)\right)\left(\bar{u}_1^2 + u_1\right)} = f_{ca}(u_1) \ . \tag{27}$$

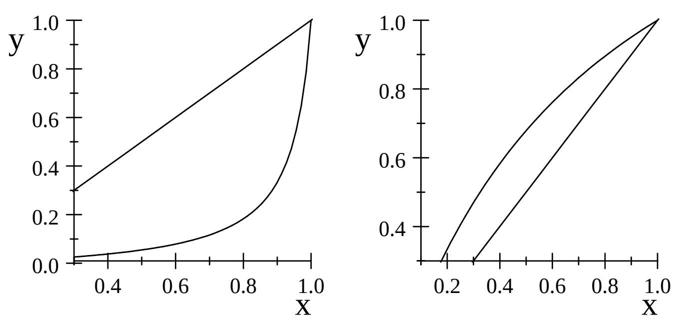
Thus to be out of this set you must have $u_2 \geq \min\{f_r(u_1), f_{ca}(u_1)\}$ and to avoid being in the complimentary set you need to have $u_2 \leq \max\{f_r(u_1), f_{ca}(u_1)\}$ or if $CH(U)$ is the convex hull of U then you need to be in:

$$U^* = \{u \in CH(U) \,|\, u_2 \in [\min\{f_r(u_1), f_{ca}(u_1)\}, \max\{f_r(u_1), f_{ca}(u_1)\}]\}. \tag{28}$$

Of course when optimizing on a finite grid the solution might also be only near U^*. Say that u Pareto dominates u' if $u \geq\geq u'$ and either $u_1 > u_1'$, $u_2 > u_2'$ or both, u is Pareto efficient if it is not Pareto dominated in U.

Proposition 5 *If u is stochastically stable it is either strongly Pareto efficient and in U^* or one of the two Pareto efficient contracts in $U \backslash U^*$ that is nearest to the set U^*.*

The usefulness of this results depends on how large the space U^* is. These two function intersect when $u_1 = \bar{u}_1^1$ and $u_1 = \bar{u}_1^2 \frac{\bar{u}_2^1}{\bar{u}_2^2} \leq \bar{u}_1^2$, and in the relevant range ($u_1 \in \left[\bar{u}_1^2, \bar{u}_1^1\right]$) if $r\left(\bar{u}^2\right) \geq r\left(\bar{u}^1\right)$ then $f_r(u_1) \geq f_{ca}(u_1)$. The functions are the same if $r\left(\bar{u}^2\right) = r\left(\bar{u}^1\right)$. To see how the size of U^* changes with the extreme contracts $\left(\bar{u}^1, \bar{u}^2\right)$ one can assume without loss of generality that $\bar{u}_1^1 = \bar{u}_2^2 = 1$, and I have found that in order to maximize the difference between these two functions $\min\left\{\bar{u}_2^1, \bar{u}_1^2\right\}$ must be nearly zero. Below I graph one extreme case and one more intermediate case.



Graph D $\left(\bar{u}_2^1, \bar{u}_1^2\right) = (.01, .3)$ Graph E $\left(\bar{u}_2^1, \bar{u}_1^2\right) = (.3, .1)$

However when the space U^* is large generally it will be one of the contracts nearest to $f_r(u_1)$ that will be important. Notice that

$$ca(u) = \min\left\{\frac{\bar{u}_2^1}{\bar{u}_2^1 + u_2} - r\left(\bar{u}^1\right), \frac{\bar{u}_1^2}{\bar{u}_1^2 + u_1} - r\left(\bar{u}^2\right)\right\} \leq \min\left\{\frac{\bar{u}_2^1}{\bar{u}_2^1 + u_2}, \frac{\bar{u}_1^2}{\bar{u}_1^2 + u_1}\right\} \leq \min\left\{\bar{u}_2^1, \bar{u}_1^2\right\} \tag{29}$$

which converges to zero. Thus in Graph D for a contract to be stochastically stable it must be near $f_r(u_1)$ or simply maximize the radius. Even in Graph E if the set of feasible contracts satisfy $u_1 + u_2 \leq 1.3$ then the maximum difference in the radius of two contracts is .30 while the maximum difference in the core attraction rate is .06.

Young essentially uses this fact to provide limiting results. He completely characterizes the stochastically stable contract when the set of contracts is dense in some convex utility space U that satisfies free disposal, or if $u \in U$ and $u' \leq\leq u$ then $u' \in U$. As the set of contracts becomes dense in U, $\min\left\{\bar{u}_2^1, \bar{u}_1^2\right\} \to 0$ and the only thing that matters is maximizing the radius. This coincides with the Kalai-Smordinsky solution.

This is also what occurs on a fixed grid if $\min\left\{\bar{u}_2^1, \bar{u}_1^2\right\} \leq 0$, or one the extreme contracts is not a limit set. Notice that $\min\left\{\bar{u}_2^1, \bar{u}_1^2\right\} \leq 0$ is true for an open set of payoffs, thus these games are generic. In this case one can show that the radii of the limit sets is still given by equation 22, however now $\bar{u}^1$ and $\bar{u}^2$ are not limit sets. Thus if any group of players coordinate on some $u >> 0$ then u will be established as the new contract. Thus there is an emergent seed with each limit set in the core, and the game can be solved by core-dominance.

4.2 Evolution in a Gift Giving Game

Johnson, Levine and Pesendorfer [11] is interesting for three reasons. First it is one of the few outside of Elision [6] to use the radius/coradius method. Second, since the radius/coradius method is a sufficient method that paper only find the stochastically stable strategy if a key parameter—β—is less than two or greater than four. I am able to characterize the stochastically stable strategy for all β. Finally it is a game with a large number of strategies and limit sets. This would suggest that the radius/coradius method would be best, but in the end the emergent seeds have a simple form.

In this model players live for two periods, in period t they are young and in period $t+1$ they are old. When a player is young he can either give a gift (1) or not (0). The game is:

	young 1	0
old	$\alpha, -1$	$0, 0$

Since the population of players is large and matched by equal likelihood if a player does not have a "reputation" then they have no incentive to give the gift. A player's reputation is established by use of *social norms* (Okuno-Fujiwara and Postlewaite [15]). Every old person has a *social status*, either r or g. Given this social status the young person follows a *social standard of behavior*, $a : \{r, g\} \to \{1, 0\}$, and then the transition rule determines his social status next period $\tau : \{r, g\} \times \{1, 0\} \to \{r, g\}$. A strategy is a pair: $\sigma = \{a, \tau\}$.

Notice that if people are using different strategies they will have different ways of evaluating a person's past behavior. There are 8 different transition rules. In order to keep track of all of these possibilities, each person has a *social status vector*, $f \in \{r, g\}^8 = \underbrace{\{r, g\} \times ... \times \{r, g\}}_{8}$. A given strategy uses only f_m—the m'th element of this vector $m \in \{1, 2, 3, ..., 8\}$. Write $a(f)$ for $a(f_m)$, and $\tau_m(a, f) = \tau_m(a, f_m)$.

A state in this model is a distribution over strategies (σ) and social status vectors (f). The distribution over f is secondary to the analysis, all that is critical is that young players believe they can meet any of the 256 possible social status vectors with positive probability. I achieve this by taking the the true distribution over f (Φ_t^0) and then replacing each old person's social status vector by a random one with probability $\eta > 0$, resulting in the final distribution (Φ_t). Young players will choose their strategy based on the "expected" distribution of f, $E\left(\Phi_t | \Phi_t^0\right)$, notice that for any Φ_t^0 the probability of any f given $E\left(\Phi_t | \Phi_t^0\right)$ is strictly greater than zero. I will write $\phi(f)$ for the probability of f given $E\left(\Phi_t | \Phi_t^0\right)$. One important implication of this is that controlling for these perturbations the expected benefit of giving the gift is $\beta \equiv (1 - \eta)\alpha$, assume that $\beta \geq 1$.

Thus young players choose σ as a best response to $E\left(\Phi_t | \Phi_t^0\right)$ and the distribution of strategies. They will next be matched with a random member of the old population; observe that person's f; and the action they take is determined by σ. This will determine their f, and then the next period they will be an old person.

There are 32 strategies in this model, however 8 of these are the *selfish strategies*—$a(r) = a(g) = 0$—and 8 more are the *generous strategies*—$a(r) = a(g) = 1$. Of the 16 remaining 8 are constructed from the other 8 by changing the "language" of the strategy. In one group green is good—$a(g) = 1$ and $a(r) = 0$—and in the other red is good ($a'(f) = 1 - a(f)$ $f \in \{r, g\}$). Looking at the "green" strategies ($a(g) = 1$ and $a(r) = 0$) an equilibrium must reward giving the gift to a green status player ($\tau(g, 1) = g$) and punish not giving a gift to a green status player ($\tau(g, 0) = r$), leaving four strategies

$\tau(g,1)$	$\tau(g,0)$	$\tau(r,1)$	$\tau(r,0)$	Strategy's Name
g	r	r	g	team
g	r	g	g	weak team
g	r	r	r	insider
g	r	g	r	tit for tat

these are all Nash equilibria when $\eta = 0$ but only the first three are Nash equilibria when $\eta > 0$. One can show that for all of the others strategies there is a sequence of best responses that leads to the selfish strategies, thus the game is *acyclic* (Young [17]) and only Nash equilibria are limit sets.

To find the emergent seed I will use the *gain function* from Johnson, Levine and Pesendorfer [11]. Letting ϕ be the distribution over social status vectors and μ be the distribution over strategies the *gain function* is the difference between using some strategy σ and some alternative strategy $\tilde{\sigma} = \{\tilde{a}, \tilde{\tau}\}$:

$$\begin{aligned}\Gamma(\sigma, \tilde{\sigma}, \phi, \mu) &= \Sigma_{f'}\left[a(f') - \tilde{a}(f')\right]\phi(f') \\ &+ \Sigma_{f'}\Sigma_{\sigma'=\{a',\tau'\}}\beta\left[a'(\tau'[\tilde{a}(f'), f']) - a'(\tau'[a(f'), f'])\right]\mu(\sigma')\phi(f')\end{aligned}$$

To find the radius one wants to find the distribution over strategies that is closest to $\mu(\sigma) = 1$ such that $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu) \leq 0$.

Lemma 10 *In the emergent seeds, the direct successor of the team, weak team, and insider strategies is the selfish strategies, the direct successor of the selfish strategies is either the team, weak team, or insider. The radius of the selfish strategies is* $\frac{1}{\beta}$*, of the team is* $\frac{\beta-1}{2\beta}$*, and of the weak team or insider is* $\min\left\{\frac{1}{\beta}, \frac{\beta-1}{2\beta}\right\}$.

Proof. In the proof assume that green is the good social status, or $a(g) \geq a(r)$. Let $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = \psi)$ be the gain given $f'_m = \psi \in \{r, g\}$, then clearly:

$$\Gamma(\sigma, \tilde{\sigma}, \phi, \mu) \geq \min\left\{\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = g), \Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = r)\right\}$$

and when f'_m is the element associated with σ's transition rule then σ is not a best response if the right hand side is weakly negative for some $\tilde{\sigma}$. The distribution I will analyze will have $\mu(\sigma) = 1 - \rho$ and $\mu(\sigma') = \rho$, σ' will be the invading strategy and $\sigma' \neq \tilde{\sigma}$ is possible.

If σ is a selfish strategy, let $\tilde{\sigma} = \sigma'$ and $\tilde{\sigma}$ be either the team, weak team, or insider strategy. Then without loss of generality I can let f'_m be the flag associated with $\tilde{\sigma}$ since the selfish strategy is independent of social status. Now clearly $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = r) = 0$ since both strategies call for the same action if the social status is r. Then $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = g) = 1 - \beta\rho$ and the selfish strategy is a worse response than that cooperative strategy when $\rho \geq \rho^*_g = \frac{1}{\beta}$.

From this case one can develop the key insights for the rest of the proof. First notice that if $a'(f') = a(f')$ then this does not impact the sign of $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu)$. Thus from now on I look at σ' where for $\psi \in \{g, r\}$ $a'(\psi) = 1 - a(\psi)$. One wants the invading strategy to reward $\tilde{\sigma}$ and punish σ_g thus $a'(f') = 1 - a(f')$ for every f', notice this implies that the invading strategy uses the same transition rule as σ.

Given these insights if σ is the team, weak team, or insider strategy then $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = g) = -1 + \beta(1-\rho) - \beta\rho$ and this is negative when $\rho \geq \rho^{**}_g = \frac{\beta-1}{2\beta}$. If $a'(g) = a'(r) = 0$ then $\tilde{\sigma}$ is a selfish strategy, and letting $P(f'_m = g)$ be the probability that f'_m is g then $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu) = P(f'_m = g) * \Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = g)$. This is non-negative when $\rho \geq \rho_g^{**}$ and a selfish strategy is a best response.

I can use the same method when $f'_m = r$, in this case $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = r) = 1 + \beta(1-\rho)V - \beta\rho$. In this expression $V = a(\tau(0,r)) - a(\tau(1,r))$, or it is the reaction of σ to players taking the wrong action at social status r. Again $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = r)$ is negative if $\rho \geq \rho_r^*(V) = \frac{\beta V+1}{\beta(V+1)}$. Consider the strategy with $a'(g) = a'(r) = 1$ (a generous strategy). If $P(f'_m = r)$ is the probability that f'_m is r then $\Gamma(\sigma, \tilde{\sigma}, \phi, \mu) = P(f'_m = r) * \Gamma(\sigma, \tilde{\sigma}, \phi, \mu | f'_m = r)$ which is non-negative if $\rho \geq \rho_r^*$ and thus a generous strategy is a best response. If all players choose a generous strategy a selfish strategy is the unique best response and the system is in the basin of attraction of the selfish strategies.

For the team strategy $V = 1$ and $\frac{\beta+1}{2\beta} > \frac{\beta-1}{2\beta}$, for the weak team or insider strategy $V = 0$ and the radius is $\min\left\{\frac{\beta-1}{2\beta}, \frac{1}{\beta}\right\}$. ■

In total there are six emergent seeds, the difference being which cooperative strategy is in the core. Since all seven limit sets are in a core the game can be solved by core-dominance.

Proposition 6 *If $\beta \leq 3$ selfish strategies are stochastically stable. If $\beta \geq 3$ then the team strategies are stochastically stable. In the latter case players are equally likely to use the green team and the red team strategy. If $\beta = 3$ then the weak team and insider strategies are also stochastically stable.*

5 Limitations in Application.

While it is hoped that analysts will find this methodology useful I should be clear that it may not always be the best strategy. The simplest reason is that calculating cost functions itself can be costly, and calculating optimized cost functions is also costly. Indeed if you have a problem that reaches the theoretic maximum number of levels before you find the emergent seed then this methodology is not better than others in the literature (see Chu and Liu [3]; Edmonds [4], and Tarjan [16]). Even then it is theoretically possible that given the emergent seed one has to unravel the entire structure to find the core attraction rate of the stochastically stable limit set(s). In contrast methodologies like half dominance only rely on local analysis, indeed you only need to calculate the radius of one limit set. Thus it is not universally an optimal methodology.

For example, Kandori and Rob [13] analyzes super-modular games, and use tree surgery to show that when one limit set has a sufficiently high radius it will be stochastically stable. Since the payoffs and thus the cost function are not generally specified in that game it is hard to use techniques like the emergent seed, and indeed those results can not be improved by using any of the general methods found in the literature.

Considering this counter-example also suggests general classes of problems where the emergent seed may be useful. Notice that in all of the applications above the space of payoffs was very simple. In general I conjecture that in this class of games one may find the emergent seed has a simple form and may simplify analysis. Consider, for example, the class of Network games as analyzed in Jackson and Watts [8]. In a network game the payoffs usually have a very simple functional form and the natural payoff from not interacting is zero. Due to this simplicity it may be simple to find general results in such games. Another characteristic that suggests the emergent seed will be useful is if for many limit sets the same limit set(s) determine their radii. This characteristic is what made it easy to find the emergent seed in the contract and gift giving games above.

A general methodology clearly seems to be to first check for half-dominance. In the process of checking for this you will naturally find the radii of the limit sets, and then finding the base of the emergent seed is essentially costless. This may be sufficient for you to characterize the stochastically stable limit set, if it is not then predecessor dominance may rule out many limit sets. At this point one may need to reevaluate whether to continue or not, if the number of limit sets with regards to the first difference cost function is large you may find it better to try sufficient methodologies like the radius/(modified) coradius. On an analytical level this is a representation theorem for stochastic evolution, and while I have found several examples where it greatly facilitates analysis one may find that for your problem it does not.

6 Conclusion

This paper finds a fundamental underlying structure in stochastic evolution—the emergent seed. When the stochastic potential of a limit set is written relative to this structure, the potential is seen to be an additive combination of the radius and the core attraction rate. Thus being evolutionarily successful can be seen to be determined by a combination of a local characteristic and a linear characteristic.

A secondary benefit of this methodology is that it can find clear and simple results where other methods have failed or provided partial results. While this is clearly not true in every problem it is significant that I can extend the analysis in several of the more complicated applications in the literature.

A final benefit of this analysis is that it helps clarify the general process of evolution. Instead of trying to find the stochastically stable limit set consider the steady state distribution over limit sets when the probability of mutations is small. This paper shows that the most likely successor to every limit set in this distribution is the limit set that determines its radius. If this does not occur then the next most likely transition is to a limit set that determines a first difference radius, and so on until finally we reach the core of the emergent seed. It may be that no limit set in the core is the most likely limit set (or stochastically stable), but evolution will pass through these states frequently.

Stochastic evolution is a structural and viable alternative to equilibrium analysis. While it has not been studied for long, this methodology shows great promise. By replacing the assumption of equilibrium with long run analysis of "limitedly rational" decision making it provides a methodology based only on the axiom of rationality. It is hoped that the emergent seed methodology increases the analytic clarity of this analysis and provides a new window of opportunity for applications.

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