ECON 439
Midterm: Normal Form Games

Kevin Hasker

This exam will start at about 20:40 and will end around 22:20

Points will only be given for work shown.

1. *(6 points)* **Honor Statement:** Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I will also not offer assistance to others. Finally I will not use a calculator or other electronic aid for calculation during this test.

   Name and Surname: _
   Student ID: _
   Signature:

   _

   Remark 1 *In retrospect, I think this question is the most important on the exam. It is quite common to observe systems that are not in equilibrium, less common to observe systems where common knowledge of rationality is rejected, and even less common to observe rejections of rationality. (Though all of the above happen.)*

   Knowing what an observation means is critical to understanding what we are doing.

2. *(12 points)* Assume we observe a large group of homogeneous people playing a game with a unique Nash equilibrium.

   (a) *(6 points)* If we observe they are not playing the equilibrium, does this mean they are not rational? What does it imply?

   Solution 2 *It is hard to express how disappointed I am at your answers. It is obvious you paid no attention to our analysis of dominated strategies and iterated deletion of dominated strategies.*

   A Nash equilibrium has two requirements. First players have to be rational, second they need to have correct expectations.

   Even though the game has a unique Nash equilibrium, this does not mean that rational agents have to play it. If they do not have correct expectations they could be rational (best respond) but be surprised by the outcome.

   Of course if the game does have a unique Nash equilibrium you would expect play to converge over time, but even if this is common knowledge it does not mean that people should play their equilibrium strategy. Indeed in such an environment you can often do better by best responding to what you believe the current distribution of play is.

Say, for example, that you are playing rock/paper/scissors. In this case if you are convinced your opponent will play rock you should play paper. The equilibrium is to play each strategy with equal likelihood, but essentially none of actual use this rule because in any given round we believe the other player is not equally likely to use each strategy. So I guess none of us are rational.

(b) *(3 points)* What would we need to observe in order to believe they do not share a common knowledge of rationality?

Solution 3 *This is the hardest question of the three, common knowledge of rationality will result in them playing only* rationalizable *strategies. I.e. the strategies left after iterated removal of (mixed) strategies that are never weak best responses. I would, though, also accept the* undominated *strategies. I.e. those that survive iterated removal of dominated strategies.*

(c) *(3 points)* What would we need to observe in order to believe they are not rational?

Solution 4 *That they play a strategy that is never a weak best response*[1]*, but of course like before I will also accept a dominated strategy.*

3. *(28 points total)* About Dominance and Iterated removal of Dominated Strategies.

(a) *(4 points)* What does it mean for strategy a to dominate strategy b?

Definition 5 *It means that for all strategies of the other players,* s_{-i}, $u_i(a, s_{-i}) > u_i(b, s_{-i})$.

(b) *(2 points)* What does it mean for strategy a to weakly dominate strategy b?

Definition 6 *It means that for all strategies of the other players,* s_{-i}, $u_i(a, s_{-i}) \geq u_i(b, s_{-i})$ *and for at least one* $\hat{s}_{-i}$ $u_i(a, \hat{s}_{-i}) > u_i(b, \hat{s}_{-i})$.

Consider the following game:

	α	β	$\delta(e, \alpha/\chi)$	χ
A	$11; 5^1$	$6; 10^{12}$	$5; 2$	$6; 5^1$
$B(f, A)$	$9; 7$	$5; 3$	$10; 5^1$	$4; 10^2$
$C(d, B)$	$4; 8^2$	$2; 4$	$5; -4$	$1; 5$
D	$11; 6^{12}$	$5; 5$	$5; 0$	$5; 1$

[1] Please note that this means "never a weak best response to any **mixed** strategy."

	α	$\beta\,(e,\delta)$	δ	χ
A	$11;10^{12}$	$3;5$	$11;7^{1}$	$9;3^{1}$
$B\,(d,C)$	$5;6$	$7;-1$	$4;5$	$1;7^{2}$
$C\,(f,A)$	$7;2$	$10;3^{1}$	$6;10^{2}$	$4;8$
D	$7;3$	$3;0$	$7;4$	$9;7^{12}$

	α	β	$\delta\,(e,\alpha/\beta)$	χ
$A\,(d,B)$	$3;7$	$2;9$	$8;-4$	$4;13^{2}$
$B\,(f,D)$	$9;9^{2}$	$8;6$	$9;1^{1}$	$5;3$
C	$8;3$	$8;1$	$1;0$	$7;4^{12}$
D	$11;8^{1}$	$11;9^{12}$	$1;2$	$7;1^{1}$

	α	$\beta\,(f,\alpha)$	$\delta\,(e,\beta/\chi)$	χ
$A\,(f,D)$	$5;2$	$3;9^{2}$	$9;4^{1}$	$10;7$
$B\,(d,A)$	$3;3$	$1;7$	$5;-6$	$6;9^{2}$
C	$5;4$	$5;1$	$4;0$	$13;5^{12}$
D	$6;9^{12}$	$6;5^{1}$	$4;3$	$13;4^{1}$

(c) *(10 points)* Find all pure strategy best responses. You may mark them on the table above but you will loose four points if you do not explain your notation below.

Solution 7 *Like always I mark a 1 (2) in the upper right hand corner if it is a best response for player 1 (2). Notice that always against one strategy their are two best responses. Finding the second best response is worth two points.*

(d) *(4 points)* Find a strategy for player 1 (the player choosing the row) that is dominated, carefully explain below what it is dominated by and why it is dominated.

Solution 8 *On the games above I write (d,X) by a strategy that is dominated by the strategy X. Notice that their is only one strategy for player 1 that is not a best response, and thus a candidate to be dominated. Let me give my detailed explanation for:*

	α	β	δ	χ
A	$11;10^{12}$	$3;5$	$11;7^{1}$	$9;3^{1}$
$B\,(d,C)$	$5;6$	$7;-1$	$4;5$	$1;7^{2}$
C	$7;2$	$10;3^{1}$	$6;10^{2}$	$4;8$
D	$7;3$	$3;0$	$7;4$	$9;7^{12}$

$u_1(C,\alpha) = 7 > 5 = u_1(B,\alpha)$, $u_1(C,\beta) = 10 > 7 = u_1(B,\beta)$, $u_1(C,\delta) = 6 > 4 = u_1(B,\delta)$, *and finally* $u_1(C,\chi) = 4 > 1 = u_1(B,\chi)$.

Remark 9 *IT'S A TRAP! Oh yes, we can all see that in each game there is a* weakly *dominated strategy. Hear A weakly dominates D. I would be disappointed in myself if I didn't first make you* define *a* *weakly dominated strategy, and then* ask you to remove dominated—not weakly dominated—strategies. *Oh, and then there's question g, that fairly clearly indicates you should* ***not*** *remove weakly dominated strategies.*

If you got caught in this trap... well please cancel your application to the rebel alliance.

(e) *(2 points)* Remove all dominated strategies, for each one indicate what dominates it.

Solution 10 *Since their is only one strategy that is not a best response for player 1, there can be no more to remove for player 1. For player 2 I mark the strategy that is dominated with* (e, X) *where* X *is what dominates it. In this case (depending on the game) there are sometimes two correct options, I denote this as* $(e, X/Y)$ *which means that either strategy* X *or strategy* Y *dominates the given strategy.*

Remark 11 *A lot of you assumed* common knowledge *of rationality at this stage, or iterated the process. This left you with a problem for the next question. As long as you didn't remove weakly dominated strategies I just counted the points below.*

(f) *(2 points)* After removing the dominated strategies, are their any strategies left that can be removed? Iterate this process until you have found the set of undominated strategies.

Solution 12 *Let me copy and past the games with the correct rows and columns deleted.*

	α	β	χ
A	$11;5^1$	$6;10^{12}$	$6;5^1$
$B\,(f,A)$	$9;7$	$5;3$	$4;10^2$
D	$11;6^{12}$	$5;5$	$5;1$

	α	δ	χ
A	$11;10^{12}$	$11;7^1$	$9;3^1$
$C\,(f,A)$	$7;2$	$6;10^2$	$4;8$
D	$7;3$	$7;4$	$9;7^{12}$

	α	β	χ
$B\,(f,D)$	$9;9^2$	$8;6$	$5;3$
C	$8;3$	$8;1$	$7;4^{12}$
D	$11;8^1$	$11;9^{12}$	$7;1^1$

	α	$\beta\,(f,\alpha)$	χ
$A\,(f,D)$	$5;2$	$3;9^2$	$10;7$
C	$5;4$	$5;1$	$13;5^{12}$
D	$6;9^{12}$	$6;5^1$	$13;4^1$

I mark the strategies that can be deleted now with (f, X) like before. I notice that for a couple of games you could actually remove two strategies using this process, since finding the second was relatively easy I decided I should not give more points for that.

(g) *(4 points)* What problem does this game illustrate about the removal of weakly dominated strategies?

Solution 13 *What? There is no problem. Who cares if we remove a Nash equilibrium? Who cares if part of the definition of a Nash equilibrium is rationality? Obviously that is just a silly stupid criterion that someone thought up. Remove weakly dominated strategies to your heart's content!*

Sorry, couldn't stop myself. The fact that in each of these games their is a Nash equilibrium that is in weakly dominated strategies clearly indicates that removing weakly dominated strategies can not be an implication of rationality. As I said, rationality is one of the two assumptions that are needed for Nash equilibrium. If you advocate removing some of them because they are "not rational," well, your thought process is messed up.

And if you basically said there was no problem and in the last question asserted that not playing equilibrium was a sign people were not rational... well... I am glad you won't be using what I teach you in the real world.

A	c	B	d	$\bar{t}_\alpha$	$\bar{t}_\omega$
64	1	50	2	8	5
108	3	100	1	6	10
50	2	64	1	5	8
98	2	75	3	7	5

4. *(20 points total)* Two countries—α and ω—are fighting to control a port. Their strategy is how long they are going to fight, for country α this is $t_\alpha \in \{0, 1, 2, ...\}$ and for country ω this is $t_\omega \in \{0, 1, 2, ...\}$. The victory condition is that if $t_\alpha > t_\omega$ then country α wins the port, if $t_\omega > t_\alpha$ then country ω wins the port. Notice that if $t_\alpha = t_\omega$ then no one wins. Their payoff functions are:

$$u_\alpha(t_\alpha, t_\omega) = \begin{cases} A - c(t_\omega)^2 & \text{if } t_\alpha > t_\omega \\ -c(t_\alpha)^2 & \text{if } t_\alpha \leq t_\omega \end{cases}$$

$$u_\omega(t_\alpha, t_\omega) = \begin{cases} B - d(t_\alpha)^2 & \text{if } t_\alpha < t_\omega \\ -c(t_\omega)^2 & \text{if } t_\alpha \geq t_\omega \end{cases}$$

(a) *(2 points)* For $i \in \{\alpha, \omega\}$ define $\bar{t}_i$ as the maximal number of periods country i can fight before winning is not worth it. Find this for both countries.

$$\begin{aligned} A - c\left(t_\omega\right)^2 &= 0 \\ t_\omega &= \sqrt{\frac{A}{c}} = \bar{t}_\alpha \end{aligned}$$

$$\begin{aligned} B - d\left(t_\alpha\right)^2 &= 0 \\ t_\alpha &= \sqrt{\frac{B}{d}} = \bar{t}_\omega \end{aligned}$$

Remark 14 *Some of you got worried about whether this should be $= 0$ or < 0, it doesn't actually matter but your best responses and equilibria will depend on your choice there. For future reference, I will not get that technical about your answers.*

(b) *(6 points)* Find the best response of both countries, be sure to recognize that often their is more than one best response.

Solution 15 *Using my definition of $\bar{t}_\alpha$ and $\bar{t}_\omega$:*

$$t_\alpha = BR_\alpha\left(t_\omega\right) = \begin{cases} \{t_\omega + 1, t_\omega + 2, t_\omega + 3...\} & \text{if } \; t_\omega < \bar{t}_\alpha \\ \{0\} \cup \{t_\omega + 1, t_\omega + 2, t_\omega + 3...\} & \text{if } \; t_\omega = \bar{t}_\alpha \\ \{0\} & \text{if } \; t_\omega > \bar{t}_\alpha \end{cases}$$

$$t_\omega = BR_\omega\left(t_\alpha\right) = \begin{cases} \{t_\alpha + 1, t_\alpha + 2, t_\alpha + 3...\} & \text{if } \; t_\alpha < \bar{t}_\omega \\ \{0\} \cup \{t_\alpha + 1, t_\alpha + 2, t_\alpha + 3...\} & \text{if } \; t_\alpha = \bar{t}_\omega \\ \{0\} & \text{if } \; t_\alpha > \bar{t}_\omega \end{cases}$$

(c) *(4 points)* Find the set of Nash equilibria.

Solution 16 *By intersecting these best responses we see that they are:*

$$\begin{aligned} t_\alpha &= 0,\ t_\omega \geq \bar{t}_\alpha \\ t_\omega &= 0,\ t_\alpha \geq \bar{t}_\omega \end{aligned}$$

(d) *(4 points)* Now assume that for $i \in \{\alpha, \omega\}$, $t_i \leq \bar{t}_i$. What is the new set of Nash equilibria?

Solution 17 *Assume that $\bar{t}_\alpha > \bar{t}_\omega$ then this means that we can never have $t_\omega \geq \bar{t}_\alpha$ and in equilibrium $t_\omega = 0$. I.e. the weaker side never wins.*

(e) *(4 points)* Based on recent world history, what characteristic of every Nash equilibrium does not seem to be satisfied? How could we change our model to explain this?

Solution 18 *These points will be given if you recognize the problem is that* $\min(t_\alpha, t_\omega) = 0$ *and discuss a minimal amount about this.*

This would mean, in short, that we would never have wars. Wouldn't that be great.

I believe the reason we do see wars is because while fighting each party updates their beliefs about the parameters of the model. Let me use the current invasion of Ukraine as an example. When Russia first invaded everyone believed that they would conquer Ukraine within a week. When that did not occur Russia still believed that they would win quickly, and to this day I suspect they believe that "the west will loose interest and then we will finish the invasion quickly." On the other hand Ukrainians believe that Russia is wrong, that they will in the end win (and probably even if the west did not help them) so they see no reason to stop fighting.

In other words I assert that the key difference between the model and the real world is that the payoff functions are uncertain in reality. In order to learn about each other's payoff functions they engage in some "preliminary skirmishes" that we call war.

But of course you can hold other opinions, and I'll give you points here as long as you don't seem too silly.

5. *(21 points)* Consider the following normal form game.

	α	β	δ
A	$7; 11^{12}$	$4; 6$	$6; 5$
B	$5; -2$	$5; 6^{1}$	$1; 10^{2}$
C	$2; 0$	$2; 8^{2}$	$7; 6^{1}$

	α	β	δ
A	$7; 5^{12}$	$7; 2$	$7; 3$
B	$3; -4$	$8; 2^{1}$	$6; 6^{2}$
C	$4; 0$	$4; 6^{2}$	$8; 2^{1}$

	α	β	δ
A	$4; 4^{12}$	$5; 3$	$7; 1$
B	$1; 0$	$6; 3^{1}$	$5; 12^{2}$
C	$3; -1$	$3; 6^{2}$	$8; 3^{1}$

	α	β	δ
A	$8; 10^{12}$	$9; 3$	$4; 7$
B	$7; -1$	$10; 3^{1}$	$2; 4^{2}$
C	$1; 0$	$1; 4^{2}$	$5; 3^{1}$

(a) *(6 points)* Find all the best responses, you may mark them on the table but you will loose two points if you do not explain your notation below.

Solution 19 *They are the same for every variation, I have marked the best responses for player 1 (2) with a 1 (2) in the upper right hand corner of the appropriate square.*

(b) *(3 points)* Find all the pure strategy Nash equilibria. For one of them explain why it is a Nash equilibrium.

Solution 20 *Yes yes, I knew there was only one. A bit of misdirection if you will. In every variant* (A, α) *is the unique pure strategy Nash equilibrium. It is a Nash equilibrium because if player 1 believes player 2 will play* α *then the only sensible thing to do is play* A*. Likewise if 2 believes 1 will play* A *then* α *is the best response.*

(c) *(2 points)* Write down the cycle in best responses.

Solution 21 *It is:* $(B, \beta) \to (B, \delta) \to (C, \delta) \to (C, \beta) \to (B, \beta)$

(d) *(6 points)* Find a candidate for a mixed strategy Nash equilibrium where only strategies in the cycle have a positive probability.

Solution 22 *I am going to solve this for every game:*

	α	β	δ
A	$7; 11^{12}$	$4; 6$	$6; 5$
B	$5; -2$	$5; 6^{1}$	$1; 10^{2}$
C	$2; 0$	$2; 8^{2}$	$7; 6^{1}$

$$\begin{aligned}
U_1(B, q) &= 5q + (1-q)1 = 2q + 7(1-q) = U_1(C, q) \\
q &= \frac{2}{3} \\
U_2(p, \beta) &= 6p + 8(1-p) = 10p + 6(1-p) = U_2(p, \delta) \\
p &= \frac{1}{3}
\end{aligned}$$

	α	β	δ
A	$7; 5^{12}$	$7; 2$	$7; 3$
B	$3; -4$	$8; 2^{1}$	$6; 6^{2}$
C	$4; 0$	$4; 6^{2}$	$8; 2^{1}$

$$\begin{aligned}
U_1(B, q) &= 8q + 6(1-q) = 4q + 8(1-q) = U_1(C, q) \\
q &= \frac{1}{3} \\
U_2(p, \beta) &= 2p + 6(1-p) = 6p + 2(1-p) = U_2(p, \delta) \\
p &= \frac{1}{2}
\end{aligned}$$

	α	β	δ
A	$4;4^{12}$	$5;3$	$7;1$
B	$1;0$	$6;3^{1}$	$5;12^{2}$
C	$3;-1$	$3;6^{2}$	$8;3^{1}$

$$
\begin{aligned}
U_1(B,q) &= 6q+(1-q)5 = 3q+8(1-q) = U_1(C,q) \\
q &= \frac{1}{2} \\
U_2(p,\beta) &= 3p+6(1-p) = 12p+3(1-p) = U_2(p,\delta) \\
p &= \frac{1}{4}
\end{aligned}
$$

	α	β	δ
A	$8;10^{12}$	$9;3$	$4;7$
B	$7;-1$	$10;3^{1}$	$2;4^{2}$
C	$1;0$	$1;4^{2}$	$5;3^{1}$

$$
\begin{aligned}
U_1(B,q) &= 10q+2(1-q) = q+5(1-q) = U_1(C,q) \\
q &= \frac{1}{4} \\
U_2(p,\beta) &= 3p+4(1-p) = 4p+3(1-p) = U_2(p,\delta) \\
p &= \frac{1}{2}
\end{aligned}
$$

(e) *(4 points)* Is this a Nash equilibrium? Why or why not?

Solution 23 *In all cases the answer is no, because we have not made sure that the strategies not in the cycle (A and α) are not best responses at the distribution we have found. Looking at the game:*

	α	β	δ
A	$7;11^{12}$	$4;6$	$6;5$
B	$5;-2$	$5;6^{1}$	$1;10^{2}$
C	$2;0$	$2;8^{2}$	$7;6^{1}$

it seems obvious that α is not a best response, so we want to compare A to (either) B or C when $q=\frac{2}{3}$

$$
\begin{aligned}
U_1(A,q) &= 4q+(1-q)6 = 4\left(\frac{2}{3}\right)+\left(1-\frac{2}{3}\right)6 = \frac{14}{3} \\
U_1(B,q) &= 5q+(1-q)1 = 5\left(\frac{2}{3}\right)+\left(1-\frac{2}{3}\right)1 = \frac{11}{3} \\
q &= \frac{2}{3}
\end{aligned}
$$

Let me repeat these calculations for each game:

	α	β	δ
A	$7;5^{12}$	$7;2$	$7;3$
B	$3;-4$	$8;2^{1}$	$6;6^{2}$
C	$4;0$	$4;6^{2}$	$8;2^{1}$

$$\begin{aligned} U_1(A,q) &= 7q+7(1-q)=7 \\ U_1(B,q) &= 8q+6(1-q)=8\left(\frac{1}{3}\right)+6\left(1-\frac{1}{3}\right)=\frac{20}{3}<7 \\ q &= \frac{1}{3} \end{aligned}$$

	α	β	δ
A	$4;4^{12}$	$5;3$	$7;1$
B	$1;0$	$6;3^{1}$	$5;12^{2}$
C	$3;-1$	$3;6^{2}$	$8;3^{1}$

$$\begin{aligned} U_1(A,q) &= 5q+7(1-q)=5\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)7=6 \\ U_1(B,q) &= 6q+(1-q)5=6\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)5=\frac{11}{2} \\ q &= \frac{1}{2} \end{aligned}$$

:

	α	β	δ
A	$8;10^{12}$	$9;3$	$4;7$
B	$7;-1$	$10;3^{1}$	$2;4^{2}$
C	$1;0$	$1;4^{2}$	$5;3^{1}$

$$\begin{aligned} U_1(A,q) &= 9q+4(1-q)=9\left(\frac{1}{4}\right)+4\left(1-\frac{1}{4}\right)=\frac{21}{4}=5.25 \\ U_1(B,q) &= 10q+2(1-q)=10\left(\frac{1}{4}\right)+2\left(1-\frac{1}{4}\right)=4 \\ q &= \frac{1}{4} \end{aligned}$$

6. *(9 points total)* Let σ^* be a Nash equilibrium:

   (a) *(6 points)* Assume that $\sigma_i^*(s_i)>0$ and $\sigma_i^*(\hat{s}_i)>0$ for $s_i \in S_i$ and $\hat{s}_i \in S_i \backslash s_i$, if $\sigma_{-i}^* = \sigma^* \backslash \sigma_i$ prove that $u_i\left(s_i,\sigma_{-i}^*\right)=u_i\left(\hat{s}_i,\sigma_{-i}^*\right)$.

Proof. Assume by contradiction $u_i\left(s_i, \sigma_{-i}^*\right) > u_i\left(\hat{s}_i, \sigma_{-i}^*\right)$, then $\sigma_i\left(s_i\right) = \sigma_i^*\left(s_i\right) + \sigma_i^*\left(\hat{s}_i\right)$, $\sigma_i\left(\hat{s}_i\right) = 0$ will give a strictly higher payoff, and σ^* can not be a Nash equilibrium.

An alternative proof is derived by considering the objective function of this person:

$$\max_{\tau}\left(\sum_{\tilde{s}_i \in S_i \backslash \{s_i, \hat{s}_i\}} \sigma_i^*\left(\tilde{s}_i\right) u_i\left(\tilde{s}_i, \sigma_{-i}^*\right) + \left[\sigma_i^*\left(s_i\right) + \sigma_i^*\left(\hat{s}_i\right)\right]\left(\tau u_i\left(s_i, \sigma_{-i}^*\right) + (1-\tau) u_i\left(\hat{s}_i, \sigma_{-i}^*\right)\right)\right)$$

now we are told their is an interior solution for $\tau = \sigma_i^*\left(s_i\right) / \left(\sigma_i^*\left(s_i\right) + \sigma_i^*\left(\hat{s}_i\right)\right)$ thus the first order condition must be binding with regards to τ or:

$$u_i\left(s_i, \sigma_{-i}^*\right) - u_i\left(\hat{s}_i, \sigma_{-i}^*\right) = 0$$

■

(b) *(3 points)* Why is this result strange? What does it imply about σ_i^*?

Solution 24 *The weird part about this result is that it implies σ_i^* is chosen to make* other *players indifferent between all strategies they are supposed to play with strictly positive probability.*

Another way of seeing this is noting that if $u_i\left(s_i, \sigma_{-i}^*\right) - u_i\left(\hat{s}_i, \sigma_{-i}^*\right) = 0$ then in the formulation above every $\tau \in [0,1]$ gives the same expected utility—or their is a range of best responses—the fact that their is an optimal τ must be determined by other player's incentives.