ECON 439

Quiz 3

Dr. Kevin Hasker

1. *(2 Points)* Please read and sign the following statement:

   I promise that my answers to this test are based on my own work without reference to any notes, books, or the assistance of any other person during the test. I will also not assist others nor use any electronic device during this test.

   Name and Surname: _

   Student ID: _

   Signature: _

α	β	μ	χ	$BR_a(q_b)$	$BR_b(p_a)$	p_a^*	q_b^*	π_a^*	π_b^*
8	3	$\frac{2}{3}$	2	$5 - \frac{1}{5}q_b$	$p_a + 1$	4	5	$\frac{20}{3}$	$\frac{25}{3}$
10	6	$\frac{2}{3}$	2	$\frac{7}{2} - \frac{1}{10}q_b$	$2p_a - 1$	3	5	$\frac{10}{3}$	$\frac{25}{6}$
17	3	$\frac{2}{3}$	5	$11 - \frac{1}{5}q_b$	$p_a + 1$	9	10	$\frac{80}{3}$	$\frac{100}{3}$
16	6	$\frac{2}{3}$	2	$5 - \frac{1}{10}q_b$	$2p_a + 2$	4	10	$\frac{40}{3}$	$\frac{50}{3}$

2. *(18 points total)* There are two firms that supply a given market, the demand curve for each is:

$$q_i = \alpha - \beta p_i + \mu\beta p_j$$

for $i \in \{a, b\}$ and $j \neq i$. They have the cost function $c(q) = \chi q_i$. However firm a maximizes over price, while firm b maximizes over quantity. For our purposes it is convenient to have firm a use the demand curve:

$$q_a = \alpha + \alpha\mu - \beta p_a + \beta\mu^2 p_a - \mu q_b$$

while firm b uses the inverse demand curve:

$$p_b = \frac{1}{\beta}(\alpha - q_b + \beta\mu p_a)$$

(a) *(5 points total)* Analyzing firm a:

i. *(2 points)* Set up the objective function of firm a, who maximizes profits over price.

$$\pi_a = (p_a - \chi)\left(\alpha + \alpha\mu - \beta p_a + \beta\mu^2 p_a - \mu q_b\right)$$

ii. *(3 points)* Find the best response of firm a.

$$\left(\alpha + \alpha\mu - \beta p_a + \beta\mu^2 p_a - \mu q_b\right) + \left(-\beta + \beta\mu^2\right)(p_a - \chi) = 0$$

$$p_a = \frac{1}{2\beta - 2\beta\mu^2}\left(\alpha + \chi\left(\beta - \beta\mu^2\right) + \alpha\mu - \mu q_b\right)$$

(b) *(5 points total)* Analyzing firm b:

i. *(2 points)* Set up the objective function of firm b, who maximizes profits over quantity.

$$\pi_b = \frac{1}{\beta}\left(\alpha - q_b + \beta\mu p_a\right) q_b - \chi q_b$$

ii. *(3 points)* Find the best response of firm b.

$$\frac{1}{\beta}\left(\alpha - q_b + \beta\mu p_a\right) - \frac{1}{\beta} q_b - \chi = 0$$

$$q_b = \frac{1}{2}\alpha - \frac{1}{2}\beta\chi + \frac{1}{2}\beta\mu p_a$$

(c) *(8 points)* Find the Nash equilibrium and the profits of each firm.

$$\begin{aligned}
p_a &= \frac{1}{2\beta - 2\beta\mu^2}\left(\alpha + \chi\left(\beta - \beta\mu^2\right) + \alpha\mu - \mu\left(\frac{1}{2}\alpha - \frac{1}{2}\beta\chi + \frac{1}{2}\beta\mu p_a\right)\right) \\
p_a &= \frac{1}{4\beta - 4\beta\mu^2}\left(2\alpha + \alpha\mu + 2\beta\chi - 2\beta\mu^2\chi - \beta\mu^2 p_a + \beta\mu\chi\right)
\end{aligned}$$

$$p_a = \frac{1}{4\beta - 3\beta\mu^2}\left(2\alpha + \alpha\mu + 2\beta\chi - 2\beta\mu^2\chi + \beta\mu\chi\right)$$

$$\begin{aligned}
q_b &= \frac{1}{2}\alpha - \frac{1}{2}\beta\chi + \frac{1}{2}\beta\mu\left(\frac{1}{4\beta - 3\beta\mu^2}\left(2\alpha + \alpha\mu + 2\beta\chi - 2\beta\mu^2\chi + \beta\mu\chi\right)\right) \\
&= -\frac{1}{3\mu^2 - 4}\left(2\alpha + \alpha\mu - 2\beta\chi - \alpha\mu^2 + 2\beta\mu^2\chi - \beta\mu^3\chi + \beta\mu\chi\right)
\end{aligned}$$

$$\pi_a^* = -\frac{1}{\beta\left(3\mu^2 - 4\right)^2}\left(\mu + 2\right)^2\left(\mu^2 - 1\right)\left(\alpha - \beta\chi + \beta\mu\chi\right)^2$$

$$\pi_b^* = \frac{1}{\beta\left(3\mu^2 - 4\right)^2}\left(\alpha - \beta\chi + \beta\mu\chi\right)^2\left(-\mu^2 + \mu + 2\right)^2$$

These solutions are a mess, almost impossible to interpret, however in the course of solving for the equilibrium ($p_a = \rho$, $q_b = \tau$) I let:

$$\begin{aligned}
\alpha &= \frac{1}{-\mu^2 + \mu + 2}\left(2\tau + \tau\mu + 2\beta\rho - 2\tau\mu^2 - 2\beta\mu^2\rho + \beta\mu^3\rho - \beta\mu\rho\right) \\
\chi &= -\frac{1}{-\beta\mu^2 + \beta\mu + 2\beta}\left(2\tau + \tau\mu - 2\beta\rho + \beta\mu^2\rho - \beta\mu\rho\right)
\end{aligned}$$

and then

$$\begin{aligned}
\pi_a &= \frac{1}{\beta}\tau^2\left(-\frac{\mu - 1}{\left(\mu + 1\right)\left(\mu - 2\right)^2}\left(\mu + 2\right)^2\right) \\
\pi_b &= \frac{1}{\beta}\tau^2
\end{aligned}$$

and it is fairly easy to prove that

$$\begin{aligned}
\left(-\frac{\mu-1}{(\mu+1)(\mu-2)^2}(\mu+2)^2\right) &< 1 \\
(1-\mu)(\mu+2)^2 &< (\mu+1)(\mu-2)^2 \\
0 &< (\mu+1)(\mu-2)^2-(1-\mu)(\mu+2)^2 \\
0 &< 2\mu^3
\end{aligned}$$

thus we know that in the symmetric problem the firm choosing quantity makes a higher profit. What more would we have to know to find the Nash equilibrium (or equilibria) of the super game where firms choose whether to choose price or quantity? (In the symmetric game with linear demand and constant marginal costs.)