Handout on Agenda Control and Strategic Voting—Two Models of Agenda Voting

ECON 439
Kevin Hasker

1 The Primitives:

There are I people with strict preferences over a set of outcomes X. $|I|$ must be odd and greater than 1. We can describe their preferences as a table, with individuals across the top and the options ranked from best to worst below them. For example if $X = \{x, y, z\}$ and $|I| = 3$ then we could have:

1	2	3
x	z	y
y	x	z
z	y	x

(1)

this means that 3 thinks y is strictly better than z which is strictly better than x. 1 thinks x is better than y is better than z. Or we could have:

1	2	3
x	z	y
y	y	z
z	x	x

(2)

2 Two Agenda Games and why we assume Weakly Undominated Strategies.

In both of these games you order the options in X, for example it could be (z, x, y). In game 1 you just reject or accept each option in sequence. In game 2 you consider the first pair, select a winner, and then compare the winner to the next option.

Example 1 (Standard Model) *1. In round k for $|X| - 1 > k \geq 1$ all committee members vote to accept (A) or reject (R) the k'th option. If the majority votes A then the game ends, otherwise round $k + 1$.*

2. In the final round you vote over the two remaining options in the agenda. The winner is determined by the first accepted option.

Example 2 (Status Quo Model) *1. In round one you vote over the first two options in the agenda, the one that receives the majority of the votes becomes the* status quo.

2. In round k for $|X| - 1 \geq k > 1$ you vote for a new option against the status quo. The one that receives the majority of votes becomes the new status quo.

the winner is the status quo in the last period.

In both games no matter what the preferences there is a subgame perfect equilibrium where any $x \in X$ is accepted. For example if preferences are given by 2 the winner could be the outcome x, which is the worst outcome for two of the three people. The strategy that makes this an outcome is: everyone votes for x when ever it is an option, and votes to reject every other option. This is an equilibrium because of the *problem of zero*. Any one person's action won't affect the outcome so voting like this is a best response, thus it is an equilibrium. This is a "problem of zero" because the problem is that every action has the same impact, zero. Most refinements can be understood as tie breaking rules when this type of problem occurs. Here we adopt the tie breaking rule of using weakly undominated strategies. To make the definition simple, we call a voter *pivotal* if their vote will determine the outcome.

Assumption When indifferent, all voters will vote as if they are pivotal.

Notice that I was very harsh on the idea of removing weakly dominated strategies *as an implication of rationality* however here what we are doing is to (sensibly) restrict the set of *equilibria*. There is no contradiction here, equilibrium already is a stronger assumption than rationality, we are merely restricting our attention to a subset of equilibria.

To clarify, the problem with using weakly undominated strategies as an implication of rationality is that it required some secondary assumptions. In general we have to know what order weakly undominated strategies would be removed. Equilibrium is one example of such a secondary assumption (expectations will be correct). Strengthening this assumption to assume others will also use weakly undominated strategies is nothing more than strengthening it. All refinements require such strengthening of the equilibrium assumption, and like other refinements this one is perfectly justified by the problem of zero.

3 Strategic Voting in the Status Quo Model.

The point of this section is to show how strategic behavior can be important, and how it can improve someone's outcome.

Definition 3 *Someone uses* naive voting *if they always vote for their most preferred outcome of those in front of them.*

Consider the preferences 1 and the agenda (x, y, z). In the first round the vote is between (x, y) and so if players use naive voting x will win. Then in the next round the vote is between x and z, and here z wins so the outcome will be z. But is 1 acting optimally? No they are not. In the first round they voted for x, if they switched their vote to y then in the second round y would beat z and 1 would get a better outcome. Committee members who do not think about the final outcome when voting will not get what they want.

Consider the preferences 2 and the agenda (x, y, z). In the equilibrium y is the outcome. In the second round y beats z and z beats x. Then in the first round y beats x, but now let's look at the individual strategies in the first round. Player 1 votes for y even though she prefers x to y, Player 2 votes for x even though he prefers y to x. In other words both of them vote for exactly the wrong outcome. If everyone used naive voting then the outcome would be the same since y beats x, but in equilibrium and in a naive voting situation the votes of 1 and 2 are reversed.

4 Equilibrium Outcomes for Different Agendas

First of all a rather obvious result is that a Condorcet Winner always win, regardless of the agenda.

Definition 4 *x is a Condorcet winner if in every pairwise competition it wins the majority of votes.*

Proposition 5 *If x is a Condorcet winner then it is always the outcome in models 1 and 2.*

Proof. Consider the model 1 and that option x comes up in round k. Then whatever is the outcome after period k at least half the voters prefer x to that option, and x will be accepted. If $k > 1$ then in round $k-1$ the implicit choice is between accepting some other option and rejecting it and accepting x, x wins that competition so it wins that round. One can iterate this to find out that it always wins.

Consider the model 2, then in any pairwise competition in the last round x will win. Thus in round $|X| - 2$ voting for some of the options is equivalent to voting for x, and since x is a Condorcet winner it will win that competition. Iterating this results in the conclusion that x will win no matter what the agenda. ■

But what are the equilibria if there is no Condorcet winner? Then it depends on the agenda. It is necessary that something be in the *top cycle*. To define this write $x \rhd y$ if a majority of voters prefer x to y—this is "x beats y." Write $x \blacktriangleright y$ if there is a sequence of options $\{u_k\}_{k=1}^{K}$ such that $x \rhd u_1$, for $k < K$ $u_k \rhd u_{k+1}$, and $u_K \rhd y$—this is "x indirectly beats y."

Definition 6 *x is in the* top cycle *if for every $y \in X$ $x \blacktriangleright y$*

How do we find the top cycle? By finding the largest possible cycle in the "beats" relationship such that nothing outside of the cycle beats anything within it. Consider the following example:

1	2	3
x	y	w
y	w	z
w	z	x
z	x	y

(3)

We can see that $w \rhd z, x$; $x \rhd y$; $y \rhd w, z$; $z \rhd x$. So we can write the cycle $w \rhd z \rhd x \rhd y \rhd w$ and we have found a cycle with every option in it. So the entire set is the top cycle. Notice that w is always preferred to z so it is not even *Pareto efficient.*

Definition 7 *$x \in X$ is Pareto efficient if there is nothing which everyone prefers to x.*

This is a very weak criterion of what is "good." Consider another example:

1	2	3
x	z	y
y	x	z
v	v	w
w	w	v
z	y	x

$x \rhd y, v, w$; $y \rhd v, w, z$; $z \rhd v, w, x$; $v \rhd w$. Now obviously w can not be in the cycle since it beats nothing, neither can v since it only beats w. So our top cycle is $x \rhd y \rhd z \rhd x$.

Now for model 1 anything in the top cycle is the equilibrium outcome for some agenda. How do we see this? Well consider z with the preferences in 3. How can we support it? Well notice that $z \rhd x \rhd y \rhd w$, so y will always beat w, so let them be the last two options. Just before them? How about x? Since y has beaten w in the last comparison x will win this comparison, and then let z be first and z will beat x.

Proposition 8 *If x is in the top cycle then it is an equilibrium of some agenda in model 1.*

Proof. Since it is in the top cycle there is a $\{u_k\}_{k=1}^{|X|-1}$ so that $x \rhd u_1$ and for $|X| - 1 > k \geq 1$ $u_k \rhd u_{k+1}$, let the agenda be $\left(x, u_1, u_2, u_3, ..., u_{|X|-1}\right)$ then in the last period $u_{|X|-2} \rhd u_{|X|-1}$. Thus in the next to the last $u_{|X|-3}$ will be accepted because $u_{|X|-3} \rhd u_{|X|-2}$. Iterating back in the first period the vote will be between accepting x and accepting u_1 in the next period. Clearly x will be accepted and we are done.[1] ■

However for the alternative model, model 2, this result does not hold, or to be precise we have to also have x be Pareto Efficient.

Proposition 9 *If x is the equilibrium outcome for some agenda in model 2 then it must be Pareto efficient and in the Top Cycle.*

Proof. First assume that x is not Pareto efficient, to be specific assume that y Pareto dominates it. This implies that $y \rhd x$. But this also means that if $x \rhd z$, $y \rhd z$ and that if $w \rhd y$, $w \rhd x$ for arbitrary z and w in X. Now

[1] I will not extend this to the case where the top cycle has "branches." For example $x \blacktriangleright v, w$, $v \blacktriangleright y$ and $w \blacktriangleright y$, however the proof can be extended to handles such cases.

clearly x must win in the last round (against the final element of the agenda) to be selected, but that means that y must win. Going to the previous round something that leads to x must win, but then something that leads to y must win since if x beats the option in that round so must y. Iterating at some point we must be choosing between something that eventually leads to x and something that eventually leads to y, and y must win this vote.

Now assume that x is not in the top cycle. In order to be in the top cycle set all that is necessary is that $x \rhd y$ for some y in the top cycle set. Thus for every y in the top cycle set $y \rhd x$ thus in the first stage that some element of the top cycle set is considered x will be eliminated from contention. ■

Model 1 is a standard model of agenda formation, but model 2 is a reasonable alternative—and it has a smaller set of equilibria. Which one to analyze depends on the rules of the committee. If the agenda is structured like in model 1 that is the appropriate one to use. If it is like the agenda in model 2 then that one is the appropriate one to use. When analyzing a committee and deciding on which rule to use it is important to realize the difference between the models.

Don't be confused, with either model you get a large range of outcomes, it is just that with model 2 you can be sure that the outcome will be Pareto Efficient. While a popular characterization of welfare in Economics in general it is not that interesting.

I should mention that I have neglected to prove one direction that is probably true in both statements. I have not proven that in model 1 everything that is not in the top cycle is not an equilibrium. This is probably fairly easily done. I have also not proven that all Pareto efficient outcomes in the top cycle are equilibria outcomes for model 2. I expect this proof to be harder. I conjecture both proofs are true—-if you doubt me I would like to see your counter example.