An Analysis of The Bertrand Game with Asymmetric Constant Marginal Costs and Discrete Prices

By Kevin Hasker, based on an insight by a Student of ECON 439, Fall 2010-2011

When I presented the Bertrand Game with Asymmetric Marginal Costs I did not work through it carefully, I did it quickly just to provide the basic insight. A student of ECON 439 in Fall of 2010-2011 pointed out my error and so I looked at the problem more carefully. In this handout I carefully derive the best responses and find the full set of equilibria in this model.

It is best to analyze the Bertrand game in a discrete environment because otherwise the best response is not well defined everywhere. This is due to an open set problem, if the other firm is pricing above my marginal cost then I will make more profit by pricing at $p_{-i}-\varepsilon$ for small $\varepsilon>0$. But it is not a best response to stop at any $\varepsilon>0$ because $\frac{1}{2}\varepsilon$ will produce a higher profit. Thus the best response is not well defined, p_{-i} is not a best response and neither is $p_{-i}-\varepsilon$ for any $\varepsilon>0$. This is annoying, and a type of problem that should be removed by improving the model. Thus I present the model using discrete price intervals. When the two firms have different marginal costs one can easily show that the game has a large set of equilibria, all which have the characteristic that the firm with the lower marginal cost sells all output, but the price at which it sells the output can be in a large range. We can then use a questionable refinement to remove all but one of these equilibria, but refinements should always be questioned carefully, and this one is only somewhat trustworthy.

1 Model

In a Bertrand model I firms choose their price, and meet demand at that price. The only differences between this model and the standard one is that the firms have different constant marginal costs and the price must be in discrete increments. For simplicity let $I=2$.

Thus there are two firms who each choose a price $p_i\in t\kappa$ for $t\in\mathbb{N}$ and must meet all demand at that price. Their costs are $c_1(q_1,q_2)=c_1q_1$ and $c_2(q_1,q_2)=c_2q_2$ with $c_1<c_2$. The total demand at a price of p is $D(p)$, and we assume that $D'<0$ and $D''\leq 0$. Firm i's demand ($i\in\{1,2\}$, $-i=\{1,2\}\setminus i$) is:

$$d_i(p_i,p_{-i})=\begin{cases}0 & p_i>p_{-i}\\ \frac{1}{2}D(p_i) & p_i=p_{-i}\\ D(p_i) & p_i<p_{-i}\end{cases}$$

Let $p_2^m=\arg\max_p(p-c_2)D(p)$ and $p_1^m=\arg\max_p(p-c_1)D(p)$ (these are the monopoly prices for the two firms) and assume that $\max\{c_1,c_2\}<\min\{p_1^m,p_2^m\}$.

We assume that $\{c_1,c_2,p_1^m,p_2^m\}$ are all in κ units, and assume that κ is small enough that its size does not matter. In other words it is always worth cutting the price to capture all of the demand.

2 Best Responses:

Firm i's profits will be:

$$\pi_i(p_i, p_{-i}) = \begin{cases} 0 & p_i > p_{-i} \\ \frac{1}{2} D(p_i)(p_i - c_i) & p_i = p_{-i} \\ D(p_i)(p_i - c_i) & p_i < p_{-i} \end{cases}$$

and its best response will be:

$$BR_i(p_{-i}) = \begin{cases} p_i^m & p_{-i} > p_i^m \\ p_{-i} - \kappa & p_i^m \geq p_{-i} > c_i + \kappa \\ p_{-i} & p_{-i} = c_i + \kappa \\ x & p_{-i} = c_i,\ x \geq c_i \\ y & p_{-i} \leq c_i - \kappa,\ y \geq p_{-i} + \kappa \end{cases} .$$

We will show this case by case. First if $p_{-i} > p_i^m$ then clearly choosing p_i^m will give all the demand at the monopoly price to firm i, so that is optimal. Next consider $p_i^m \geq p_{-i} > c_i + \kappa$, then in this case the profits at $p_{-i} - \kappa$ will be:

$$D(p_{-i} - \kappa)(p_{-i} - \kappa - c_i) \geq \frac{1}{2} D(p_i)(p_i - c_i) > \frac{1}{2} D(c_i)(c_i - c_i) = 0$$

so $p_{-i} - \kappa$ is a better response than p_{-i}. Clearly since firm i is capturing all of the demand $p_{-i} - \kappa$ is a better response than $p_{-i} - t\kappa$, for $t > 1$, thus $p_{-i} - \kappa$ is the best response.[1] Now what about the case where $p_{-i} = c_i + \kappa$? In this case cutting price to c_i will produce zero profit. Raising it to $c_i + t\kappa$ for $t > 1$ will produce zero profit. But if $p_i = p_{-i} = c_i + \kappa$ then we get:

$$\pi_i(c_i + \kappa, c_i + \kappa) = \frac{1}{2} D(c_i + \kappa)(c_i + \kappa - c_i) = \frac{1}{2} D(c_i + \kappa)\kappa > 0$$

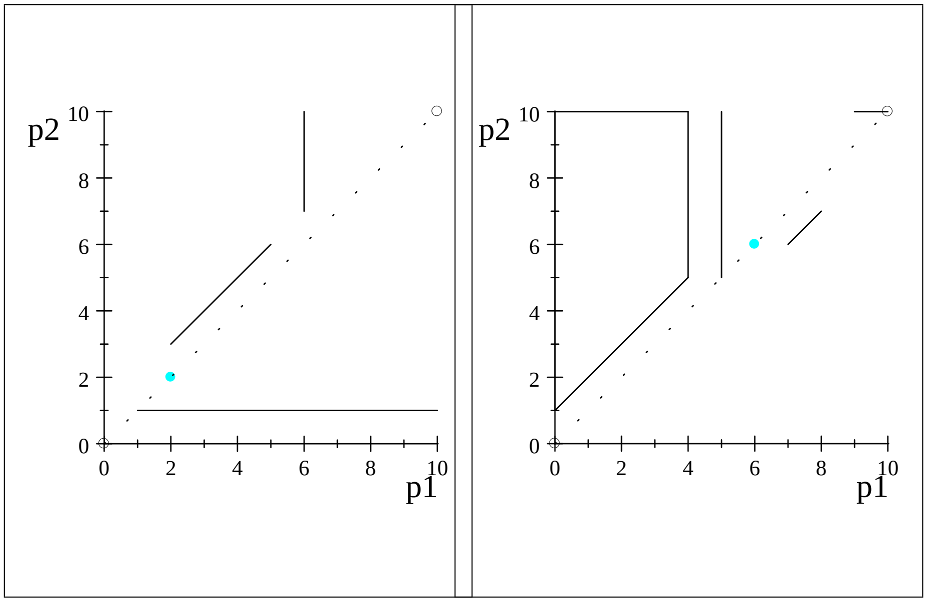
so it is optimal to produce at the same price. If $p_{-i} = c_i$ then any price weakly greater than p_{-i} produces zero profit, any price lower produces negative profit, so any price greater than marginal cost is a best response. Now if $p_{-i} \leq c_i - \kappa$ then pricing at a price equal to or lower than p_{-i} will produce negative profit, but any price that is strictly higher will produce zero profit, so the best response is as given.

3 Equilibria

Now let us draw the best responses to have a concrete case for our analysis. Let $c_1 = 1$ and $c_2 = 5$, $D(p) = 11 - p$, and finally let $\kappa = 1$. (Note that κ may be too large, this example is purely for illustration.) Given these facts $p_1^m = 6$ and

[1] Remember that we are below the monopoly price, so cutting price always decreases profits.

$p_2^m = 8$. Now we can graph the two best responses.



Best Response, Firm 1 Best Response, Firm 2

The faint dotted line is the 45 degree line, where $p_1 = p_2$. It may help as well to see the best responses written down:

$$BR_1(p_2) = \begin{cases} 6 & p_2 > 6 \\ p_2 - 1 & 6 \geq p_2 > 2 \\ 2 & p_2 = 2 \\ x & p_2 = 1,\, x \geq 1 \\ y & p_2 \leq 0,\, y \geq p_2 + 1 \end{cases}, BR_2(p_1) = \begin{cases} 8 & p_1 > 8 \\ p_1 - 1 & 8 \geq p_1 > 6 \\ 6 & p_1 = 6 \\ x & p_1 = 5,\, x \geq 5 \\ y & p_1 \leq 4,\, y \geq p_1 + 1 \end{cases}.$$

Using either method we can see that the intersection of the best response are at $p_1 \in \{2, 3, 4, 5\}$ $p_2 = p_1 + 1$. Generalizing this it seems that $p_1 \in \{c_1 + \kappa, c_1 + 2\kappa, c_1 + 3\kappa, ..., c_2\}$ and $p_2 = p_1 + \kappa$.

To prove that these are equilibria, first the highest profit that firm two can make by lowering their price in any of these equilibria is zero, when $p_1 = c_2$. Otherwise lowering their price will produce strictly negative profit. On the other hand, increasing their price will always produce zero profit. Thus $p_2 = p_1 + \kappa$ is *a* best response. There are others, but it is a best response to price in this manner. For firm one, they are already getting all of the demand and a strictly positive profit. If they lower their price they will make a strictly lower profit, if they raise their price they will make zero profit. Thus $p_1 = p_2 - \kappa$ is a best response in this region.

Furthermore the upper and lower bounds are strict. If $p_2 = c_1 + \kappa$ then the best response for firm one is $p_1 = p_2$ because pricing at $p_1 = c_1$ produces zero profit, but then $p_2 = p_1$ is not a best response for firm two. If $p_1 = c_2 + \kappa$ then $p_2 = p_1$ is a best response for firm two, but then firm one will want to cut their price.

So we have a large span of equilibria, with very different implications for how much profit firm one will make. This is disturbing, is there an acceptable way to get rid of some of them?

3.1 Refining away these equilibria:

A general problem with Nash Equilibria I call "the problem of zero." If an event has zero probability then crazy things can happen. This is no problem for the equilibrium because the event has a zero probability and thus what happens does not affect profit or utility. But it can be unsatisfying. An entire branch of the game theory literature is devoted to pointing out that some Nash equilibria are not "reasonable" because something crazy is happening at a zero probability event. Some of these refinements have been generally accepted, but they all should be carefully questioned. Perhaps Nash Equilibrium is too weak, but any refinement argument must be based on *more* than "I just don't like some of these equilibria."

3.1.1 Weak Dominance

One generally accepted refinement is *one application* of weak dominance. Iterating it is known to be a failed concept, but surely one application is not a problem? Well in other handouts I have shown that this in itself is not always straightforward in application, and that it can remove Pareto Efficient equilibria, but in games where it does not, why not use it? This is a refinement that people use with a slight sense of embarrassment, but they still do. And it does limit our set of equilibria quite nicely.

In this game one can show that it is weakly dominated to have $p_i \leq c_i$. This is easy to show, by pricing at c_i or less you can make at most zero profit, by pricing higher you can sometimes make a positive profit. This results in a unique equilibrium, $p_2 = c_2 + \kappa$, $p_1 = c_2$. You should verify that if $c_1 = c_2$ then the equilibrium is $p_1 = p_2 = c + \kappa$.

4 Comparing to the Continuous Strategy game.

You may wonder why I insist on analyzing this game in a discrete space. To understand this drive κ to zero in the equilibrium that survived our refinement. In that case $p_1 = p_2 = c_2$, but this is not an equilibrium because firm one wants to cut its price so that it can capture all of the demand. To solve this conundrum you have to mess with the fundamentals of the game, the way demand is shared when firms have the same price. Let $\alpha \in [0,1]$, then you have to change it to:

$$d_1(p_1,p_2) = \begin{cases} 0 & p_1 > p_2 \\ \alpha D(p_1) & p_1 = p_2 \\ D(p_1) & p_1 < p_2 \end{cases}, d_2(p_2,p_1) = \begin{cases} 0 & p_2 > p_1 \\ (1-\alpha) D(p_2) & p_2 = p_1 \\ D(p_2) & p_2 < p_1 \end{cases}$$

and you have to set $\alpha = 1$. In other words you have to change the sharing rule so that it coincides with the equilibria! This is an absolute mess, and really shows that the model has been misspecified. Furthermore notice that this equilibrium is in weakly dominated strategies, firm two prices at its marginal cost. So is the refinement still a good idea when it removes all of the equilibria? Most people would say no.

You should be able to verify that the full set of equilibria is $p_1 = p_2 \in [c_1, c_2]$.