A Handout on The Differentiated Bertrand and Cournot Models

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1 The Cournot Model:

In this model firms choose output, and price is determined to clear market. For comparison with the Bertrand Model below we will assume $Q = a - bP$, or $P = \frac{a}{b} - \frac{Q}{b}$. We will work with the standard costs of $c(q) = cq$.

$$\max_{q_2} \left(\frac{a}{b} - \frac{q_1 + q_2}{b} \right) q_2 - cq_2$$

$$\begin{aligned} -\frac{1}{b} q_2 + \left(\frac{a}{b} - \frac{q_1 + q_2}{b} \right) - c &= 0 \\ \frac{a}{b} - c - \frac{1}{b} q_1 - \frac{2}{b} q_2 &= 0 \\ \frac{1}{2}(a - bc) - \frac{1}{2} q_1 &= q_2 \end{aligned}$$

Notice that if $q_1 = 0$ this firm will sell the monopoly output at the monopoly price: $P_m = \frac{a}{b} - \frac{\frac{1}{2}(a-bc)}{b} = \frac{1}{2}\frac{a}{b} + \frac{1}{2}c$. In the Cournot Equilibrium they will produce $q_1 = q_2 = q$

$$\begin{aligned} \frac{1}{2}(a - bc) - \frac{1}{2} q &= q \\ \frac{1}{3}(a - bc) &= q_c \end{aligned}$$

$$\begin{aligned} P_c &= \frac{a}{b} - \frac{2\left(\frac{1}{3}(a - bc)\right)}{b} \\ P_c &= \frac{1}{3}\frac{a}{b} + \frac{2}{3}c \end{aligned}$$

$$\pi_c = (P_c - c) q_c = \left(\frac{1}{3}\frac{a}{b} + \frac{2}{3}c - c \right) \frac{1}{3}(a - bc) = \frac{1}{9b}(a - bc)^2$$

1.1 The Stackleberg Model

Now we have firm 2 choose their output after firm 1 does. By the normal arguments we still have:

$$\frac{1}{2}(a - bc) - \frac{1}{2} q_1 = q_2$$

but now firm 1 takes this into consideration when choosing their output

$$\max_{q_1} \left(\frac{a}{b} - \frac{q_1 + q_2(q_1)}{b}\right) q_1 - cq_1$$
$$\max_{q_1} \left(\frac{a}{b} - \frac{q_1 + \frac{1}{2}(a - bc) - \frac{1}{2}q_1}{b}\right) q_1 - cq_1$$
$$\max_{q_1} \frac{1}{2b} q_1 (a - bc - q_1)$$

$$\begin{aligned}
a - bc - 2q_1 &= 0 \\
q_1^s &= \frac{1}{2}(a - bc) \\
q_2^s &= \frac{1}{2}(a - bc) - \frac{1}{2}q_1 \\
&= \frac{1}{2}(a - bc) - \frac{1}{2}\left(\frac{1}{2}(a - bc)\right) \\
&= \frac{1}{4}(a - bc) \\
P_s &= \frac{a}{b} - \frac{\frac{1}{4}(a - bc) + \frac{1}{2}(a - bc)}{b} \\
&= \frac{1}{4}\frac{a}{b} + \frac{3}{4}c
\end{aligned}$$

$$\begin{aligned}
\pi_1^s &= \left(\frac{1}{4}\frac{a}{b} + \frac{3}{4}c - c\right)\frac{1}{2}(a - bc) = \frac{1}{8b}(a - bc)^2 \\
\pi_2^s &= \left(\frac{1}{4}\frac{a}{b} + \frac{3}{4}c - c\right)\frac{1}{4}(a - bc) = \frac{1}{16b}(a - bc)^2
\end{aligned}$$

Notice that

$$\begin{aligned}
\pi_1^s &> \pi_c > \pi_2^s \\
\frac{1}{8b}(a - bc)^2 &> \frac{1}{9b}(a - bc)^2 > \frac{1}{16b}(a - bc)^2
\end{aligned}$$

this is because q_1 and q_2 are *strategic substitutes*, or that $\frac{\partial q_1(q_2)}{\partial q_2} < 0$.

2 Differentiated Bertrand

Now firms choose price, and quantity is:

$$\begin{aligned}
q_1 &= a - bp_1 + p_2 \\
q_2 &= a - bp_2 + p_1
\end{aligned}$$

$$\max_{p_2} (p_2 - c)(a - bp_2 + p_1)$$

$$
\begin{aligned}
(p_2 - c)(-b) + (a - bp_2 + p_1) &= 0 \\
a + bc + p_1 - 2bp_2 &= 0 \\
\frac{1}{2}c + \frac{1}{2}\frac{a}{b} + \frac{1}{2b}p_1 &= p_2
\end{aligned}
$$

Notice that if $p_1 = 0$ that this is the monopoly price in the Cournot model, but that in general p_2 will be *higher* than the monopoly price. The equilibrium is where $p = p_1 = p_2$.

$$
\begin{aligned}
\frac{1}{2}c + \frac{1}{2}\frac{a}{b} + \frac{1}{2b}p &= p \\
p_b &= \frac{1}{2b-1}(a + bc) \\
q_b &= a - b\left(\frac{1}{2b-1}(a + bc)\right) + \left(\frac{1}{2b-1}(a + bc)\right) \\
&= \frac{b}{2b-1}(a + c - bc) \\
\pi_b &= \left(\frac{1}{2b-1}(a + bc) - c\right)\frac{b}{2b-1}(a + c - bc) \\
&= \frac{b}{(2b-1)^2}(a + c - bc)^2
\end{aligned}
$$

2.1 A "Stackleberg" Variation on the Bertrand model.

Now we will, like before, have firm 2 choose their price after firm 1.

$$
\max_{p_1}(p_1 - c)(a - bp_1 + p_2(p_1))
$$

$$
\max_{p_1}(p_1 - c)\left(a - bp_1 + \frac{1}{2}c + \frac{1}{2}\frac{a}{b} + \frac{1}{2b}p_1\right)
$$

$$
\begin{aligned}
(p_1 - c)(-b) + (p_1 - c)\left(\frac{1}{2b}\right) + \left(a - bp_1 + \frac{1}{2}c + \frac{1}{2}\frac{a}{b} + \frac{1}{2b}p_1\right) &= 0 \\
a + \frac{1}{2}c + bc + \frac{1}{2}\frac{a}{b} - \frac{1}{2b}c - 2bp_1 + \frac{1}{b}p_1 &= 0 \\
\frac{1}{4b^2 - 2}\left(a - c + 2b^2c + 2ab + bc\right) &= p_1^s \\
\frac{1}{2}c + \frac{1}{2}\frac{a}{b} + \frac{1}{2b}\left(\frac{1}{4b^2 - 2}\left(a - c + 2b^2c + 2ab + bc\right)\right) &= p_2^s \\
\frac{1}{8b^3 - 4b}\left(4ab^2 - c - a + 2b^2c + 4b^3c + 2ab - bc\right) &= p_2^s
\end{aligned}
$$

$$
\begin{aligned}
q_1^s &= a - b\frac{1}{4b^2-2}\left(a-c+2b^2c+2ab+bc\right)+\frac{1}{8b^3-4b}\left(4ab^2-c-a+2b^2c+4b^3c+2ab-bc\right)\\
&= \frac{(2b+1)}{4b}(a+c-bc)\\
q_2^s &= a - b\frac{1}{8b^3-4b}\left(4ab^2-c-a+2b^2c+4b^3c+2ab-bc\right)+\left(\frac{1}{4b^2-2}\left(a-c+2b^2c+2ab+bc\right)\right)\\
&= \frac{\left(4b^2+2b-1\right)}{4\left(2b^2-1\right)}(a+c-bc)
\end{aligned}
$$

Notice that firm 1 is charging the higher price:

$$
\begin{aligned}
p_1^s &> p_2^s\\
\frac{a+c-bc}{(2b^2-1)\,4b} &> 0
\end{aligned}
$$

and selling the lower quantity. However if you calculate their profit:

$$
\begin{aligned}
\pi_1^{sb} &= \left(\frac{1}{4b^2-2}\left(a-c+2b^2c+2ab+bc\right)-c\right)\frac{(2b+1)}{4b}(a+c-bc)\\
&= \frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)}(a+c-bc)^2\\
\pi_2^{sb} &= \frac{1}{16b}\frac{\left(4b^2+2b-1\right)^2}{\left(2b^2-1\right)^2}(a+c-bc)^2
\end{aligned}
$$

we see that:

$$
\begin{aligned}
\pi_2^{sb} &> \pi_1^{sb} > \pi_b\\
\frac{1}{16b}\frac{\left(4b^2+2b-1\right)^2}{\left(2b^2-1\right)^2}(a+c-bc)^2 &> \frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)}(a+c-bc)^2 > \frac{b}{(2b-1)^2}(a+c-bc)^2\\
\frac{1}{16b}\frac{\left(4b^2+2b-1\right)^2}{\left(2b^2-1\right)^2} &> \frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)} > \frac{b}{(2b-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
\pi_2^{sb} &> \pi_1^{sb}\\
\frac{1}{16b}\frac{\left(4b^2+2b-1\right)^2}{\left(2b^2-1\right)^2} &> \frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)}\\
\frac{1}{16b}\frac{\left(4b^2+2b-1\right)^2}{\left(2b^2-1\right)^2}16b\left(2b^2-1\right)^2 &> \frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)}16b\left(2b^2-1\right)^2\\
\left(4b^2+2b-1\right)^2 &> 2\,(2b+1)^2\left(2b^2-1\right)\\
16b^4+16b^3-4b^2-4b+1 &> 16b^4+16b^3-4b^2-8b-2\\
4b &> -3
\end{aligned}
$$

$$
\begin{aligned}
\pi_1^{sb} &> \pi_b \\
\frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)} &> \frac{b}{(2b-1)^2} \\
\frac{1}{8b}\frac{(2b+1)^2}{(2b^2-1)}8b\,(2b-1)^2\left(2b^2-1\right) &> \frac{b}{(2b-1)^2}8b\,(2b-1)^2\left(2b^2-1\right) \\
\left(4b^2-1\right)^2 &> 8b^2\left(2b^2-1\right) \\
16b^4-8b^2+1 &> 16b^4-8b^2 \\
1 &> 0
\end{aligned}
$$

So again, all of the statements I made in class are generally true. The fundamental reason for this is because p_1 and p_2 are *strategic compliments* or $\frac{\partial p_1(p_2)}{\partial p_2} > 0$ in the best response.

To get a solvable problem it is actually easiest to set $b = 1$. Then we get:

$$p_b = a + c;\ q_b = a;\ \pi_b = a^2$$

$$
\begin{array}{cc}
p_1^s = \frac{3}{2}a + c & p_2^s = \frac{5}{4}a + c \\
q_1^s = \frac{3}{4}a & q_2^s = \frac{5}{4}a \\
\pi_1^{sb} = \frac{9}{8}a^2 = 1.125a^2 & \pi_2^{sb} = \frac{25}{16}a^2 = 1.5625a^2
\end{array}
$$

which makes it easy for you to verify what took several pages of math above.