

A Handout on The Differentiated Bertrand and Cournot Models

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In this handout I will try to explain both intuitively and mathematically the difference between the differentiated Bertrand and the Cournot model. On the face of it the two models seem identical. After all in both you have two firms that are trying to maximize their profits, so why should there be any difference at all?

This would be true if the firms were working together, or they were a cartel or simply a monopoly, but it will not be true when they behave strategically. Why? Well, it's almost obvious when you think about it. A equilibrium in an oligopoly must take into consideration how others will react to what you do and in the two different models the way your competitors react differs dramatically, so the equilibrium should differ.

Indeed this is a key lesson in Game theory, just because objectives are the same in two similar situations it doesn't mean the equilibrium will be. What matters is the way people react to each other, and if this differs then the equilibrium probably will as well.

It is hard to understand that there is a difference between the two models unless one makes a direct head to head comparison. So consider the following standard Differentiated Bertrand model:

$$\begin{aligned}q_1 &= a - bp_1 + \alpha bp_2 \\q_2 &= a - bp_2 + \alpha bp_1\end{aligned}$$

where the costs of the firms are $c_1(q_1) = cq_1$ and $c_2(q_2) = cq_2$. (Remember that $\alpha \in [0, 1)$, $a > 0, b > 0, c \geq 0$). Now in order to solve this as a Cournot model we have to properly invert the two equations for quantity. We can do this two ways, first by substituting out q_2 and then solving for p_1 :

$$\begin{aligned}p_2 &= \frac{1}{b}(a - q_2 + b\alpha p_1) \\q_1 &= a - bp_1 + \alpha b \left(\frac{1}{b}(a - q_2 + b\alpha p_1) \right) \\p_1 &= \frac{1}{b(1 - \alpha^2)}((1 + \alpha)a - q_1 - \alpha q_2)\end{aligned}$$

or the more mathematically elegant method of solving the system of equations

for the price vector given the quantity vector.

$$\begin{aligned} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} &= a \begin{bmatrix} 1 \\ 1 \end{bmatrix} - b \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= \left(b \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix} \right)^{-1} \left(a \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right) \\ p_1 &= \frac{1}{b(1-\alpha^2)} ((1+\alpha)a - q_1 - \alpha q_2) \\ p_2 &= \frac{1}{b(1-\alpha^2)} ((1+\alpha)a - q_2 - \alpha q_1) \end{aligned}$$

Either way results in the same solution.

We will first solve for the Monopoly benchmark, and then solve the Cournot and Differentiated Bertrand models and then show graphically how the two situations differ.

1 Monopoly

In this section we will show that either way that you solve the problem results in the same solution. To do this we will first find the optimal price and quantity by maximizing over price and then by maximizing over quantity. In both models the profits of each firm are:

$$\pi_i = p_i q_i - c_i(q_i)$$

and our objective is to maximize joint profits:

$$\begin{aligned} \pi &= \pi_1 + \pi_2 \\ &= p_1 q_1 - c_1(q_1) + p_2 q_2 - c_2(q_2) \end{aligned}$$

the difference between the methodologies is merely whether we substitute for quantity using the demand curve or price using the inverse demand curve.

1.1 Optimizing over Price

The objective function is

$$\begin{aligned} \pi &= p_1(a - bp_1 + \alpha bp_2) - c(a - bp_1 + \alpha bp_2) + \\ &\quad p_2(a - bp_2 + \alpha bp_1) - c(a - bp_2 + \alpha bp_1) \\ &= (p_1 - c)(a - bp_1 + \alpha bp_2) + (p_2 - c)(a - bp_2 + \alpha bp_1) . \end{aligned}$$

Given the symmetry of the problem we should look for a symmetric solution, which means we only need to maximize over one price and this greatly decreases our work.

$$\frac{\partial \pi}{\partial p_1} = (a - bp_1 + \alpha bp_2) - b(p_1 - c) + \alpha b(p_2 - c) = 0 \quad (1)$$

Notice that the difference between this and our competitive solution is the second term, which should be positive so basically this will make the firm set prices higher. As an unnecessary intermediate step let's solve for the "reaction function" for p_1 . This is not a true reaction function because this is a joint optimization problem, but it will be instructive to compare it to the true best responses we will find when looking at the competitive solution.

$$\frac{1}{2b}(a + bc - bc\alpha) + \alpha p_2 = p_1 \quad (2)$$

And from this we can find the equilibrium, which will have $p_1 = p_2 = p_m$

$$\frac{1}{2b}(a + bc - bc\alpha) + \alpha p_m = p_m$$

$$p_m = \frac{1}{b} \frac{a + (1 - \alpha)bc}{2 - 2\alpha}$$

And for comparative reasons let's find the quantity. It will be the same for both firms since the price both charge is the same.

$$\begin{aligned} q_m &= a - b \left(\frac{1}{2b} \frac{a + bc - bc\alpha}{1 - \alpha} \right) + \alpha b \left(\frac{1}{2b} \frac{a + bc - bc\alpha}{1 - \alpha} \right) \\ &= \frac{1}{2}(a - (1 - \alpha)bc) . \end{aligned}$$

1.2 Optimizing over Quantity

Now let's switch the problem around and look at optimization over quantity.

$$\begin{aligned} \pi &= \pi_1 + \pi_2 \\ &= \left(\frac{1}{b(1 - \alpha^2)} ((1 + \alpha)a - q_1 - \alpha q_2) \right) q_1 - cq_1 + \left(\frac{1}{b(1 - \alpha^2)} ((1 + \alpha)a - q_2 - \alpha q_1) \right) q_2 - cq_2 . \end{aligned}$$

Again exploiting the symmetry for all it's worth:

$$\begin{aligned} \frac{\partial \pi}{\partial q_1} &= \left(\frac{1}{b(1 - \alpha^2)} ((1 + \alpha)a - q_1 - \alpha q_2) \right) - \frac{1}{b(1 - \alpha^2)} q_1 - c - \frac{\alpha}{b(1 - \alpha^2)} q_2 = 0 \\ q_1 &= \frac{1 + \alpha}{2}(a - (1 - \alpha)bc) - \alpha q_2 \\ q_m &= \frac{1 + \alpha}{2}(a - (1 - \alpha)bc) - \alpha q_m \\ q_m &= \frac{1}{2}(a - (1 - \alpha)bc) , \end{aligned} \quad (3)$$

As you should expect this results in the same quantity.

In this problem methodology is clearly just a matter of convenience. Obviously the proper structural form of the problem is the same, you should set it up as a Lagrangian with six choice variables (prices, quantities, and the multipliers on the constraints). But couldn't the same thing be said about the two

competitive problems? No, because the critical difference is that the impact of the other firm's competitive variable on yours will have a different sign.

By the way, just for the fun of it, you could also solve for one price and the other quantity. It wouldn't matter, of course you'd have no more symmetry. You have to solve a system of two equations with two unknowns.

2 Bertrand Competition:

Now we must set up the problem by substituting out for q_1 .

$$\max_{p_1} (a - bp_1 + \alpha bp_2) (p_1 - c)$$

Again we don't need to solve firm 2's problem by symmetry.

$$(a - bp_1 + \alpha bp_2) - b(p_1 - c) = 0 \tag{4}$$

$$\frac{a + bc}{2b} + \frac{\alpha}{2} p_2 = p_1 \tag{5}$$

Now compare equation 2 and equation 5, the Bertrand best response. Both the intercept and the slope are different, the reason for this is that in equation 1 there is one more term than in 4: $\alpha b(p_2 - c) = b\alpha p_2 - bc\alpha$, which captured the impact of increasing the price of firm 1 on firm 2's profits. The second term is in the intercept, the first term is why the slope is twice as high as in this case.

Now we can use symmetry again and we get:

$$\begin{aligned} \frac{a + bc}{2b} + \frac{\alpha}{2} p_b &= p_b \\ p_b &= \frac{1}{b} \frac{a + bc}{2 - \alpha} \end{aligned}$$

Finally we will solve for the quantity each firm produces to make comparison with the other solutions easier.

$$\begin{aligned} q_b &= a - b \left(\frac{1}{b} \frac{a + bc}{2 - \alpha} \right) + \alpha b \left(\frac{1}{b} \frac{a + bc}{2 - \alpha} \right) \\ q_b &= q_1 = q_2 = \frac{1}{2 - \alpha} (a - (1 - \alpha) bc) \end{aligned}$$

3 Cournot Competition:

In this problem we have non-standard demand equations to make the comparison with the Differentiated Bertrand model clearer, but we can still solve it in exactly the same manner.

$$\max_{q_1} \left(\frac{1}{b(1 - \alpha^2)} ((1 + \alpha)a - q_1 - \alpha q_2) \right) q_1 - cq_1$$

The first order condition and the reaction function are:

$$-\frac{1}{b(1-\alpha^2)}q_1 + \left(\frac{1}{b(1-\alpha^2)}((1+\alpha)a - q_1 - \alpha q_2) \right) - c = 0$$

$$\frac{(1+\alpha)}{2}(a - (1-\alpha)cb) - \frac{\alpha}{2}q_2 = q_1. \quad (6)$$

Again it is worthwhile to compare this best response with the monopoly "reaction function," or equation 3. Here the missing term is simply αq_2 , and this why the slope here is again half what it was for the monopolist. Again by symmetry we can guess there is a symmetric solution.

$$q_c = q_1 = q_2$$

$$\frac{1}{2}(1+\alpha)(a - bc(1-\alpha)) - \frac{\alpha}{2}q_c = q_c$$

$$\frac{1+\alpha}{\alpha+2}(a - (1-\alpha)bc) = q_c.$$

and just for the fun of it we can also find the equilibrium price:

$$p_c = \frac{1}{b(1-\alpha^2)} \left((1+\alpha)a - \left(\frac{1+\alpha}{\alpha+2}(a - (1-\alpha)bc) \right) - \alpha \left(\frac{1+\alpha}{\alpha+2}(a - (1-\alpha)bc) \right) \right)$$

$$= \frac{1}{b(1-\alpha^2)} \frac{1+\alpha}{\alpha+2} (a + (1-\alpha^2)bc).$$

4 Comparison of Equilibria.

First of all, there is no reason to think that for general demand curves and general costs the results here will hold true, but let's compare the output under the three different scenarios.

$$q_m = \frac{1}{2}(a - (1-\alpha)bc)$$

$$q_b = \frac{1}{2-\alpha}(a - (1-\alpha)bc)$$

$$q_c = \frac{1+\alpha}{\alpha+2}(a - (1-\alpha)bc)$$

The difference in these quantities all depends on the size of the initial coefficients, the term $(a - (1-\alpha)bc)$ is common to them all. So $q_m < q_b$ because $\frac{1}{2} < \frac{1}{2-\alpha}$ since $\alpha < 1$. $q_m < q_c$ because $0 < \alpha$. (I will show this just for completeness).

$$\frac{1}{2} < \frac{1+\alpha}{\alpha+2}$$

$$\alpha+2 < 2(1+\alpha)$$

$$0 < \alpha$$

and in this class of problems $q_b > q_c$:

$$\begin{aligned} \frac{1}{2-\alpha} &> \frac{1+\alpha}{\alpha+2} \\ \alpha+2 &> (2-\alpha)(1+\alpha) \\ \alpha+2 &> -\alpha^2 + \alpha + 2 \\ 0 &> -\alpha^2 . \end{aligned}$$

Notice that if $\alpha = 0$ then all three quantities are the same, as they should be because then there is no interaction between the two demand curves and in all three cases we're just finding the monopoly quantity. Now the first comparisons ($q_m < q_b$ and $q_m < q_c$) should be obvious because of the nature of competition, the latter one ($q_b > q_c$)? Well I've never seen it proven, and I would have to see the proof to believe it.

If more output is produced that means that the price is lower, and this means that consumers are happier, so consumers would like the Bertrand competition more than the Cournot and either more than the Monopoly. And what about profits? Well, generally speaking over the relative range a higher quantity means a lower profit, and that does hold. You can calculate the profits and the order of the profits, but I'm not going to show the math.

$$\begin{aligned} \pi_m &= \frac{1}{4b(1-\alpha)} (a - (1-\alpha)bc)^2 > \\ \pi_c &= \frac{1}{b(1-\alpha)(\alpha+2)^2} (a - (1-\alpha)bc)^2 > \\ \pi_b &= \frac{1}{b(\alpha-2)^2} (a - (1-\alpha)bc)^2 \end{aligned}$$

5 The Reason for the Difference, Strategic Substitutes versus Strategic Compliments

So why is there this difference? Well it can be explained by looking at the second derivative of profits with respect to both choice variables (own and other firm's) or the derivative of the best response. In the Bertrand model these cross derivatives are:

$$\begin{aligned} \frac{\partial \pi_1(p_1, p_2)}{\partial p_1 \partial p_2} &= \alpha b > 0 \\ \frac{\partial p_1}{\partial p_2} &= \frac{\alpha}{2} > 0 . \end{aligned}$$

Since these are positive prices are *strategic compliments* in this model. This means that if your opponent increases his strategic variable you will increase your own. You get a positive feedback loop, and in order for there to be an equilibrium this feedback needs to dampen down, or $\frac{\alpha}{2} < 1$, We generally assume the stronger condition that $\alpha < 1$.

Notice you only need to check one of these two conditions, if one is true then the other is true. When you find the equilibrium the second condition will be easier to check, if you are just looking at an abstract problem then the first one might be easier to check.

In the Cournot model these cross derivatives are:

$$\frac{\partial \pi_1(q_1, q_2)}{\partial q_1 \partial q_2} = -\frac{\alpha}{b(1-\alpha^2)} < 0$$

$$\frac{\partial q_1}{\partial q_2} = -\frac{\alpha}{2} < 0$$

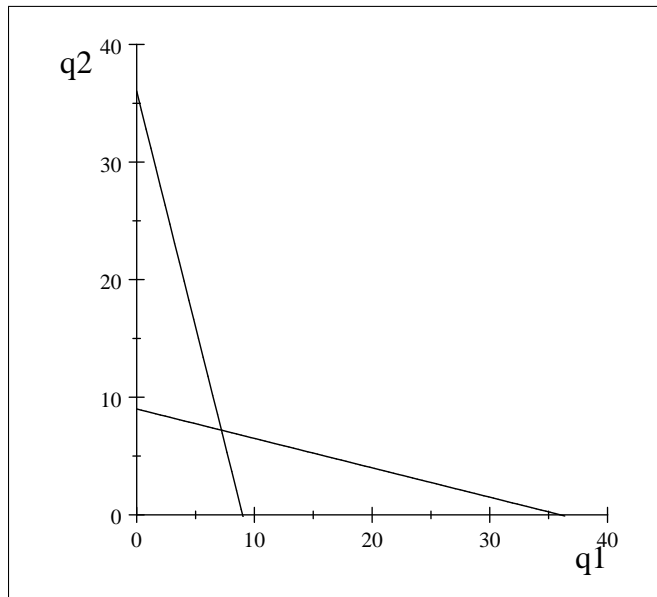
so these are *strategic substitutes*. This means that if your opponent increases his strategic variable then you will decrease your own. There is negative feedback here so generally there will always be an equilibrium no matter what α is. However notice that in this model if $\alpha > 1$ then the objective function is convex in quantity, and this means that the "best response" you found is actually a worst response—the optimal output is either zero or diverges to infinity. So the model only makes sense if $\alpha < 1$.

You can also see the difference graphically by substituting in some values for (a, b, c, α) , let $(a, b, c, \alpha) = (14, 1, 4, \frac{1}{2})$, in this case:

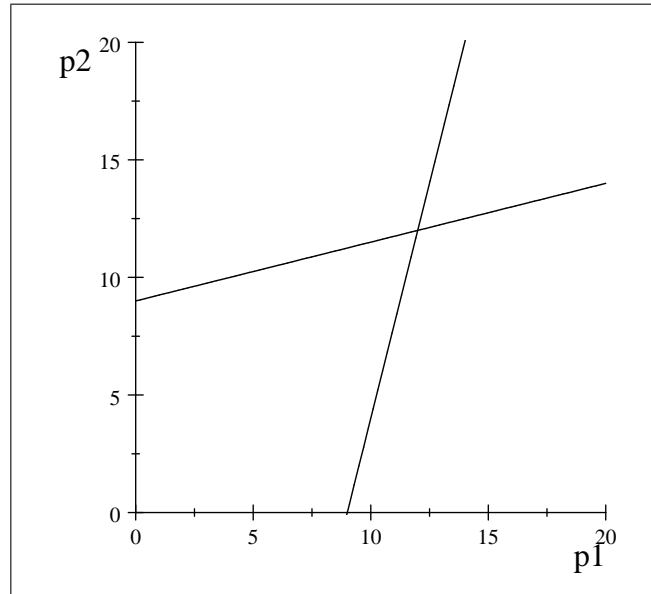
$$p_1 = 9 + \frac{1}{4}p_2$$

$$q_1 = 9 - \frac{1}{4}q_2$$

And a plot of q_1 against q_2 looks like:



The best response of firm 1 is the steeper line. Thus you can see graphically that when q_2 increase q_1 decreases. A plot of p_1 against p_2 looks like:



so here you can clearly see that the more p_2 increases the more p_1 will increase. Equilibrium in the Bertrand game only requires that $\alpha < 2$, but in general we restrict α to be less than one.

One implication of the difference between strategic compliments and strategic substitutes becomes clear in the sequential game. In a sequential game if the two choice variables are strategic substitutes then the person who goes first gets a higher profit than the person who goes second. If the two are strategic compliments the person who goes first gets a lower profit than the person who goes second. Notice that in both cases the first person does better than in these simultaneous choice models, but if the two variables are strategic compliments the second mover is happy to let the first mover go first.

6 A Cautionary note: The Hybrid Model

It is possible that I should not even include this section. One warning you often receive is that you shouldn't show students the wrong way to solve a problem. This section is about a wrong way of solving the problem. But I am including it for three reasons. First, it is a third theoretically possible way to solve this problem. Second, it is similar to what a lot of students do on exams. Finally, it creates a model where one variable is a substitute for the other while the other is a compliment for that variable.

One thing that I often see people do when I give them a Bertrand model and tell them to solve it is that they will invert for the quantity of one firm

and then try to solve that problem. Technically speaking there really isn't a problem with solving this hybrid problem, but it is not one that I would ever ask you to solve for (without some extreme and weird reason) and it takes a lot more work.

The main reason it takes a lot more work is because symmetry no longer holds. Look at the two objective functions below, if you switch the indices does one objective function look like the other? No, then symmetry no longer holds and you can't assume symmetry to solve the problem.

But let's do it anyway. Say that we solve for p_2 from the demand curve for firm 2:

$$\begin{aligned} p_2 &= \frac{1}{b}(a - q_2 + b\alpha p_1) \\ &= \frac{a}{b} + \alpha p_1 - \frac{1}{b}q_2 . \end{aligned}$$

Now the first logical mistake people do is that they don't take this through logically. Logically they should take this value for p_2 and substitute it into the demand for firm 1:

$$\begin{aligned} q_1 &= a - bp_1 + \alpha b \left(\frac{1}{b}(a - q_2 + b\alpha p_1) \right) \\ &= (1 + \alpha)a - (1 - \alpha^2)bp_1 - \alpha q_2 . \end{aligned}$$

If you did this correctly then you will have a model where firm 1 acts like a Bertrand competitor (optimizing over price) and firm 2 acts like a Cournot competitor (optimizing over quantity).

Solving firm 1's problem:

$$\max_{p_1} (p_1 - c) \left((1 + \alpha)a - (1 - \alpha^2)bp_1 - \alpha q_2 \right) ,$$

$$\begin{aligned} ((1 + \alpha)a - (1 - \alpha^2)bp_1 - \alpha q_2) - (1 - \alpha^2)b(p_1 - c) &= 0 \\ \frac{1}{2b(1 - \alpha)}(a + (1 - \alpha)bc) - \frac{1}{2b} \frac{\alpha}{1 - \alpha^2}q_2 &= p_1 . \end{aligned}$$

Now solving firm 2's problem:

$$\max_{q_2} \left(\frac{a}{b} + \alpha p_1 - \frac{1}{b}q_2 \right) q_2 - cq_2 ,$$

$$\begin{aligned} \frac{a}{b} + \alpha p_1 - \frac{1}{b}q_2 - \frac{1}{b}q_2 - c &= 0 \\ \frac{1}{2}(a - bc) + \frac{b\alpha}{2}p_1 &= q_2 . \end{aligned}$$

To find the equilibrium we have to substitute q_2 into firm 1's objective function:

$$\frac{1}{2b(1 - \alpha)}(a + (1 - \alpha)bc) - \frac{1}{2} \frac{\alpha}{b - b\alpha^2} \left(\frac{1}{2}(a - bc) + \frac{b\alpha}{2}p_1 \right) = p_1$$

$$\begin{aligned}
p_1^h &= \frac{1}{b} \frac{1}{4-3\alpha^2} (a(\alpha+2) + bc(2(1-\alpha^2) + \alpha)) \\
q_2^h &= \frac{1}{2} (a-bc) + \frac{b\alpha}{2} \left(\frac{1}{b} \frac{1}{4-3\alpha^2} (a(\alpha+2) + bc(2(1-\alpha^2) + \alpha)) \right) \\
&= \frac{2 + (1-\alpha)\alpha}{4-3\alpha^2} (a - (1-\alpha)bc)
\end{aligned}$$

From these we can find out the two missing variables, but we will only solve for q_1 :

$$\begin{aligned}
q_1^h &= (1+\alpha)a - (1-\alpha^2)b \left(\frac{1}{4b-3b\alpha^2} (a(\alpha+2) + bc(2(1-\alpha^2) + \alpha)) \right) - \alpha \left(\frac{2 + (1-\alpha)\alpha}{4-3\alpha^2} (a - (1-\alpha)bc) \right) \\
&= \frac{(1-\alpha^2)(\alpha+2)}{4-3\alpha^2} (a - (1-\alpha)bc) .
\end{aligned}$$

Now firm 2 (who was maximizing over quantity) will end up producing more output, or one can easily show that $q_2^h > q_1^h$. Firm 2 will actually end up producing more output than in the Bertrand model, $q_2^h > q_b$. While firm 1 will end up producing less output than in the Cournot model, $q_1^h < q_c$. Oddly enough (and for this I have no explanation) if $\alpha > 0.78078$ $q_1^h < q_m$. Theoretically this should not be possible and I suspect this means there is an error in my math. Please feel free to find it and bring it to my attention.

Notice that in this problem:

$$\begin{aligned}
\frac{\partial q_2}{\partial p_1} &= \frac{b\alpha}{2} > 0 \\
\frac{\partial p_1}{\partial q_2} &= -\frac{1}{2b} \frac{\alpha}{1-\alpha^2} < 0 .
\end{aligned}$$

So this tells us that p_1 is a strategic complement for q_2 and q_2 is a strategic substitute for p_1 . While this is a weird model there are probably other more natural models where something of this sort was true. However I promise you that if I do give you such a problem I would present it in its primitive form and not ask you to solve for a choice variable (like we solved for p_2 and q_1 above.)