On the Private Provision of a Public Good

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9 April, 2019

One of the classic problems in public goods is the *free rider problem.* Those who benefit from the public good do not need to provide it, and thus they can take advantage of the people do provide it or "take the ride" for free. By happenstance I came across a simple model where we find absolute free riding. Notice that absolute free riding means that you will contribute nothing. This is a *corner solution,* and that gives rise to some weird mathematics that you might have forgotten. The primary reason for this handout is to discuss this further.

Before getting started I want to point out that free riding is not always absolute. Sometimes everyone will contribute a little bit to the public good, and its just that the good will be under provided. For example in the mixed Nash equilibrium of the reporting a crime game each person sometimes will report the crime, or provide the public good, but only sometimes. There is always a chance that no one will report the crime, even though each and every person would report it if it was left to them. This is not Pareto efficient. Just to remind you if the game is:

	R	N
R	$1-c_1; 1-c_2$	$1-c_1; 1$
N	$1; 1-c_2$	$0; 0$

where c_1 and c_2 are both strictly between 0 and 1. Remember that the mixed strategy equilibrium is also the *no coordination* equilibrium. In other words it best represents a real world situation where people observe something that needs to be reported, but there is no way to coordinate on who will report it. Let $p_i = \Pr(\text{person } i \text{ choose } R)$, then the mixed strategy equilibrium is $p_2 = 1 - c_1$ and $p_1 = 1 - c_2$. Thus both contribute sometimes which is the same as partial free riding. Still with probability $(1-p_1)(1-p_2) = c_1 c_2 > 0$ the crime is not reported, even though both would rather have it reported than not.

1 The Mathematics of Corner Solutions and the Best Response.

Absolute free riding means that I contribute nothing, or in other words that my provision is at it's lower bound. Let's look at the problem we are going to solve in this handout. Assume the benefit function is:

$$B_i(Q) = \begin{cases} \alpha_i Q - \frac{\beta_i}{2} Q^2 & Q \le \frac{\alpha_i}{\beta_i} \\ \frac{1}{2}\frac{\alpha_i^2}{\beta_i} & Q \ge \frac{\alpha_i}{\beta_i} \end{cases}$$

where $\alpha_i > 0$, $\beta_i > 0$, and $Q = Q_{-i} + q_i$, with Q_{-i} being the amount *others* contribute and q_i being the amount i contributes. First of all, notice that this is

a function of the *total* quantity provided because this is a non-rival good. We all consume what any one person contributes. Second, when $Q > \frac{\alpha_i}{\beta_i}$ we basically are saying that there is no marginal benefit from the good, the value $\frac{1}{2}\frac{\alpha_i^2}{\beta_i}$ is just convenient because then $B_i(Q)$ is continuous and differentiable. Notice that this is a non-rival good, so each and every person benefits from the total quantity. Assuming $Q_{-i} < \frac{\alpha_i}{\beta_i}$, then i's utility function will be:

$$
\begin{aligned}
u_i(Q, q_i) &= B_i(Q) - pq_i \\
u_i(Q, q_i) &= \alpha_i(Q_{-i} + q_i) - \frac{\beta_i}{2}(Q_{-i} + q_i)^2 - pq_i
\end{aligned}
$$

Now let's get parametric, let $p = 2$, $\beta_i = 1$, $\alpha_i = 5$ and graph this for $Q_{-i} \in \{0, 2, 4\}$:
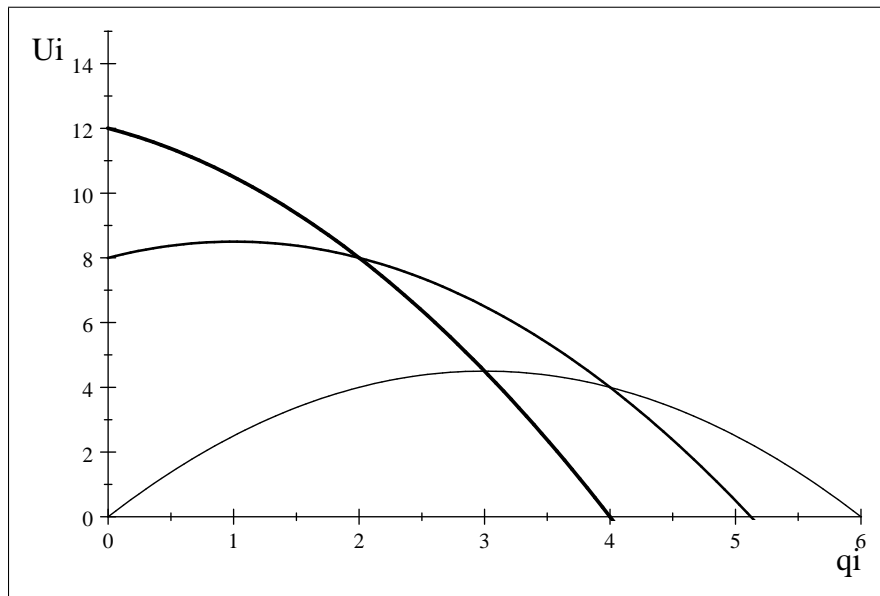
The thin line is when $Q_{-i} = 0$, or the others are not contributing, then in this case person i will want to contribute 3 units. The medium thickness line is the case where $Q_{-i} = 2$. This crosses the y axis higher than when $Q_{-i} = 1$, achieves its maximum earlier ($q_i = 1$) and—most importantly—is flatter than when $Q_{-i} = 0$. When $Q_{-i} = 4$ we have the thick line. It crosses the y axis the highest, and the critical point is that it is **always downward sloping**. What is a sensible person to do in this case? If they contribute anything it will just make them less happy. So they should not contribute any of the public good and just free ride off of the others.

Let's re-think this by looking at the Marginal benefit and marginal cost. In this problem:

$$
\begin{aligned}
MB_i &= \alpha_i - \beta_i Q \\
&= \alpha_i - \beta_i(Q_{-i} + q_i)
\end{aligned}
$$

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Let's redo our example above analyzing the marginal benefit at $Q_{-i} \in \{0, 2, 4\}$.

$$MB_i = \begin{cases} 5 - (0 + q_i) = 5 - q_i & \text{if } Q_{-i} = 0 \\ 5 - (2 + q_i) = 3 - q_i & \text{if } Q_{-i} = 2 \\ 5 - (4 + q_i) = 1 - q_i & \text{if } Q_{-i} = 4 \end{cases}$$

in all cases the marginal cost is two. It is clear that in the first case the maximum is $q_i = 3$, in the second it is $q_i = 1$, and in the third? Again, something weird has happened. The marginal benefit is always strictly below two! What is a sensible person to do if the marginal benefit is strictly below the marginal cost? Well, isn't it obvious? You don't want to contribute anything, or the optimal q_i is $q_i = 0$.

Do you remember Kuhn-Tucker conditions from Math for Economists (ECON 225)? I bet you don't, I bet you studied them for the final and then promptly forgot them. Well.... now you need to remember them again. There are always *complementary slackness* conditions for maximization. You can either have:

$$q_i \geq 0, \ \frac{\partial U_i}{\partial q_i} = 0$$

or you can have:

$$q_i = 0, \ \frac{\partial U_i}{\partial q_i} \leq 0 \ .$$

The second condition is exactly the case we have above. If Q_{-i} is too high then the utility function is always strictly decreasing in q_i ($\frac{\partial U_i}{\partial q_i} < 0$) and you want to choose the corner or $q_i = 0$.

Now having figured out this insight, let's find the best responses.

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} &= MB_i - MC \\ &= \alpha_i - \beta_i (Q_{-i} + q_i) - p \end{aligned}$$

The possible solutions are:

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} &= 0 = \alpha_i - \beta_i (Q_{-i} + q_i) - p \\ q_i &= \frac{\alpha_i - p}{\beta_i} - Q_{-i} \ . \end{aligned}$$

Or, we can have $\frac{\partial U_i}{\partial q_i} \leq 0$ and $q_i = 0$:

$$\begin{aligned} \frac{\partial U_i}{\partial q_i} &= \alpha_i - \beta_i (Q_{-i} + 0) - p \leq 0 \\ Q_{-i} &\geq \frac{\alpha_i - p}{\beta_i} \ . \end{aligned}$$

And we notice these cases nest perfectly (as they should). Therefore:

$$q_i = BR_i (Q_{-i}) = \begin{cases} \frac{\alpha_i - p}{\beta_i} - Q_{-i} & Q_{-i} \leq \frac{\alpha_i - p}{\beta_i} \\ 0 & Q_{-i} \geq \frac{\alpha_i - p}{\beta_i} \end{cases}$$

2 An Example:

Let us go through the analysis of one simple case. Assume that the price of providing the public good is 2 ($p = 2$), and let person 1 be the person we just analyzed ($\alpha_1 = 5$, $\beta_1 = 1$). Assume that for person 2 $\alpha_2 = 4$, $\beta_2 = \frac{1}{2}$, then the best responses of the two people are:

$$q_1 = \begin{cases} 3 - q_2 & q_2 \leq 3 \\ 0 & q_2 \geq 3 \end{cases}$$

$$q_2 = \begin{cases} 4 - q_1 & q_1 \leq 4 \\ 0 & q_1 \geq 4 \end{cases}$$

And let's find the equilibrium by simply starting with a guess and then iterating. Letting person 1 optimize, then person 2, and so on. Let's start at $q_1 = q_2 = 1$. Then person 1 will choose $q_1 = 3 - q_2$ (notice $q_2 = 1 < 4$) or $q_1 = 2$. Then person 2 will choose $q_2 = 4 - q_1 = 4 - 2 = 2$. Person 1 will then choose $q_1 = 3 - 2 = 1$, person 2 will choose $q_2 = 4 - 1 = 3$. Now $q_1 = 3 - 3 = 0$ so $q_2 = 4 - 0 = 4$, and since $q_2 = 4 > 3$ we know person 1's best response is zero and we have reached an equilibrium.

But are there others? Well, you know iteration is impractical, but in this case I think it worth going through a few more sequences. I'll start with only a guess about q_2 and in the table below iterate the values of q_1 and q_2 until we reach an equilibrium. The superscript will indicate the iteration we are on.

q_2^0	q_1^1	q_2^1	q_1^2	q_2^2	q_1^3	q_2^3	q_1^4	q_2^4
0	3	1	2	2	1	3	0	4
1	2	2	1	3	0	4		
2	1	3	0	4				
3	0	4						
4	0							

It seems we only have one equilibrium, and it is $q_1 = 0$, $q_2 = 4$.

3 Meta Analysis to find the Nash equilibria of any game.

Let's ask what seem to be an arbitrary question, and then look at how it simplifies our analysis. What would person i contribute if he was the only one providing? I.e. if $Q_{-i} = 0$? This is quite simple to find:

$$\alpha_i - \beta_i (0 + q_i) - p = 0$$
$$q_i = \frac{\alpha_i - p}{\beta_i}$$

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Let's call this his *stand alone quantity*, denoted q_i^s. But now let's look at his best response again. If we replace $\frac{\alpha_i - p}{\beta_i}$ with q_i^s it becomes:

$$q_i = BR_i(Q_{-i}) = \begin{cases} q_i^s - Q_{-i} & Q_{-i} \leq q_i^s \\ 0 & Q_{-i} \geq q_i^s \end{cases}.$$

This makes it transparent that as far as i is concerned he will always contribute up to q_i^s and will not if the total supplied is more than that. An obvious guess at the equilibrium is then $Q^* = \max_i q_i^s$, and in fact it's fairly simple to verify that this guess is true.

Proposition 1 *In this public good game in equilibrium $Q^* = \max_i q_i^s$ and if $q_j^s < \max_i q_i^s$ then $q_j^* = 0$.*
Proof. *Let's first show that $Q^* = \max_i q_i^s$ is correct. If we had $Q^* > \max_i q_i^s$ then anyone who is not contributing zero will want to reduce their contribution, thus this can't be an equilibrium. If we had $Q^* < \max_i q_i^s$ then anyone for whom $q_j^s = \max_i q_i^s$ will want to increase their contribution. Thus this can not be equilibrium. This verifies that $Q^* = \max_i q_i^s$.*

Finally a little bit of paperwork to figure out what we can about the equilibrium. If $Q^* = \max_i q_i^s$ and $q_j^s < \max_i q_i^s$ then the best response of person j is clearly zero. If $q_j^s = \max_i q_i^s$ then this person will contribute up to $\max_i q_i^s$ to provide the public good, but obviously if there are multiple people in this group then who will contribute how much can not be decided. Thus this is the class of equilibria.[1] ∎

This equilibrium exhibits extreme *free riding.* Only a select subgroup (basically those who value the public good the most) contribute anything, everyone else just benefits from the public good without contributing anything. This explains why many suburbs (sitesiler) have regulations about minimal upkeep on lawns.[2] Of course in this case someone who already values a nice lawn will receive positive feedback from their neighbors, increasing their contribution even more, but essentially it is this equilibrium. Government intervention is necessary because not everyone wants to contribute, and to make sure minimal standards are met there must be regulation. When on sabbatical we stayed in a suburb with such regulations—specifically about the height of grass. The response of some of the older/less physically fit residents was to cover their lawn with flowers because flower beds were not regulated.

This also explains why for important public goods government provision is required. So let's turn to the welfare maximizing quantity next.

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[1] In any equilibrium each person provides an exact amount. If person j has $q_j^s = \max_i q_i^s$ then $q_j^* = \mu_j q_j^s$, where $\mu_j \in [0, 1]$ and the sum of such μ_j's is one. What we can not determine is how much a particular μ_j will be.

[2] I have a guess that I would like to have confirmed. Do some sitesiler in Turkey hire gardeners for all the residents? I.e. pay someone to come by and mow your lawn and tend your flowers, etc. It would seem a very Turkish way of solving the problem.

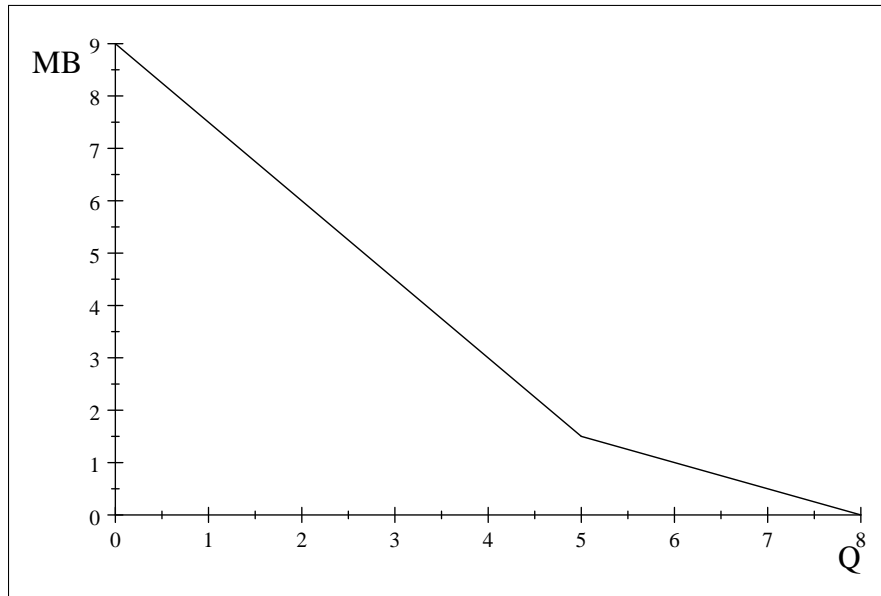# 4 The Optimal amount of Public Good.

In this environment one can show that welfare is properly defined as:

$$W(Q) = \sum_{i=1}^{n} B_i(Q) - pQ \ .$$

Just to make it simple we will assume that $Q \leq \min_i \frac{\alpha_i}{\beta_i}$, otherwise we will have to deal with kinks in the demand curve. Like usual with a public good we vertically sum the marginal benefits, or:

$$
\begin{aligned}
MB(Q) &= \sum_{i=1}^{n} MB_i(Q) \\
&= \sum_{i=1}^{n} (\alpha_i - \beta_i Q)
\end{aligned}
$$

Let's graph this in our example.

As you can see, there is a kink in the aggregate demand curve when $Q = \frac{\alpha_1}{\beta_1} = 5$, at this point person 1's demand drops to zero and so the aggregate demand curve becomes person 2's demand curve. Generally we will want to ignore those kinks, and thus we analyze the case where $Q \leq \min_i \frac{\alpha_i}{\beta_i}$. Let $\bar{\alpha} = \frac{1}{n} \sum_{i=1}^{n} \alpha_i$ and $\bar{\beta} = \frac{1}{n} \sum_{i=1}^{n} \beta_i$ then

$$MB(Q) = n\bar{\alpha} - n\bar{\beta}Q$$

thus the welfare maximizing is:

$$
\begin{aligned}
MB\left(Q\right) &= p \\
n\bar{\alpha} - n\bar{\beta}Q^e &= p \\
Q^e &= \frac{1}{\bar{\beta}}\left(\bar{\alpha} - \frac{p}{n}\right)
\end{aligned}
$$

Or in other words the price for a unit of public good is essentially divided by the population size. In the end (if $\bar{\alpha}$ and $\bar{\beta}$ stay the same) the cost of providing a unit will completely disappear and Q^e will be almost $\frac{\bar{\alpha}}{\bar{\beta}}$.

This will always be larger than $\max_i q_i^s$ as long at least two people have $\frac{\alpha_j}{\beta_j} > Q^e$ for example in our problem above this is:

$$
\begin{aligned}
Q^e &= \frac{1}{\frac{1+\frac{1}{2}}{2}}\left(\frac{(5+4)}{2} - \frac{2}{2}\right) \\
&= \frac{1}{\frac{3}{4}}\left(\frac{9}{2} - \frac{2}{2}\right) \\
&= \frac{14}{3} = 4.666\,7\ .
\end{aligned}
$$

Though of course we don't have to use the formula slavishly. It is a lot simpler just to not take the averages, in which case:

$$
Q_e = \frac{1}{1+\frac{1}{2}}\left(5+4-2\right) = \frac{1}{\frac{3}{2}}\left(7\right) = \frac{14}{3}\ .
$$

If we only have one person with $\frac{\alpha_j}{\beta_j} > Q^e$ then over the relevant range the aggregate demand curve is just that person's demand curve and so society will choose $Q^e = q_j^s = \max_i q_i^s$. We always have $Q^e \geq \max_i q_i^s$, and they are only equal when one person simply wants the good a whole lot more than everyone else. In this unique case they would be equal and private provision is as good as public provision.