The Hotelling Linear City model with discrete locations.

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1 Introduction

One of the most important choices of a firm is where to locate their business. In the city center? At a shopping mall? Which floor? As an interesting example of this, when I first moved to Ankara my wife needed new shoes. She had been told that all of the shoe sellers were located on Tunalı Caddesi, so we went downtown. And yes, all of the shoe sellers had small shops on Tunalı Caddesi. We went from shop to shop and quickly realized something else. Not only where the shoe sellers in the same physical location but also the same taste location—all of the shoes were the same. They looked nice, but they weren't comfortable. How can we explain this?

We want to analyze firm's choice of location, and thus we want to abstract from all other concerns. Thus we will assume that all firms are producing the same good and charging the same price. This price will be fixed above marginal cost and thus firms will always want more customers.

2 The Model.

There will be two firms and a finite set of locations, $L = \left\{1, 2, 3, ..., \bar{l}\right\}$, a firm will choose $l_i \in L$ with the goal of maximizing their expected sales.

There will be n customers, each endowed with a location $l_c \in L$, each who will purchase one unit of the good. They will go to firm 1 if $(l_c - l_1)^2 < (l_c - l_2)^2$, firm 2 if $(l_c - l_1)^2 > (l_c - l_2)^2$ and if $(l_c - l_1)^2 = (l_c - l_2)^2$ then they will choose which firm to buy from by flipping a coin—i.e. half the time they will go to firm 1 and half the time they will go to firm 2. (Or, in other words, if their are two customers at a given location one will go to firm 1 and the other to firm 2. For this reason there will never be an odd number of customers with a given location.) Let C_l be the number of customers with location l.

3 Analysis

At this point it's best to just write down an example and think about things a bit. Obviously what we really care about is the number of customers at each location, so a model is a set of locations and a number of customers at each location. For example:

L	1	2	3	4	5	6
C_l	2	8	6	4	2	20

.

From this we can see that $n = \sum_{l=1}^{6} C_l = 42$. Now assume that firm 2 is at location 2, could the best response of firm 1 be to locate at location 6? That makes sense, after all that's where the most customers are... except. These customers are going to buy from one of these two firms, so if firm 1 moves to location 5 all of the customers at location 6 would go to them. They also would compete better for the customers in between the firms. If firm 1 is at location 6 half the customers at location 4 will got to firm 1, if they move to location 5 these customers would all now come to firm 1. So 5 must be a better response to 2 than 6. But the same reasoning tells us that 4 would be better than 5, the customers at 5 and 6 are captured and by moving to location 4 they will get three customers from location 3. And again, 3 must be better than 4 because now all the customers at 4,5, and 6 are captured and those at location 3 must all go to firm 1. This insight obviously generalizes, and gives us the following Lemma:

Lemma 1 (Business Stealing) $BR_i(l_j) \in \{l_j - 1, l_j, l_j + 1\}$

Proof. *What we have ruled out is being more than one space from the other firm. Say, for example, that firm i choose $l_j + 2$ instead of $l_j + 1$. With this choice half the customers at $l_j + 1$ will go to firm j, a clear loss with no benefit. Likewise this holds for and $l_j + k$ for $k > 1$ and by symmetry it would hold for $l_j - k$.* ■

In order to solve this problem let's calculate the best responses for the problem above. To do this we need to find the profits from being located one down, being at the same location, and being located one up. Notice that being located at the same location always nets the same profit—$n/2$. So let me fill out the table.

L	1	2	3	4	5	6
C_l	2	8	6	4	2	20

If $l_2 =$	1	2	3	4	5	6
$\Pi_1(l_2 - 1, l_2) =$	NA	2	10	20	20	22
$\Pi_1(l_2, l_2) =$	21	21	21	21	21	21
$\Pi_1(l_2 + 1, l_2) =$	40	32	26	22	20	NA
$BR_1(l_2) =$	2	3	4	5	5	5

Notice that the best responses of firm 2 will be the same because of symmetry. Since in a Nash equilibrium we have to have $l_1 = BR_1(BR_2(l_1))$ we can immediately see that in this economy the Nash equilibrium is $l_1^* = l_2^* = 5$.

Can we generalize these results? It is quite simple to do so. Notice that one can always get half the business if they choose the same location, so the goal of the firm can be re-written as getting at least half the business. What do they have to do in order to be sure of this, move toward the *medial location.*

Definition 2 *The* median location *(denoted l_m) is the location where more than half the customers are at that location or above and more than half are at that* *location or below. Mathematically* $\sum_{l_m}^{\bar{l}} C_l \geq \frac{n}{2}$ *and* $\sum_{1}^{l_m} C_l \geq \frac{n}{2}$.

This might not be unique but in any problem I assign it will be. For example in our example $n = 42$ so $n/2 = 21$. The number of customers at locations 5 and 6 is 22 which is strictly above 21, and those located at 5 or lower is 22. Thus $l_m = 5$. Generally we can conclude:

Proposition 3 *Assume that there is a unique median location, then the best response in the Hotelling linear model is:*

$$BR_i(l_j) = \begin{cases} l_j + 1 & \text{if} \quad l_j < l_m \\ l_j & \text{if} \quad l_j = l_m \\ l_j - 1 & \text{if} \quad l_j > l_m \end{cases}$$

Proof. *If* $l_j < l_m$ *then by locating at* $l_j + 1$ *you will get more than* $n/2$ *customers, while locating at* l_j *will give you only* $n/2$ *and locating to the right must give you less than* $n/2$ *since now firm* j *is closer to* l_m*. A symmetric argument handles* $l_j > l_m$*. If* $l_j = l_m$ *then if* $l_i \neq l_j$ *firm* i *must be getting strictly less than* $n/2$ *by definition.* ■

Note that if there are two median locations the only difference is that the notation gets messier. The location of the other firm must now be strictly below or above both of the median locations, and if it is one of them then the best response can be either.

This, of course, tells us immediately that:

Theorem 4 *If there is one median location in the Hotelling linear model then the unique Nash equilibrium is to have both firms locate at the median location.*