On the Relationship between Mixed Strategies and Asymmetric Information

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1 The mixed strategy equilibrium of Reporting a Crime

In games like Meeting in New York:

	S	G
Statue of Liberty (S)	$8;8$	$0;0$
Grand Central Station (G)	$0;0$	$1;1$

there are two symmetric pure strategy equilibria and a symmetric mixed strategy equilibrium. For many reasons, the mixed strategy equilibrium is of little interest.

However consider a game where all pure strategy equilibria are asymmetric, like the reporting a crime game. When a crime occurs, or someone needs help, we all agree that it would be better if *someone* reported it, if *someone* helped the person out—but why should it be you? Why should you bear the cost when everyone gets the benefit? For simplicity let's consider a two player version, normalize the benefit of reporting the crime to one and let the cost of reporting the crime be c, $0 < c < 1$. Then the game is:

	R	N
Report the crime (R)	$1-c;1-c$	$1-c;1$
Not report the crime (N)	$1;1-c$	$0;0$

Here you should be able to show that the two pure strategy Nash equilibria are (R,N) and (N,R). (In the n person version, the equilibria are exactly one person reports.) But which equilibrium are we going to play? I think it's obviously the one where you report, oh, you disagree do you? Well I'm the teacher so there!

But what do we do when such communication is not possible? It should be obvious that we will fail to coordinate one of the n equilibria. This leads to the appeal of *symmetric* equilibria.

Definition 1 *A* symmetric equilibrium *is one where every player uses the same (possibly mixed) strategy. To be precise, I can write the equilibrium strategy without reference to the person using it.*[1]

[1] In games of incomplete information, my strategy might depend on some unknown parameter (for example maybe you don't know my value for c) but I can still write it as one function of that parameter for all people.

Remark 2 *A symmetric equilibrium is appealing because no communication between parties is ever necessary. One just does what is expected.*

Unfortunately, in many games, the symmetric equilibrium is in mixed strategies, for example here let the probability you report be p ($\Pr(R) = p$), then we can find an equilibrium in symmetric strategies. The expected utilities are:

$$\begin{aligned} U(R,p) &= 1-c \\ U(N,p) &= \Pr(\text{someone else reports}), \end{aligned}$$

here I have written the second one in such a general form because we will make use of this later. Obviously in our two player game $\Pr(\text{someone else reports}) = p$.

Then we make use of the Proposition that in a mixed strategy equilibrium all strategies have the same expected utility to conclude:

$$\begin{aligned} U(N,p) &= U(R,p) \\ p^* &= 1-c. \end{aligned}$$

Notice one of the unfortunate implications of this is that now there is a positive probability that no one reports:

$$\Pr(\text{no reports}) = (1-p^*)^2 = (1-(1-c))^2 = c^2 > 0.$$

1.1 The 3 (n) Player Version

We now want to extend to 3 (n) players. Even in the three player version $\Pr(\text{someone else reports})$ is becoming very complicated. If you are player 1, then this is the sum of three different events: Player 2 reports but 3 does not; Player 3 reports but 2 does not; and both other players report. Instead of figuring out this complicated sum there is a simpler way to proceed. We look at the complimentary event to $\Pr(\text{someone else reports})$:

$$\Pr(\text{someone else reports}) + \Pr(\text{no one else reports}) = 1$$

by definition, and $\Pr(\text{no one else reports}) = (1-p)^{n-1}$ for any n. (Notice you have decided not to report, so that only leaves $n-1$ to report.) So:

$$\begin{aligned} \Pr(\text{someone else reports}) &= 1 - \Pr(\text{no one else reports}) \\ &= 1-(1-p)^{n-1} \end{aligned}$$

Thus in the three player version:

$$\begin{aligned} U(N,p) &= U(R,p) \\ 1-(1-p)^2 &= 1-c \\ c &= (1-p)^2 \\ c^{\frac{1}{2}} &= 1-p \\ p &= 1-c^{\frac{1}{2}}. \end{aligned}$$

Let $p^*(n)$ be the equilibrium with n players, then $p^*(n) = 1 - c^{\frac{1}{n-1}}$.

Notice first of all that $p^*(2) > p^*(3)$ because:

$$\begin{aligned} p^*(2) &= 1 - c > 1 - c^{\frac{1}{2}} = p^*(3) \\ 1 - c &> 1 - c^{\frac{1}{2}} \\ c^{\frac{1}{2}} &> c \\ c &> c^2 \end{aligned}$$

and this is true because $c < 1$. Intuitively that should not be too surprising, each person is less likely to report the crime if there are more people around. However, what is $\Pr(\text{no reports}|n)$?

$$\begin{aligned} \Pr(\text{no reports}|n) &= (1 - p^*(n))^n \\ &= \left(1 - \left(1 - c^{\frac{1}{n-1}}\right)\right)^n \\ &= c^{\frac{n}{n-1}} \end{aligned}$$

and with some work one can see that this is *increasing* in n. This is easiest to see when $n = 3$;

$$\begin{aligned} \Pr(\text{no reports}|3) &= (1 - p^*(3))^3 \\ &= \left(1 - \left(1 - c^{\frac{1}{2}}\right)\right)^3 \\ &= c^{\frac{3}{2}} \end{aligned}$$

and indeed

$$\begin{aligned} \Pr(\text{no reports}|3) &> \Pr(\text{no reports}|2) \\ c^{\frac{3}{2}} &> c^2 \\ c^3 &> c^4 \\ 1 &> c. \end{aligned}$$

So... the situation is getting worse and worse. The more people are around, the less likely that anyone will report.

1.2 Empirical Examples of the Reporting a Crime Game

Analysis of this game was kicked off by the murder of Kitty Genovese on March 13, 1964. Over the course of one and a half hours somewhere between 12 and 38 people heard the slow murder of Catherine ("Kitty") Genovese. She was screaming, begging for help, and yet not a single person helped her. The press went into an uproar "people in cities are just no good," etcetera, etcetera. However I ask you, take one of those people to a National park—where they know they are the only one to hear the crime—do you still think they wouldn't report it? Wouldn't try to intervene themselves?

The problem with this situation, of course, is that we have no way of knowing how many people actually heard the murder. There were no video cameras. However on 13 October 2011 near Foshan, China there was a video camera. Baby Yue-Yue was the child of people who sold goods in an open air bazaar. While her parents were setting up the stand she wandered off—during rush hour. Yes, brace yourself, she is going to die. But that wasn't the worst bit. The worst bit was that after she was hit by a car 18 people drove past her body laying in the street and didn't do anything. These people often had to carefully drive around her body in the intersection, slowing down to walking pace, but went on their way. In their defence it was rush hour, and who knows how their boss would have responded to them being late to work because some kid got hit by a car? She was eventually rescued by a "scavenger" which I believe means a trash picker—someone who collects things like plastic bottles and takes them to a recycling center. Chen Xianmei (58) helped her out and got her to a hospital—but as I said the doctors did not succeed.[2]

CNN had experts on, of course, and the experts said two things. First, if you are hurt you better hope not many people are around. Second, you better hope that people are not busy (for example, rushing to work). In other words, the top sociological experts basically believe that the equilibrium we have just specified is the best explanation of how these things occur in the real world.

2 Reporting a Crime when other's costs are unknown—Asymmetric Information.

I submit to you, however, that a mixed strategy is not a very good way to describe what you would actually do in that situation. I personally know that I would consider my personal cost of reporting (at that date and time) and if it was low enough I would report.[3] Formalizing this strategy would be something like:

$$S(c_i) = \begin{cases} R & \text{if } c_i \le c^* \\ N & \text{if } c_i \ge c^* \end{cases},$$

notice that if $c_i = c^*$ then I would be indecisive, but this is rarely going to happen. This is called a *cut-off strategy* in economics. The problem, of course, is that I do not know your costs. At most I might have some expectation of them. If the costs have the distribution F ($F(x) = \Pr(c_j \le x)$) then now:

$$U(N, c^*) = F(c^*).$$

At this point we reach the critical lesson in this handout, the critical thing you need to recognize.

[2] I should mention that China does not have "good samaritan" laws. These laws protect someone who is trying to help someone else out from being criminally liable if something goes wrong. Thus Chen Xianmei was technically putting herself at risk.

[3] Obviously, my moral feelings would affect this cost.

Remark 3 *In a world of asymmetric information—where you do not know other's payoffs—it is* as if *your opponents are using a mixed strategy.*

In this game you can see this by recognizing that previously the probability someone else reported was p^*, now it is $F(c^*)$. Furthermore, previously we used a proposition to say that at p^* you would be indifferent between reporting and not reporting, now we simply use the fact that at c^* both must be best responses:

$$\begin{aligned} U(N, c^*) &= U(R, c^*) \\ F(c^*) &= 1 - c^*. \end{aligned}$$

Which will always result in a value for c^*. For example if c is distributed uniformly over $[a, b]$ $(b > a)$ then $F(c^*) = \frac{c^*-a}{b-a}$,

$$\begin{aligned} \frac{c^* - a}{b - a} &= 1 - c^* \\ c^* - a &= b - a - c^*(b - a) \\ c^* + c^*(b - a) &= (1 + b - a) c^* = b \\ c^* &= \frac{b}{b - a + 1}. \end{aligned}$$

(Note that for consistency we must have $a < 1$, which is hardly an assumption.)

3 The Flirting Game (Battle of the Sexes) with Asymmetric Information

I am sure you know that the main question when you are flirting with someone is whether that other person likes you. For simplicity, let's assume she or he is sure you like them. As far as you are concerned, there are two states of the world. The good one (γ) where she or he likes you and the bad one (β) where she or he doesn't.

As usual we will assume their are two coffee shops (Starbucks, S, or Mozarts, M) and that each of you gets a utility of one from going to your favorite coffee shop. You always get a utility of 2 from being with that special someone, and in the good state of the world (γ) she or he feels the same—but in the bad state of the world (β) she or he gets a utility of two from not meeting you. If you like Starbucks and the other person likes Mozarts then the game would be:

γ

	M	S
M	$2;3$	$0;0$
S	$1;1$	$3;2$

β

	M	S
M	$2;1$	$0;2$
S	$1;3$	$3;0$

,

Pr p $\qquad$ $(1 - p)$

Notice in order to complete the model we need to know the probability of γ and β, here $\Pr(\gamma) = p$.

Now how do we proceed? What we are going to do is basically find the equilibria by iterating best responses. First we try out each of your strategies, then find the other person's best response (which will depend on the state of the world) and then we find out if your original strategy is a best response to the best response of the other person. Let me lay it out in a table:

s_1	$BR_2(s_1)$	$s_1 = BR_1(BR_2(s_1))$?
M	$(M(\gamma), S(\beta))$	(to be completed)
S	$(S(\gamma), M(\beta))$	(to be completed)

To complete the table we have to find out the expected utilities of the two strategies of Player 1 in response to Player 2:

$$\begin{aligned} U_1(M,(M(\gamma),S(\beta))) &= pU_1(M,M) + (1-p)U_1(M,S) \\ &= p(2) + (1-p)(0) = 2p. \end{aligned}$$

Notice how this is essentially identical to saying that player 2 will play M with probability p and S with probability $1-p$. Now the expected utility of S:

$$\begin{aligned} U_1(S,(M(\gamma),S(\beta))) &= pU_1(S,M) + (1-p)U_1(S,S) \\ &= p(1) + (1-p)(3) = 3-2p. \end{aligned}$$

In order for this to be an equilibrium we need M to be the best response, or:

$$\begin{aligned} U_1(M,(M(\gamma),S(\beta))) &\geq U_1(S,(M(\gamma),S(\beta))) \\ 2p &\geq 3-2p \\ 4p &\geq 3 \\ p &\geq \frac{3}{4}. \end{aligned}$$

Thus if $p \geq \frac{3}{4}$ there is a pure strategy equilibrium (M), $(M(\gamma), S(\beta))$.

Now we do the same thing for the other potential equilibrium:

$$\begin{aligned} U_1(S,(S(\gamma),M(\beta))) &= pU_1(S,S) + (1-p)U_1(S,M) \\ &= p(3) + (1-p)(1) = 2p+1, \end{aligned}$$

$$\begin{aligned} U_1(M,(S(\gamma),M(\beta))) &= pU_1(M,S) + (1-p)U_1(M,M) \\ &= p(0) + (1-p)(2) = 2-2p, \end{aligned}$$

$$\begin{aligned} U_1(S,(S(\gamma),M(\beta))) &\geq U_1(M,(S(\gamma),M(\beta))) \\ 2p+1 &\geq 2-2p \\ 4p &\geq 1 \\ p &\geq \frac{1}{4}. \end{aligned}$$

And we can finalize our table:

s_1	$BR_2(s_1)$	$s_1 = BR_1(BR_2(s_1))$?
M	$(M(\gamma), S(\beta))$	If $p \geq \frac{3}{4}$
S	$(S(\gamma), M(\beta))$	If $p \geq \frac{1}{4}$

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Finally, it's time for my favorite question. What if $p < \frac{1}{4}$? What happens then? Is there a Nash equilibrium? Yes, there is always a Nash equilibrium. Is it in pure strategies? No, we've checked the two possibilities in pure strategies. Thus it must be in mixed strategies. In other words, player 1 (who doesn't know if P2 likes him or her) randomizes over the two coffee shops in order to trick P2 into showing up and flirting with him or her. Just a suggestion, but perhaps you should find someone else to flirt with?

Generally speaking, from now on we will not be interested in mixed strategy equilibria, instead nature will make our universe random and we will just try to find a simple path through (a pure strategy). If you find a region where there is only a mixed strategy equilibrium, just note it in your answer and move on.[4]

[4] Unless your evil teacher explicitly tells you to find it, of course. Man, he's such a meany.