1 Domination and Iterated Domination.

Just to remind you of some basic terminology, a game in strategic or normal form is a triplet, $G = \{I, A, u\}$ where $I = \{1, 2, 3, ...I\}$ is a finite set of players $A = \times_{i \in I} A_i$ is a finite set of action profiles, and $u : A \to R^I$ is a payoff or utility function. Note that player i chooses an action in the set A_i and gets the payoff $u_i \in u$.

1.1 Dominant and Dominated Strategies.

Perhaps one of the reasons that the Prisoner′s Dilemma is so compelling is that it has a dominant strategy.

Definition 1 *Player $i \in I$ has a* dominant strategy *if no matter what actions his opponents take his optimal action is the same.*

The strategy therefore is "dominant" because it always is better than every other strategy. In the Prisoner′s Dilemma this strategy is "Squeal" or "Do not cooperate" in many other applications of the game. Let me give a generic version of the game:

	Cooperate	Do not Cooperate
Cooperate	8,8	-1,11[2]
Do not Cooperate	11,-1[1]	0,0[12]

The best responses of both players are marked by a 1 or 2 in the upper right hand corner of the box, and as you can see the best response does not depend on the other player′s action. Thus there is a dominant strategy. This makes $\{D, D\}$ a very compelling candidate for equilibrium. What are you going to say? That players should not take a dominant action? Clearly if they are at all rational they should be able to find a dominant action. The "dilemma" in the Prisoner′s Dilemma is that when people follow the dominant strategy it results in a socially undesirable outcome.

Now consider the following arbitrary game:

	A	B	C
A	4,4[12]	3,3[1]	0,1
B	3,3[2]	2,2	4,1[1]
C	1,0	1,4[2]	3,3

Again I have marked the best responses in the upper right hand corner, and we can see that there is no dominant strategy in this game. But notice something, C always gives a worse payoff than B in this game. So would we ever expect someone to use the action C? No, using B is always a best option. C is a *dominated* action.

Definition 2 *An action is* dominated *if there is another action that always gives a **strictly** higher payoff.*

The word “strictly” is very important in this case. There is another concept called “weak dominance.”

Definition 3 *An action is* weakly dominated *if there is another action that always gives a higher payoff (and strictly in at least one case), but no action that always gives a strictly higher payoff.*

This definition benefits from a more precise statement. $a_i \in A_i$ *is* weakly dominated *if there is an* $a_i' \in A_i$ *such that for all* $a_{-i} \in A_{-i}$, $u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i})$ *and there exists a* $a_{-i}' \in A_{-i}$ *such that* $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$.

In most cases the difference between “strict” and “weak” is not important, but it is here. There are games where the only equilibrium is in weakly dominated strategies. Consider the following simplified Bertrand game.

Example 4 (Simplified Bertrand) *Assume that there are two players and their payoffs are:*

$$\begin{aligned} p_i \text{ if } p_i &< p_j \\ \frac{1}{2}p_i \text{ if } p_i &= p_j \\ 0 \text{ if } p_i &> p_j \end{aligned}$$

Notice that for every $p_i > 0$ *the lowest the payoff can be is zero, and for some* p_j *the payoff is strictly greater than zero. Thus* $p_i > 0$ *weakly dominates* $p_i = 0$. *However one can check that the Nash Equilibrium is* $p_1 = p_2 = 0$.

This **can not** happen with dominated strategies. Since B is always *strictly* better than C there can not be a Nash Equilibrium where C is played.

1.2 Iterated Domination

Now clearly no rational person would ever use a dominated strategy, right? So a clear implication of rationality is that no one should use dominated strategies. This is rather exciting, because it gives us some more restrictions on how rational agents should behave. But consider the following game:

	A	B	C	D	E
A	$9,9^{12}$	$6,7^{1}$	1,5	2,2	2,1
B	$7,6^{2}$	5,5	$3,4^{1}$	2,3	1,0
C	5,1	$4,3^{2}$	2,2	$7,1^{1}$	1,0
D	2,2	3,2	$1,7^{2}$	4,4	$3,3^{1}$
E	1,2	0,1	0,1	$3,3^{2}$	2,2

In this game one can see that D dominates E, but every other strategy is a best response and so it can not be dominated. But now look at the game where E is removed:

	A	B	C	D
A	$9,9^{12}$	$6,7^{1}$	1,5	2,2
B	$7,6^{2}$	5,5	$3,4^{1}$	2,3
C	5,1	$4,3^{2}$	2,2	$7,1^{1}$
D	2,2	3,2	$1,7^{2}$	4,4

Now D is never a best response. Is it dominated? Yes it is, by C. Now how can we justify playing D anymore? It is no longer a best response, and it is always a worst strategy than C. Thus the same process that ruled out E rules out D. Thus a rational person should use neither strategy D or E.

But once we have done this once we should do it again, should we not? When we repeat this process:

	A	B	C
A	$9,9^{12}$	$6,7^{1}$	1,5
B	$7,6^{2}$	5,5	$3,4^{1}$
C	5,1	$4,3^{2}$	2,2

we see that we can now rule out C, and then with C deleted from the game B is now dominated, leaving only one un-dominated strategy. A, the strategy which is in the Nash Equilibrium.

This process is called the *iterated deletion of dominated strategies.*

Definition 5 *The set of* un-dominated strategies *is the largest set of strategies A^* such that every strategy in A_i^* is not dominated with respect to $A_{-i}^* = \times_{j \neq i} A_j^*$. Or for every $a_i \in A_i^*$ there is no $a_i' \in A_i^*$ such that for every $a_{-i} \in A_{-i}^*$ $u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$.*

Notice the importance of the word "largest" in this definition. Any action profile (list of one action for each person) is un-dominated by definition. You can find this set by deleting dominated strategies in any order at all.

Proposition 6 *Let A^1 be a subset of A such that at least one dominated strategy in A is not in A^1 (if possible). Let A^n be a subset of A^{n-1} such that at least one dominated strategy in A^{n-1} is not in A^n (if possible). Then:*

$$A^* = \lim_{n \to \infty} A^n$$

and the limit is unique.

I will not ask you to prove this statement, proving such statements is not really what I want you to focus on in this class. However it is true. *And this result is vitally important.*

Why is this result so important? Imagine if it was not true. Then by deleting actions in different order you could end up with a different set of strategies. This would mean that using an un-dominated strategy was not really a characteristic of rational people, instead it would be a logical conclusion similar to equilibrium.

Now before you get all excited let me show you the following sequential game (or extensive form game of perfect information), called the *centipede game*.

$$\begin{array}{cccccccccc}
 & & \mathrm{I}_1 & & & \mathrm{I}_2 & & & \mathrm{I}_1 & \\
 & 1 & \longrightarrow & & 2 & \longrightarrow & & 1 & \longrightarrow & (\text{go to } \alpha) \\
\mathrm{O}_1 & \downarrow & & \mathrm{O}_2 & \downarrow & & \mathrm{O}_1 & \downarrow & & \\
 & \{1,0\} & & & \{0,3\} & & & \{3,2\} & &
\end{array}$$

$$\begin{array}{ccccccccccc}
 & & & \mathrm{I}_2 & & & \mathrm{I}_1 & & & \mathrm{I}_2 & \\
\alpha & \longrightarrow & 2 & \longrightarrow & & 1 & \longrightarrow & & 2 & \longrightarrow & \{11,11\} \\
 & \mathrm{O}_2 & \downarrow & & \mathrm{O}_1 & \downarrow & & \mathrm{O}_2 & \downarrow & & \\
 & & \{2,6\} & & & \{9,5\} & & & \{8,12\} & &
\end{array}$$

Each person when they move has the choice of playing I_i (for *in*) or O_i (for *out*). If they choose "in" then they pass the decision to the other player. Let us solve this by backward induction. first clearly at his last decision node player 2 is going to choose O_2 because 12 is higher than 11, thus I_2 is dominated at the last decision node. Given this O_1 is the dominant strategy for player 1 at his last decision node, iterating backward we find that always playing "out" is the only un-dominated strategy. But wait a second, what if player 2 was not perfectly rational? What if player 2 played "in" at his first decision node? Then the worst player 1 could do was 3 instead of 1. Is it really rational for player 1 to always play out?

This game has been tested experimentally because the "rational" thing to do has such an awful outcome. What they find is that players choose "in" until a certain number of periods before the end. Then someone plays out and the game ends before achieving the final payoff. So what lesson should an economist take from this example? Not, as one might first think, that people are not rational. Rather they do not believe that the person they are playing with is always rational, thus they will not iterate a dominance relationship forever and ever. Thus we can not iterate a dominance relationship forever, in reality one should only apply a few iterations of dominance if one wants to be certain people are not going to violate your prediction.

Example 7 *One of the most beautiful applications of iterated dominance is our old standby the Cournot game. In this game there are two oligopolists who compete by choosing quantity. The inverse demand curve is* $P = a - bQ$ *and the cost function of both firms is* $c(q) = cq$. *The objective function of the first firm is then:*

$$\max_{q_1} R(q_1, q_2) - c(q_1)$$
$$\max_{q_1} (a - b(q_1 + q_2))\, q_1 - cq_1$$

and by finding the reaction function (or best response) one can see that:

$$q_1 = \frac{1}{2}\frac{a-c}{b} - \frac{1}{2}q_2$$

q_1 *is decreasing in* q_2. *Thus the highest value that* q_1 *can ever rationally take is found when* q_2 *is at its minimum possible value, or* 0. *This maximum is*

$q_1 = \frac{1}{2}\frac{a-c}{b}$. Thus we can conclude that q_1 should always be in the interval $\left[0, \frac{1}{2}\frac{a-c}{b}\right]$, any quantity greater than $\frac{1}{2}\frac{a-c}{b}$ is dominated by $\frac{1}{2}\frac{a-c}{b}$.

This game is symmetric, so we can also conclude the same things about q_2. But this means the lowest q_1 should be is when it is best responding to q_2s maximum value, so

$$q_1 \geq \frac{1}{2}\frac{a-c}{b} - \frac{1}{2}\left(\frac{1}{2}\frac{a-c}{b}\right) = \left(\frac{1}{2} - \frac{1}{4}\right)\frac{a-c}{b} = \frac{1}{4}\frac{a-c}{b},$$

so again by symmetry $q_2 \in \left[\frac{1}{4}\frac{a-c}{b}, \frac{1}{2}\frac{a-c}{b}\right]$, but then:

$$q_1 \leq \frac{1}{2}\frac{a-c}{b} - \frac{1}{2}\left(\frac{1}{4}\frac{a-c}{b}\right) = \left(\frac{1}{2} - \frac{1}{8}\right)\frac{a-c}{b} = \frac{3}{8}\frac{a-c}{b},$$

and $q_2 \in \left[\frac{1}{4}\frac{a-c}{b}, \frac{3}{8}\frac{a-c}{b}\right]$, then $\left[\frac{5}{16}\frac{a-c}{b}, \frac{3}{8}\frac{a-c}{b}\right]$, $\left[\frac{5}{16}\frac{a-c}{b}, \frac{11}{32}\frac{a-c}{b}\right]$, $\left[\frac{21}{64}\frac{a-c}{b}, \frac{11}{32}\frac{a-c}{b}\right]$...

To figure out if these two limits converge you just need to calculate the sequence for the maximum and the minimum.

$$\begin{aligned} m_1 &= \frac{1}{2} - \frac{1}{2}l_0 \\ l_1 &= \frac{1}{2} - \frac{1}{2}m_1 \end{aligned}$$

thus

$$\begin{aligned} m_n &= \frac{1}{2} - \frac{1}{2}l_{n-1} \\ l_n &= \frac{1}{2} - \frac{1}{2}m_{n-1} \end{aligned}$$

and in the limit when $m_n = m_{n-1} = m$ and $l_n = l_{n-1} = l$

$$\begin{aligned} m &= \frac{1}{2} - \frac{1}{2}l \\ l &= \frac{1}{2} - \frac{1}{2}m \end{aligned}$$

or

$$\begin{aligned} l &= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}l\right) \\ l &= \frac{1}{3} \\ m &= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{3} \end{aligned}$$

since the limits are equal, this game can be solved by iterated deletion of dominated strategies.

This is a truly amazing magic trick. Notice that it works for any $\{\alpha, \beta\}$ *if* $\beta + 1 \geq \alpha > \beta > 0$*, as shown below:*

$$
\begin{aligned}
m &= \alpha - \beta l \\
l &= \alpha - \beta m \\
l &= \alpha - \beta (\alpha - \beta l) \\
l &= \frac{\alpha}{\beta + 1} \\
m &= \alpha - \beta \left(\frac{\alpha}{\beta + 1} \right) = \frac{\alpha}{\beta + 1}
\end{aligned}
$$

however this does not work in most other "natural" games. Again, how did Cournot do it? Not only did he find a Nash equilibrium but he described a natural game where it could be solved by iterated deletion of dominated strategies.

1.3 What is wrong with Iterated Deletion of Weakly Dominated strategies?

Above I gave a game where the only Nash equilibrium is in weakly dominated strategies, but I would like to further explain the problem with this concept. Consider the following game

	A	B	C	D
A	0,0	$6,0^{1}$	$1,1^{12}$	$0,1^{2}$
B	$0,6^{2}$	$6,6^{12}$	$1,0^{1}$	-1,-1
C	$1,1^{12}$	$0,1^{2}$	0,0	$8,0^{1}$
D	$1,0^{1}$	-1,-1	$0,8^{2}$	$8,8^{12}$

.

In this game there are four Nash equilibria and two weakly dominated strategies, B and D. B is weakly dominated by A, D is weakly dominated by C. However if you remove D first then B is not weakly dominated in the rest of the game, if you remove B first then D is not weakly dominated in the rest of the game. If you remove both B and D simultaneously you end up with another game. Thus depending on your order of deletion you end up with different games, with different equilibria. Furthermore if you apply the (appealing but arbitrary) criteria of selecting Pareto efficient equilibria then the equilibrium you select will depend on the order of deletion.

If that is not enough to disturb you, well, I must say you have an iron constitution. One can argue that removing weakly dominated strategies is appealing, but it is not without problems. Many people (including the book) prefer equilibria which are not weakly dominated, and argue that one removal of weakly dominated strategies is not a bad idea. I do not find it that appealing, though as I will show in some examples the "intuitive" equilibrium is sometimes found by removing weakly dominated actions.

There is a very simple argument in favor of one iteration of removing weakly dominated strategies, imagine that with a trivial probability ε a player uses a strategy at random, then any weakly dominated strategy is also strictly dominated. But is that a good criterion? What about when it rules out Pareto efficient equilibria? What about when it rules out the only Nash equilibrium?

1.4 Rationalizability—the next step

Can we go further? Yes we can if we work with beliefs instead of just strategies. For any i $\Delta(A_i)$ is the set of mixtures over A_i, in other words it is the set of $p : A_i \rightarrow [0,1]^{|A_i|}$ such that $\Sigma_{a_i \in A_i} p(a_i) = 1$. Then an *independent belief* for player i is $\beta_i \in \times_{j \neq i} \Delta(A_j)$.

Definition 8 *A strategy* $\sigma_i \in \Delta(A_i)$ *is* never a weak best response *if for all* $\beta_i \in \times_{j \neq i} \Delta(A_j)$ $\sigma_i \notin \max_{\sigma_i' \in \Delta(A_i)} u_i(\sigma_i', \beta_i)$.

The set of *rationalizable strategies* is the set of strategies that survives iterated removal of strategies that are never weak best responses. One can show that this is the smallest set of "rational" strategies, but finding it is very difficult, which is why Osborne doesn't approach this subject until very late in the book. We will not cover this due to it's advanced nature.