The Market for Lemons, A precise treatment.
By Kevin Hasker

In this model the seller knows the value of the good they want to sell, we will denote this as $w \in [w^-, w^+]$ and we will assume that all consumer know is that it is uniformally distributed over this interval, so the density (or probability density function) is $\frac{1}{w^+ - w^-}$. Consumers will not know the value of the good, but their personal value is $v \in [v^-, v^+]$ and again their value is uniformally distributed over that interval with density $\frac{1}{v^+ - v^-}$. The value of a consumer who receives a car of value w is vw.

To make anaysis easier in this case we are going to treat both sellers and buyers as if there are a continuum of them. The mass of buyers will be M the mass of sellers will be N.

1   The Demand Curve:

Now what will be the demand if the average good has a value of $\bar{w}$ and they are charged a price of P? Traditionally this is:

$$D(P) = \{\text{number of consumers for whom } v\bar{w} \geq P\} \ .$$

Now in this market it is actually easier to work with the inverse demand curve:

$$P(Q) = \{\text{value of the marginal consumer for whom } v^*\bar{w} \geq P\}$$

If $Q = 0$ then this is v^+ if $Q = M$ then this is v^-, thus the we know that

$$
\begin{aligned}
P(0) &= v^+\bar{w} \\
P(M) &= v^-\bar{w}
\end{aligned}
$$

and in between due to the uniform distribution it is going to have a linear relationship, so the only solution to this problem is:

$$P(Q) = v^+\bar{w} - \frac{v^+\bar{w} - v^-\bar{w}}{M}Q \ .$$

If you check the two conditions above then this becomes obvious. Or

$$P(Q) = \left[v^+ - \frac{v^+ - v^-}{M}Q\right]E(w) \ .$$

Now in equilibirum a car will only be offered if $P \geq w$ so in equilibrium this must be:

$$P(Q) = \left[v^+ - \frac{v^+ - v^-}{M}Q\right]E(w|w \leq P) \ .$$

Now we must calculate $E\left(w|w \leq P\right)$.

$$
\begin{aligned}
E\left(w|w \leq P\right) &= \int_{w^-}^{P} \frac{\frac{1}{w^+ - w^-} w \, dw}{P\left(w \leq P\right)} \\
P\left(w \leq P\right) &= \int_{w^-}^{P} \frac{1}{w^+ - w^-} dw = \frac{P - w^-}{w^+ - w^-} \\
E\left(w|w \leq P\right) &= \int_{w^-}^{P} \frac{\frac{1}{w^+ - w^-} w \, dw}{\frac{P - w^-}{w^+ - w^-}} = \int_{w^-}^{P} \frac{1}{w^+ - w^-} \frac{w^+ - w^-}{P - w^-} w \, dw\backslash \\
&= \int_{w^-}^{P} \frac{1}{P - w^-} w \, dw \\
&= \frac{1}{P - w^-} \left[\frac{w^2}{2}\right]_{w^-}^{P} \\
&= \frac{1}{P - w^-} \frac{P^2 - \left(w^-\right)^2}{2} \\
&= \frac{P}{2} + \frac{w^-}{2}
\end{aligned}
$$

so the demand curve is:

$$
P = \left[v^+ - \frac{v^+ - v^-}{M} Q\right]\left[\frac{P}{2} + \frac{w^-}{2}\right] .
$$

2 The Supply Curve:

Like in the case of demand it is easier to figure out the marginal value of the supplier who will just enter the market:

$$
\begin{aligned}
P\left(0\right) &= w^- \\
P\left(N\right) &= w^+
\end{aligned}
$$

and the tradeoff in between is linear, so the inverse supply curve is:

$$
P\left(Q\right) = w^- + \frac{w^+ - w^-}{N} Q .
$$

3 The equilibrium:

Let

$$
v^* = v^+ - \frac{v^+ - v^-}{M} Q
$$

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then

$$P = v^* \left[\frac{P}{2} + \frac{w^-}{2} \right]$$

$$\left(1 - \frac{v^*}{2}\right) P = v^* \frac{w^-}{2}$$

$$P = \frac{v^*}{2 - v^*} w^-$$

and so:

$$w^- + \frac{w^+ - w^-}{N} Q = \frac{v^*}{(2 - v^*)} w^-$$

$$\frac{w^+ - w^-}{N} Q = \left(\frac{v^*}{(2 - v^*)} - 1\right) w^-$$

$$\frac{w^+ - w^-}{N} Q = \left(\frac{v^* - (2 - v^*)}{(2 - v^*)}\right) w^-$$

$$\frac{w^+ - w^-}{w^-} \frac{Q}{N} = \frac{2v^* - 2}{(2 - v^*)}$$

$$\frac{w^+ - w^-}{w^-} \frac{Q}{N} = \frac{2\left(v^+ - \frac{v^+ - v^-}{M} Q\right) - 2}{2 - \left(v^+ - \frac{v^+ - v^-}{M} Q\right)}$$

$$\frac{w^+ - w^-}{w^-} \frac{Q}{N} \left(2 - \left(v^+ - \frac{v^+ - v^-}{M} Q\right)\right) = 2\left(v^+ - \frac{v^+ - v^-}{M} Q - 1\right)$$

and this is a quardratic equation. I do not expect you to solve quadratic equations. However it is possible to solve it if $w^- = 0$, then in this case:

$$P = v^* \frac{P}{2}$$

$$v^* = 2$$

in this case we can figure out the quantity demanded from the demand curve,

$$2 = v^+ - \frac{v^+ - v^-}{M} Q$$

$$\frac{v^+ - v^-}{M} Q = v^+ - 2$$

$$Q = \frac{v^+ - 2}{v^+ - v^-} M$$

Notice that if $v^+ < 2$ then the market collapses, the quantity is negative. Now the final unkown is the price, it didn't matter when we calculated the marginal value, and so we haven't found it yet. We get this from the supply curve:

$$P = 0 + \frac{w^+ - 0}{N} Q$$

$$= \frac{w^+}{N} \frac{v^+ - 2}{v^+ - v^-} M \ .$$

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remember this is also the quality of the marginal car offered for sale. Notice that price is increasing in total quantity demanded (M) but decreasing in $\frac{v^+-2}{v^+-v^-}$. The reason for this is because part of the demand, $\frac{2-v^-}{v^+-v^-}$, will never be supplied so if that term increases (which would mean that $\frac{v^+-2}{v^+-v^-} = 1 - \frac{2-v^-}{v^+-v^-}$ decreases) there are more demanders who are not being supplied. It is also decreasing in supply (N) and increasing in the maximum value of a good that might be offered on the market (w^+) which of course means that the unconditional quality of the average car $(\frac{w^+}{2})$ is increasing.

4 A simplified model:

Now, how could I vary this to make it simpler for an exam? Well I could tell you that the inverse demand curve was

$$P = (a - bQ) \, E\left[w|w \leq P\right]$$

and that the supply curve was
$$P = cQ$$

and that w is distributed uniformally over $[0, w^+]$ then the equlibrium would be:

$$
\begin{aligned}
P &= (a - bQ)\frac{P}{2} \\
Q &= \frac{1}{b}(a - 2) \\
P &= \frac{c}{b}(a - 2)
\end{aligned}
$$

which is much simpler to solve, though perhaps the intuition behind it is not as rich as the fully specified model above.