1 The CES function in General Equilibrium

This handout is devoted to analysis of the CES function in General Equilibrium. You should recognize that there is relatively little difference between utility and production, the only fundamental difference is that with utility you get a budget constraint. This only matters when we actually derive the equilibrium, which isn't the subject of most of this handout, so we will ignore the difference. Actually there is one other difference, with a production function one has to keep track of the ordinal transformations that you make, so I will write this handout using the production function and you can just relabel the analysis for consumtion. The CES function is:

$$q = \begin{cases} A\left(\alpha k^{\sigma} + (1-\alpha) l^{\sigma}\right)^{\frac{\rho}{\sigma}} & \sigma \neq 0 \\ A k^{\alpha\rho} l^{(1-\alpha)\rho} & \sigma = 0 \end{cases}$$

The parameters of the function are:

- $A > 0$—total factor productivity, in essence a natural endownment.
- $\sigma \leq 1$—controls the elasticity of substitution, or how much you can trade off one good for the other. If $\sigma = 1$ you can trade them off at a constant ratio, as σ gets smaller (it can be negative, and actually converge to negative infinity) you are less and less able to trade off one good for the other and if you don't have one good you can't produce output. This also means, by the way, that the isoquant will be come more curvy. The formula for the elasticity of substitution is $\frac{1}{1-\sigma}$, but I just am telling you that for your information.
- $\alpha \in [0, 1]$—relative importance of the two goods is determined by $\frac{\alpha}{1-\alpha}$.
- $\rho > 0$—the returns of scale, if $\rho \leq 1$ you have decreasing returns to scale, which is the normal case.

Now last semester you learned that utility functions are *ordinal* or if $g' > 0$ then $u(f, c)$ and $g(u(f, c))$ where the same utility function. A similar thing is true for production functions, except that you have to keep track of what $g(q)$ is, so while this is the standard method to present the function it is almost always easier to analyze ($\sigma \neq 0$). In other words if I am looking at a production problem I can analyze:

$$\begin{aligned} \frac{1}{\sigma}\left(\frac{q}{A}\right)^{\frac{\sigma}{\rho}} &= \frac{\alpha}{\sigma} k^{\sigma} + \frac{(1-\alpha)}{\sigma} l^{\sigma} & \sigma \neq 0 \\ \left(\frac{q}{A}\right)^{\frac{1}{\rho}} &= k^{\alpha} l^{(1-\alpha)} & \sigma = 0 \end{aligned}$$

The new problem has much simpler derivatives, and is always the same as the old one because for $x > 0$, $A > 0$ and $\rho > 0$: $g(x) = \frac{x}{A}$, $g(x) = x^{\rho}$, $g(x) = \frac{1}{\sigma} x^{\sigma}$ are all monotonic function.

1.1 The Contract Curve with CES functions.

Our general function is

$$\begin{array}{ll} \frac{1}{\sigma}\left(\frac{q_1}{A_1}\right)^{\frac{\sigma}{\rho_1}} = \frac{\alpha_1}{\sigma}k_1^{\sigma} + \frac{(1-\alpha_1)}{\sigma}l_1^{\sigma} & \sigma \neq 0 \\ \left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} = k_1^{\alpha_1}l_1^{(1-\alpha_1)} & \sigma = 0 \end{array}$$

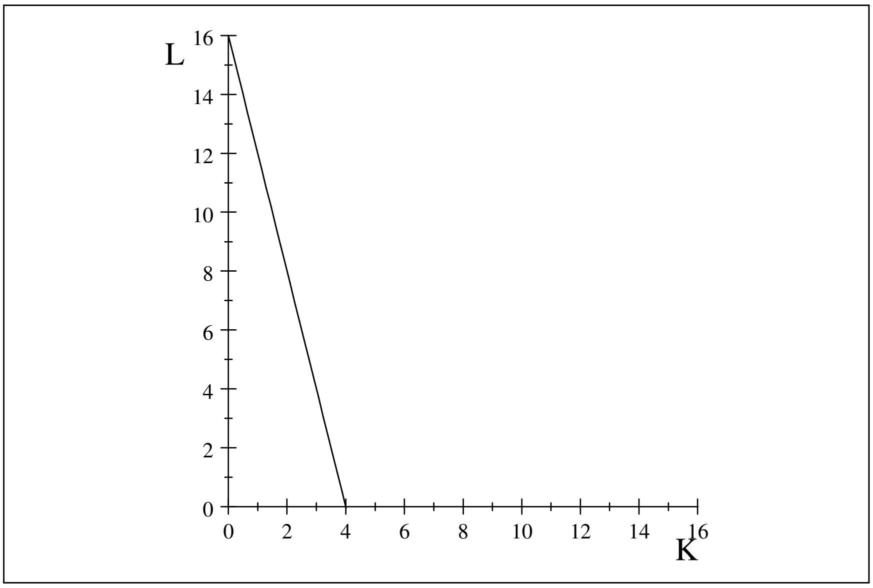
with the derivatives:

$$\begin{aligned} MU_k^1 &= \alpha_1 k_1^{\sigma-1} = \frac{\alpha_1}{k_1^{1-\sigma}} \\ MU_l^1 &= (1-\alpha_1)\, l_1^{\sigma-1} = \frac{(1-\alpha_1)}{l_1^{1-\sigma}} \\ \frac{MU_k^1}{MU_l^1} &= \frac{\frac{\alpha_1}{k_1^{1-\sigma}}}{\frac{(1-\alpha_1)}{l_1^{1-\sigma}}} = \frac{\alpha_1}{1-\alpha_1}\left(\frac{l_1}{k_1}\right)^{1-\sigma} \end{aligned}$$

which are also correct even if $\sigma = 0$, but are somewhat hard to interpret if $\sigma \to -\infty$. The first thing we want to do is get rid of the two special cases, where $\sigma = 1$ or $\sigma \to -\infty$.

1.1.1 The Perfect Substitutes Function, $\sigma = 1$

First let's look at a graph of the isuquant associated with this function:



This is for the production function where $\alpha = \frac{4}{5}$ and $\left(\frac{q}{A_1}\right)^{\frac{1}{\rho_1}} = \frac{16}{5}$. As you can see all this person cares about is the ratio at which he trades capital for labor, the marginal rate of technical substitution in this case is:

$$\frac{MU_k^2}{MU_l^2} = \frac{\alpha_2}{(1-\alpha_2)}$$

and for the other person ($\sigma_1 < 1$):

$$\frac{MU_k^1}{MU_l^1} = \frac{\frac{\alpha_1}{k_1^{1-\sigma}}}{\frac{(1-\alpha_1)}{l_1^{1-\sigma}}} = \frac{\alpha_1}{1-\alpha_1}\left(\frac{l_1}{k_1}\right)^{1-\sigma_1}$$

so the contract curve is:

$$\begin{aligned}\left(\frac{l_1}{k_1}\right)^{1-\sigma_1} &= \frac{(1-\alpha_1)\,\alpha_2}{\alpha_1\,(1-\alpha_2)} \\ \frac{l_1}{k_1} &= \left(\frac{(1-\alpha_1)\,\alpha_2}{\alpha_1\,(1-\alpha_2)}\right)^{\frac{1}{1-\sigma_1}} \\ l_1 &= \left(\frac{\alpha_2\,(1-\alpha_1)}{\alpha_1\,(1-\alpha_2)}\right)^{\frac{1}{1-\sigma_1}} k_1\end{aligned}$$

in other words the other person/firm completely determines the contract curve. Notice that the person who thinks of the goods as perfect substitutes completely determines the prices in general equilibrium.

$$\frac{MU_k^2}{MU_l^2} = \frac{\alpha_2}{(1-\alpha_2)} = \frac{r}{w}$$

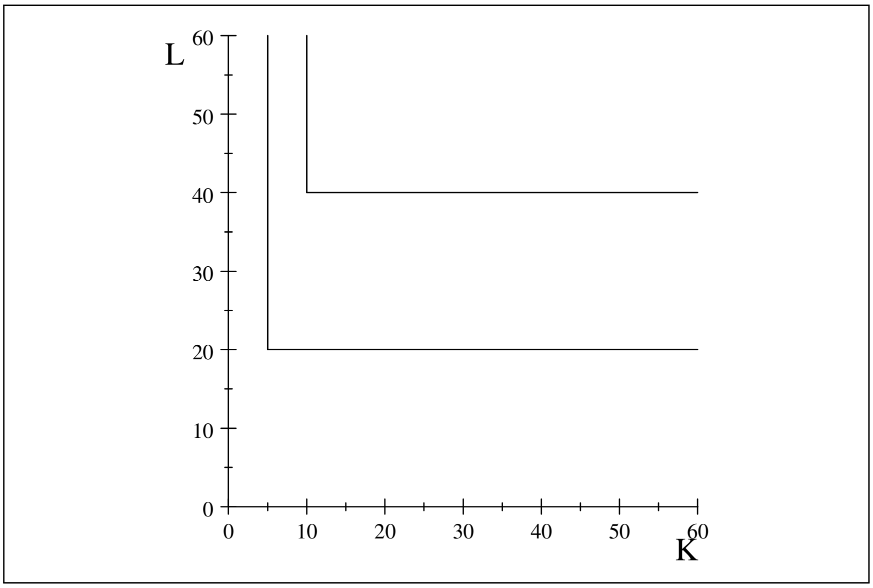
so a natural representation of prices is $r = \alpha_2$, $w = 1 - \alpha_2$.

1.1.2 The Leontief/Perfect Compliments/Fixed Proportions Function.

In this case the function is:

$$\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} = \min\left(\alpha_1 k_1, (1-\alpha_1)\, l_1\right)$$

and in order to understand what is optimal in this case let's assume that $\alpha_1 = \frac{4}{5}$ and draw a few isoquants.



You should be able to figure out from the graph that only a fool would ever produce at any point other than the corner of the isoquant, where

$$\alpha_1 k_1 = (1 - \alpha_1)\, l_1$$

if you aren't exactly clear on why think about this as right and left shoes. Would you ever go out shopping only for a left shoe? Why not? Well one reason is that you obviously are going to need a right one as well, so why not buy a pair? So if this is the case then we have the contract curve immediately, independent of what the other person wants.

$$l_1 = \frac{\alpha_1}{1 - \alpha_1} k_1$$

the prices in the general equilibrium will be completely determined by the other person. This person doesn't care about prices, they MUST have goods in that exact ratio.

1.1.3 The Contract Curve and Production Possibilities frontier when $\sigma_1 = \sigma_2 \in (-\infty, 1)$

In this case

$$\frac{MU_k^1}{MU_l^1} = \frac{\alpha_1}{1 - \alpha_1} \left(\frac{l_1}{k_1} \right)^{1-\sigma} = \frac{\alpha_2}{1 - \alpha_2} \left(\frac{l_2}{k_2} \right)^{1-\sigma} = \frac{MU_k^2}{MU_l^2}$$

$$
\begin{aligned}
\left(\frac{l_1}{k_1}\right)^{1-\sigma} &= \frac{\alpha_2\left(1-\alpha_1\right)}{\alpha_1\left(1-\alpha_2\right)}\left(\frac{l_2}{k_2}\right)^{1-\sigma} \\
\frac{l_1}{k_1} &= \left(\frac{\left(1-\alpha_1\right)\alpha_2}{\left(\alpha_1\right)1-\alpha_2}\right)^{\frac{1}{1-\sigma}}\frac{l_2}{k_2}
\end{aligned}
$$

and we now—for the first time—need to consider the resource constraints.

$$
\begin{aligned}
l_1 + l_2 &= \bar{l} \\
k_1 + k_2 &= \bar{k}
\end{aligned}
$$

and just to simplify analysis let $\chi = \left(\frac{(1-\alpha_1)\alpha_2}{(\alpha_1)1-\alpha_2}\right)^{\frac{1}{1-\sigma}}$, its just going to be some number in any problem you have to solve, so let that number be called χ.

$$
\begin{aligned}
\frac{l_1}{k_1} &= \chi\frac{\bar{l}-l_1}{\bar{k}-k_1} \\
l_1\left(\bar{k}-k_1\right) &= \chi k_1\left(\bar{l}-l_1\right) \\
l_1\bar{k} - l_1 k_1 &= \chi k_1\bar{l} - \chi k_1 l_1 \\
l_1\bar{k} - l_1 k_1 + \chi k_1 l_1 &= \chi k_1\bar{l} \\
l_1 &= \frac{\chi\bar{l}k_1}{\left(\bar{k}-k_1\right)+\chi k_1}
\end{aligned}
$$

Now if these are production functions we will want to take one final important step, we will want to find the production possibilities frontier, or the total amount that we are able to produce in this society. The first thing we do is use the resource constraints to get everything in terms of (k_1, l_1)

$$
\begin{aligned}
l_2 &= \bar{l} - l_1 \\
k_2 &= \bar{k} - k_1 \\
\frac{1}{\sigma}\left(\frac{q_2}{A_2}\right)^{\frac{\sigma}{\rho_2}} &= \frac{\alpha_2}{\sigma}\left(\bar{k}-k_1\right)^{\sigma} + \frac{(1-\alpha_2)}{\sigma}\left(\bar{l}-l_1\right)^{\sigma}
\end{aligned}
$$

Then we use the contract curve to get everything in terms of k_1.

$$
\begin{aligned}
\frac{1}{\sigma}\left(\frac{q_1}{A_1}\right)^{\frac{\sigma}{\rho_1}} &= \frac{\alpha_1}{\sigma}k_1^{\sigma} + \frac{(1-\alpha_1)}{\sigma}\left(\frac{\chi\bar{l}k_1}{\left(\bar{k}-k_1\right)+\chi k_1}\right)^{\sigma} \\
\frac{1}{\sigma}\left(\frac{q_2}{A_2}\right)^{\frac{\sigma}{\rho_2}} &= \frac{\alpha_2}{\sigma}\left(\bar{k}-k_1\right)^{\sigma} + \frac{(1-\alpha_2)}{\sigma}\left(\bar{l}-\frac{\chi\bar{l}k_1}{\left(\bar{k}-k_1\right)+\chi k_1}\right)^{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} &= \left(\alpha_1 + (1-\alpha_1)\left(\frac{\chi\bar{l}}{\left(\bar{k}-k_1\right)+\chi k_1}\right)^{\sigma}\right)^{\frac{1}{\sigma}} k_1 \\
\left(\frac{q_2}{A_2}\right)^{\frac{1}{\rho_2}} &= \left(\alpha_2 + (1-\alpha_2)\left(\frac{\bar{l}}{\left(\bar{k}-k_1\right)+\chi k_1}\right)^{\sigma}\right)^{\frac{1}{\sigma}} \left(\bar{k}-k_1\right)
\end{aligned}
$$

And in principle we can use this to draw our production possibilities frontier. But it isn't very useful, now is it? Still, it is possible, and showing that is important. We will go further in the special case when $\alpha_1 = \alpha_2$.

The General Equlibrium and the Production Possibilities Fronteir when $\alpha_1 = \alpha_2 = \alpha$ In this case we have $\chi = 1$ and:

$$l_1 = \frac{1}{\omega_1 + (1-\omega_1)} \frac{\bar{l}}{\bar{k}} k_1 = \frac{\bar{l}}{\bar{k}} k_1$$

The contract curve is a straight line, and furthermore the Marginal Rate of Technical Substitution is constant. So in the equilibrium:

$$\begin{aligned}\frac{r}{w} &= \frac{\alpha}{1-\alpha}\left(\frac{l_1}{k_1}\right)^{1-\sigma} = \frac{\alpha}{1-\alpha}\left(\frac{\frac{\bar{l}}{\bar{k}}k_1}{k_1}\right)^{1-\sigma} \\ &= \frac{\alpha}{1-\alpha}\left(\frac{\bar{l}}{\bar{k}}\right)^{1-\sigma} = \frac{\alpha \bar{l}^{1-\sigma}}{(1-\alpha)\bar{k}^{1-\sigma}}\end{aligned}$$

and the natural normalization is: $w = \frac{(1-\alpha)}{\bar{l}^{1-\sigma}}$, $r = \frac{\alpha}{\bar{k}^{1-\sigma}}$. Notice the rather surprising thing that the prices do not depend on the final allocation. Since both parties value the two goods in relatively the same way they will agree to share the goods in fixed proportions.

Notice as well what the impact of increasing the supply is on the prices:

$$\begin{aligned}\frac{\partial w}{\partial \bar{l}} &= \frac{(1-\sigma)(1-\alpha)}{\bar{l}^{2-\sigma}}, \frac{\partial r}{\partial \bar{l}} = 0 \\ \frac{\partial w}{\partial \bar{k}} &= 0, \frac{\partial r}{\partial \bar{l}} = \frac{(1-\sigma)\alpha}{\bar{k}^{2-\sigma}}\end{aligned}$$

and remember that as $\sigma \to 1$ these goods become closer and closer substitutes. If you remember I began this section by what would happen in two markets where the goods are substitutes, well now we know the answer. In the general equilibrium the impact of increasing supply on price will be *less* as goods become closer and closer substitutes. The reason is because if the two goods are very close substitutes the people will just change their consumption until the prices aren't affected very much. Wild, ehh?

Now if the production functions have the same σ and α then we can really get find a production possibilities fronteir. With these substitutions we have:

$$\begin{aligned}\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} &= \left(\alpha + (1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}} k_1 \\ \left(\frac{q_2}{A_2}\right)^{\frac{1}{\rho_2}} &= \left(\alpha + (1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}} (\bar{k} - k_1)\end{aligned}$$

$$\frac{\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}}}{\left(\alpha + (1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}}} = k_1$$

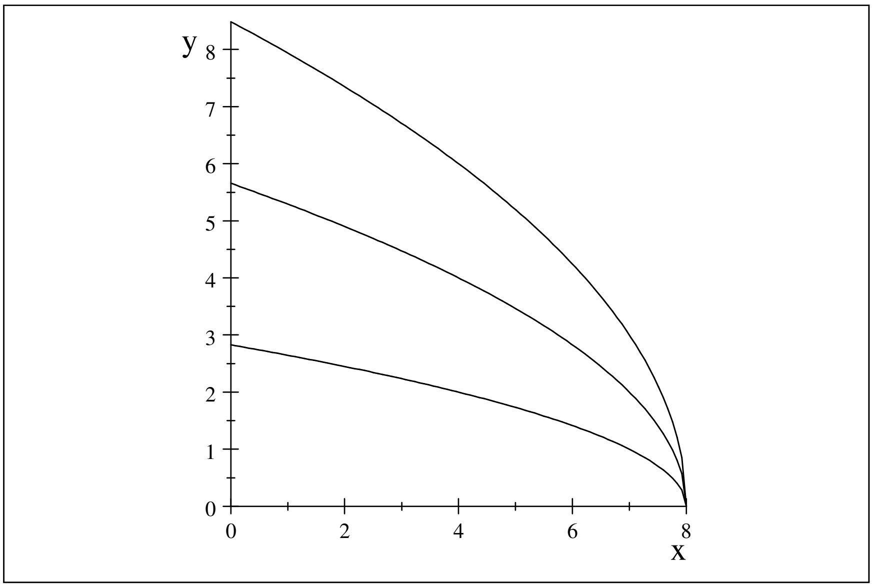
and when we do the substitution:

$$
\begin{aligned}
\left(\frac{q_2}{A_2}\right)^{\frac{1}{\rho_2}} &= \left(\alpha+(1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}}\left(\bar{k}-\frac{\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}}}{\left(\alpha+(1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}}}\right) \\
\left(\frac{q_2}{A_2}\right)^{\frac{1}{\rho_2}} &= \bar{k}\left(\alpha+(1-\alpha)\left(\frac{\bar{l}}{\bar{k}}\right)^{\sigma}\right)^{\frac{1}{\sigma}}-\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} \\
\left(\frac{q_2}{A_2}\right)^{\frac{1}{\rho_2}}+\left(\frac{q_1}{A_1}\right)^{\frac{1}{\rho_1}} &= \left(\alpha\bar{k}^{\sigma}+(1-\alpha)\,\bar{l}^{\sigma}\right)^{\frac{1}{\sigma}}
\end{aligned}
$$

Didn't that work out neatly? Now the right hand side can be thought of as sort of the "base amount that can be produced" and the left hand side indicates how we can transform that into output. Just to give an example let's assume that $\left(\alpha\bar{k}^{\sigma}+(1-\alpha)\,\bar{l}^{\sigma}\right)^{\frac{1}{\sigma}}=8$, $A_1=\rho_1=1$, $\rho_2=\frac{1}{2}$ and we will vary A_2. Then this equation is:

$$
\begin{aligned}
\left(\frac{q}{A_2}\right)^2+\frac{q_1}{A_1} &= 8 \\
q_2 &= A_2\sqrt{8-q_1}
\end{aligned}
$$

and we can graph this for $A_2 \in \{1,2,3\}$.



and this is what the production possibilities fronteir will look like.