Cournot Oligopoly

5 May 2024

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Antoine-Augustin Cournot is something of a genius in the history of economic theory. In 1838 (in Recherches sur les Principes Mathématiques de la Théorie des Richesses) he laid out what is, to this day, our benchmark model of imperfect competition.

To give you a sense of perspective, Adam Smith published what is widely considered the first book in Economics in 1776 (An Inquiry into the Nature and Causes of the Wealth of Nations)—62 years before Cournot. The book credited with founding game theory was published in 1944 (Theory of Games and Economic Behavior by Von Neumann and Morgenstern)—106 years later. He nearly comes up with the equilibrium concept now called Nash equilibrium (1950, Equilibrium points in n-person games by John Nash). That article is famous for taking less than one page to lay out one of the most important concepts in Game theory, and the reason it is so brief illustrates Cournot's achievement. In a footnote Nash thanks David Gale for recommending he cite Kakutani's fixed point theorem, published in 1941. Without Kakutani Nash would need to prove it himself (which he had done), but thanks to Kakutani his article is complete and self contained in one page. In other words his work could rely on developed mathematical theory. The first fixed point theorem was Brouwer's, published in 1912—72 years after Cournot published. Cournot did not only have to define equilibrium, he had to *develop the mathematical foundations* for his equilibrium.

1 Model

Their are $n < \infty$ firms, each of which chooses q_i. The price is then set to clear the market, using the inverse demand curve $P(Q)$ where $Q = \sum\limits_{i=1}^{n} q_i$. We assume $\frac{dP}{dQ} < 0$.

Firms choose q_i to maximize their profit, if we write Q_{-i} for the other firm's output, or $Q_{-i} = \sum\limits_{j=1, j\neq i}^{n} q_j = Q - q_i$ then this profit function is

$$\pi_i(q_i, Q_{-i}) = P(Q_{-i} + q_i) q_i - c_i(q_i) \ , \tag{1}$$

where $c_i(q_i)$ is firm i's cost of producing q_i.

Most of the following will be about the simpler model where $P(Q) = a - bQ$, $c_i(q) = c_i q_i$, this is the model analyzed in Cournot's book.

Remark 1 *Right now I feel that Game Theory is in something of a "hyper-structuralist" phase. I.e. we want to derive all the results from a detailed description of the structure of the interaction and people's preferences. There is no place for "as if" arguments for this school. To these scientists the problem with*

Cournot's model is who sets the price? *Some magical "Walrasian auctioneer" who just really wants markets to clear? This criticism is also true in competitive economics, and it bothers Game Theorists in both settings.*

2 Analyzing models with constant marginal cost

2.1 Two firms, symmetric marginal cost

This is the classic model, and the primary one Cournot focused on. In this model:

$$\pi_1(q_1, q_2) = (a - b(q_1 + q_2))\, q_1 - cq_1 \ . \tag{2}$$

The first question Cournot had to answer was "what should this firm assume about q_2?" How do I optimize when their is an unknown in my objective function? He made the deduction that it is outside of the control of firm 1, thus $\frac{\partial q_2}{\partial q_1} = 0$. Given this he can find the first order condition:

$$\frac{\partial \pi_1}{\partial q_1} = (a - b(q_1 + q_2)) - bq_1 - c = 0 \tag{3}$$

Let me point out that for the general objective function 1 this would be:

$$\frac{\partial \pi_i}{\partial q_i} = P(Q) + \frac{dP}{dQ} q_i - mc_i = 0 \tag{4}$$

and we can implicitly define the output of firm i as:

$$q_i^* = (P - mc_i) \Big/ \left| \frac{dP}{dQ} \right| \tag{5}$$

(please note that since $\frac{dP}{dQ} < 0$ $\left|\frac{dP}{dQ}\right| = -\frac{dP}{dQ}$.)

But anyway, this is still a function of q_1 and q_2, thus he decided to solve for what we would call the *best response*—q_i as a function of q_j—but he called the *reaction function*. In either case this is the same function:

$$q_1 = BR_1(q_2) = R_1(q_2) = \frac{1}{2b}(a - c) - \frac{1}{2} q_2 \ . \tag{6}$$

Notice how elegant this best response is. If $q_2 = 0$ then $q_1 = \frac{1}{2b}(a - c)$ or the monopoly quantity, and when $q_2 = \frac{1}{b}(a - c)$—or the competitive output—firm one will produce nothing.

2.1.1 Three ways to find Nash equilibrium

The first is the most direct application of the theory:

$$q_1^* = BR_1(BR_2(q_1^*))$$

we notice that obviously $BR_2(q_1) = \frac{1}{2b}(a-c) - \frac{1}{2}q_1$ so this would mean:

$$\begin{aligned} q_1 &= \frac{1}{2b}(a-c) - \frac{1}{2}\left(\frac{1}{2b}(a-c) - \frac{1}{2}q_1\right) \\ &= \frac{a}{2b} - \frac{c}{2b} - \frac{a}{4b} + \frac{c}{4b} + \frac{1}{4}q_1 \\ &= \frac{1}{4b}(a-c) + \frac{1}{4}q_1 \end{aligned}$$

$$\begin{aligned} \frac{3}{4}q_1 &= \frac{1}{4b}(a-c) \\ q_1 &= \frac{1}{3b}(a-c) \end{aligned}$$

and of course $q_2 = q_1$ in equilibrium, though we should verify this:

$$\begin{aligned} q_2 &= \frac{1}{2b}(a-c) - \frac{1}{2}\left(\frac{1}{3b}(a-c)\right) \\ &= \frac{a}{2b} - \frac{c}{2b} - \frac{a}{6b} + \frac{c}{6b} \\ &= \frac{1}{3b}(a-c) \end{aligned}$$

Notice this means that $Q = q_1 + q_2 = \frac{2}{3b}(a-c)$ and $P = a - b\left(\frac{2}{3b}(a-c)\right) = \frac{1}{3}a + \frac{2}{3}c$. Comparing this to the monopoly price: $P = \frac{1}{2}a + \frac{1}{2}c$, and the competitive price: $P = c$, we can see that this is somewhere in between.

A second method requires that we notice that the game is *symmetric*.

Definition 2 *A game is symmetric if $u_1(x,y) = \alpha u_2(y,x) + \beta$ for $\alpha > 0$ and any real number β.*

This definition is for a two player game but it can be extended to n players.

If a game is symmetric then their *may* be a symmetric equilibrium and even more rarely it *might* be unique. But applying this is simplicity itself here, if $q_1 = q_2 = q$ then:

$$\begin{aligned} q &= \frac{1}{2b}(a-c) - \frac{1}{2}q \\ \frac{3}{2}q &= \frac{1}{2b}(a-c) \\ q &= \frac{1}{3b}(a-c) \end{aligned}$$

Notice that you *can only assume symmetry after you have taken all derivatives.* It would seem a lot simpler to just assume $Q = q + q$ in the objective function, but when you take the derivative you are in effect assuming $\frac{\partial q_2}{\partial q_1} = 1$. This is simply wrong, as Cournot concluded $\frac{\partial q_2}{\partial q_1} = 0$.

A last method assumes nothing more than $q_1 > 0$ and $q_2 > 0$ in equilibrium. We notice that in this case:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_1} &= a - b(q_1 + q_2) - bq_1 - c = 0 \\
\frac{\partial \pi_2}{\partial q_2} &= a - b(q_1 + q_2) - bq_2 - c = 0
\end{aligned}$$

so obviously

$$\frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_2}{\partial q_2} = 0$$

$$\begin{aligned}
a - b(q_1 + q_2) - bq_1 - c + a - b(q_1 + q_2) - bq_2 - c &= 0 \\
2a - 2b(q_1 + q_2) - bq_1 - bq_2 - 2c &= 0
\end{aligned}$$

and we notice that $-bq_1 - bq_2 = -b(q_1 + q_2)$ and that $q_1 + q_2 = Q$, or the total quantity. Making these substitutions we see:

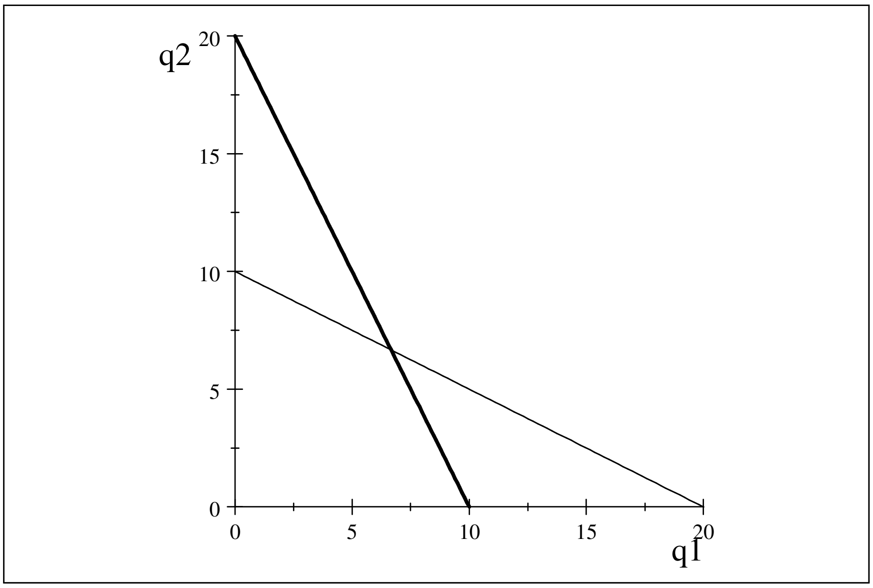
$$\begin{aligned}
2a - 2bQ - bQ - 2c &= 0 \\
2(a - c) &= 3bQ \\
Q &= \frac{2}{3b}(a - c)
\end{aligned}$$

using this we can use the first order condition for each firm to find:

$$\begin{aligned}
a - bQ - bq_i - c &= 0 \\
a - b\left(\frac{2}{3b}(a - c)\right) - c &= bq_i \\
\frac{1}{3}(a - c) &= bq_i \\
q_i &= \frac{1}{3b}(a - c)
\end{aligned}$$

like we did before.

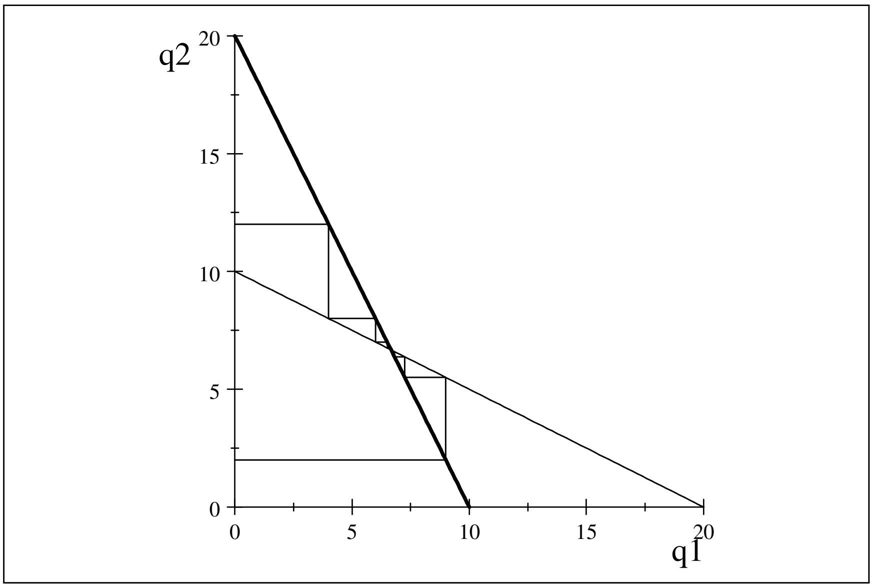
Remark 3 (Where Cournot went wrong.) *Cournot isolated the same quantities in equilibrium, but his methodology meant that he could not be sure what the equilibrium would be in a general interaction—or even if it would exist. One* *can easily draw a graph of the reaction functions:*



the dark one is $R_1(q_2)$ *and the light one is* $R_2(q_1)$. *The parameters are* $q_1 = 10 - \frac{1}{2}q_2$ *and* $q_2 = 10 - \frac{1}{2}q_1$. *He then said:*

"What if firm one believes firm two will produce 12 *units of output? Firm one would react to this by producing* 4 *units of output. But then if firm two knew this they would react by producing* 8 *units, and then firm one would react to this by producing* 6*..."*

One can see that this will always have a unique limit in this model—where the best responses intersect. Let me draw two such sequences, one starting with $q_2 = 12$ *and one starting with* $q_2 = 2$, *you will get two "cobweb" patterns leading* *to the unique Nash equilibrium, as illustrated below:*



and thus the only sensible prediction is that they will produce the limit of these sequences:

$$\begin{aligned} q_1^\infty &= \frac{1}{2b}(a-c) - \frac{1}{2}q_2^\infty \\ q_2^\infty &= \frac{1}{2b}(a-c) - \frac{1}{2}q_1^\infty \end{aligned}$$

which is the Nash equilibrium.

The problem with his approach is that he defined "equilibrium" as the limit of a sequence. As a general definition, this will not work. Will the sequence even converge? He had no way to answer this. Will the sequence always have the same limit? From our analysis of Nash equilibria we know the answer is no, but he had no way to answer this either. Fixed points are simply stable points of a system, one can describe them as potential limits of some undefined sequence but analyzing the sequence itself is unwise—the process can be complicated and if someone disagrees with your dynamic argument they can reject your result.

2.2 n firms, symmetric marginal cost.

If we have n firms then the objective can be written as:

$$\pi_i(q_i, Q_{-i}) = (a - b(q_i + Q_{-i}))\, q_i - cq_i \ . \tag{7}$$

and obviously this is symmetric, we can replace the i subscript with any j and it would be the same. The first order condition (and remember, we **must** find the first order condition before assuming symmetry) is:

$$(a - b(q_i + Q_{-i})) - bq_i - c = 0$$

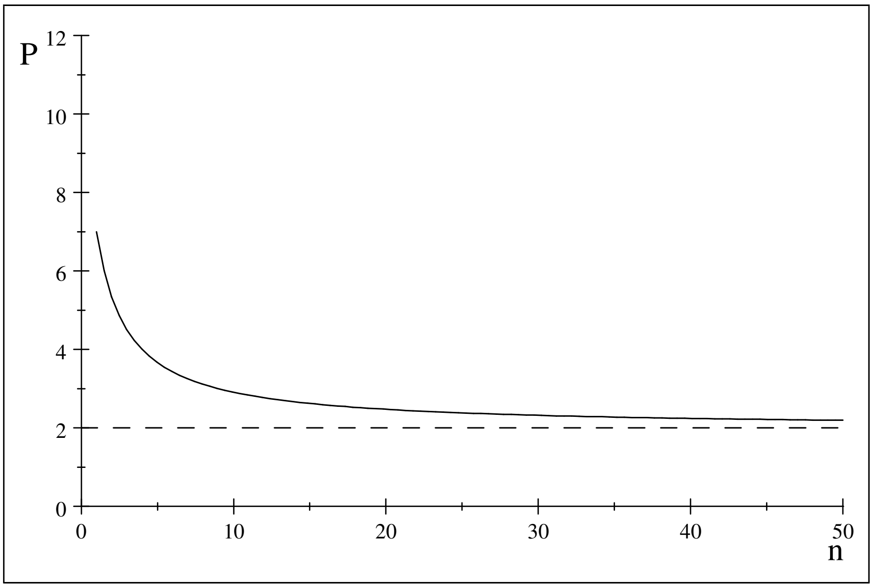
and the best response is simply:

$$q_i = \frac{1}{2b}(a-c) - \frac{1}{2}Q_{-i} \tag{8}$$

it is interesting to realize this is exactly the same as in the two firm case. Their are three ways to solve this, and one of them does not require assuming symmetry. First let us assume symmetry, or that in equilibrium $q_i = q$ and $Q_{-i} = (n-1)q$. Then:

$$\begin{aligned}
q &= \frac{1}{2b}(a-c) - \frac{1}{2}(n-1)q \\
\left(1 + \frac{1}{2}(n-1)\right)q &= \frac{1}{2b}(a-c) \\
\frac{1}{2}(n+1)q &= \frac{1}{2b}(a-c) \\
q &= \frac{1}{n+1}\frac{a-c}{b}.
\end{aligned}$$

Notice that if $n = 1$ this is the monopoly quantity, and of course it coincides with the two firm solution above when $n = 2$. This means that $Q = nq = \frac{n}{n+1}\frac{a-c}{b}$ and $P = a - b\left(\frac{n}{n+1}\frac{a-c}{b}\right) = \frac{1}{n+1}a + \frac{n}{n+1}c$. Thus the price will smoothly converge to marginal cost as the number of firms gets large. Below I graph the price when $a = 12$ and $c = 2$, the solid line is price and the dashed line is marginal cost.



Notice this model provides an elegant transition between monopoly (where $n = 1$, $P = 7$) and perfect competition (where $n = \infty$, $P = 2$). Notice as well that most of the collapse happens very quickly. Even if $n = 4$ then $P = 4$ or it is closer to marginal cost than the monopoly price, if $n = 10$ then $P = 2.9091$ and by the time $n = 20$ $P = 2.4762$.

There are two other ways to solve this model, the first one is to simply apply symmetry to the first order condition:

$$\begin{aligned}(a - b(q + (n-1)q)) - bq - c &= 0 \\ a - c &= b(n+1)q \\ q &= \frac{1}{n+1}\frac{a-c}{b}.\end{aligned}$$

However the problem with both of these methods is we need to assume symmetry. Is their a way to proceed without this assumption? Yes if all firms produce a positive amount then:

$$\frac{\partial \pi_i}{\partial q_i} = (a - bQ) - bq_i - c = 0$$

for all i, and we know that:

$$\sum_{i=1}^{n} \frac{\partial \pi_i}{\partial q_i} = 0 = n(a - bQ) - b\sum_{i=1}^{n} q_i - nc$$

and of course $\sum_{i=1}^{n} q_i = Q$, thus:

$$\begin{aligned}n(a - bQ) - bQ - nc &= 0 \\ n(a-c) &= (n+1)bQ \\ Q &= \frac{n}{n+1}\left(\frac{a-c}{b}\right)\end{aligned} \tag{9}$$

and then from any of the first order conditions we can realize that:

$$\begin{aligned}(a - b(Q)) - bq_i - c &= 0 \\ \left(a - b\left(\left(\frac{n}{n+1}\left(\frac{a-c}{b}\right)\right)\right)\right) - bq_i - c &= 0 \\ \frac{1}{n+1}a + \frac{n}{n+1}c - bq_i - c &= 0 \\ \frac{1}{n+1}a + \frac{n}{n+1}c - c &= bq_i\end{aligned}$$

$$q_i = \frac{1}{b}\left(\frac{1}{n+1}a + \frac{n}{n+1}c - c\right) = \frac{1}{n+1}\frac{a-c}{b} \tag{10}$$

Notice that since we did not assume symmetry we have shown there is a *unique* equilibrium, and that it is symmetric.

2.3 Two firms, asymmetric marginal cost.

Now we simply assume $c_i(q) = c_i q_i$, and this marginal cost can vary between firms. Thus:

$$\pi_1(q_1, q_2) = (a - b(q_1 + q_2)) q_1 - c_1 q_1 \tag{11}$$

and

$$\frac{\partial \pi_1}{\partial q_1} = (a - b(q_1 + q_2)) - bq_1 - c_1 = 0 \tag{12}$$

If you don't mind, I would like to use the adding the first order conditions method first, because the total quantity makes more sense than each firms output:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial q_1} + \frac{\partial \pi_2}{\partial q_2} &= (a - b(q_1 + q_2)) - bq_1 - c_1 + (a - b(q_1 + q_2)) - bq_2 - c_2 \\
&= 2(a - bQ) - bQ - c_1 - c_2 \\
Q &= \frac{1}{3b}(2a - c_1 - c_2) \\
&= \frac{2}{3b}(a - \bar{c})
\end{aligned}$$

where $\bar{c}$ is average marginal cost, $\bar{c} = \frac{c_1 + c_2}{2}$, likewise $P = a - b\left(\frac{2}{3b}(a - \bar{c})\right) = \frac{1}{3}a + \frac{2}{3}\bar{c}$, just like in the symmetric case. Of course now we have to worry that one of the firms might not produce output, but let's delay that discussion for now.

The best responses are very similar to before, we can derive from equation 12 that:

$$q_1 = \frac{1}{2}\frac{a - c_1}{b} - \frac{1}{2}q_2 \ .$$

Notice again if $q_2 = 0$ this firm is producing what they would as a monopolist. Obviously then:

$$q_2 = \frac{1}{2}\frac{a - c_2}{b} - \frac{1}{2}q_1$$

and in equilibrium we must have:

$$\begin{aligned}
q_1 &= \frac{1}{2}\frac{a - c_1}{b} - \frac{1}{2}\left(\frac{1}{2}\frac{a - c_2}{b} - \frac{1}{2}q_1\right) \\
&= \frac{1}{4}q_1 + \frac{1}{4}\frac{a}{b} - \frac{1}{2b}c_1 + \frac{1}{4b}c_2 \\
\frac{3}{4}q_1 &= \frac{1}{4}\frac{a}{b} - \frac{1}{2b}c_1 + \frac{1}{4b}c_2 \\
q_1 &= \frac{1}{3b}(a - 2c_1 + c_2) \\
&= \frac{1}{3b}(a - c_1 + [c_2 - c_1]) \qquad (13)
\end{aligned}$$

Notice that if we write q_1 like it is in equation 13 you can see it is like in the symmetric model, except that firm one boosts their output when $c_2 > c_1$.

Obviously:

$$q_2 = \frac{1}{3b}\left(a + c_1 - 2c_2\right) \ .$$

Notice something strange about this equilibrium, if $c_1 < c_2$ firm one is more efficient but firm two will usually produce output. We can see when they will shut down from equation 5.

$$q_2 = \frac{1}{b}\left(P - c_2\right) \ .$$

so they will shut down if $P \leq c_2$. To be precise in this model it will be if $\frac{1}{2}a + \frac{1}{2}c_1 \leq c_2$, and $\frac{1}{2}a + \frac{1}{2}c_1$ is the price firm one would charge if they were a monopolist.

In general the Cournot equilibrium is not *production efficient.*

Definition 4 *An outcome is production efficient if their is no way to produce the same output for a lower total cost. This implies that if $q_i > 0$ then $mc_i = \min_j mc_j$—or all firms either have shut down or have the lowest marginal cost.*

Thus when you have constant marginal cost it means the higher cost firm should shut down. It is actually shocking how hard it is to get production efficiency in a Cournot equilibrium.

Lemma 5 *A Cournot equilibrium is production efficient if and only if it is symmetric.*

Proof. *From the condition 5 we see that if $q_i = q_j$ then:*

$$\left(P - mc_i\right)/\frac{dP}{dQ} = \left(P - mc_j\right)/\frac{dP}{dQ}$$

which implies $mc_i = mc_j$, likewise if $mc_i = mc_j$ then $q_i = \left(P - mc_i\right)/\frac{dP}{dQ} = \left(P - mc_j\right)/\frac{dP}{dQ} = q_j$. Since production efficiency requires $mc_i = mc_j$ for all pairs of firms, if we have production efficiency we must have a symmetric equilibrium. Likewise if we have a symmetric equilibrium we must have $mc_i = mc_j$ for all pairs of firms and we have production efficiency. ■

2.4 n firms, asymmetric marginal cost.

We now only have one way to solve this model, summing the first order conditions:

$$\begin{aligned}
\frac{\partial \pi_i}{\partial q_i} &= (a - bQ) - bq_i - c_i \\
\sum_{i=1}^{n} \frac{\partial \pi_i}{\partial q_i} &= n\left(a - bQ\right) - b\sum_{i=1}^{n} q_i - \sum_{i=1}^{n} c_i \\
&= n\left(a - bQ\right) - bQ - n\bar{c} = 0
\end{aligned}$$

where $\bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i$ is average marginal cost. Thus:

$$\begin{aligned} n(a-\bar{c}) &= (n+1)\,bQ \\ Q &= \frac{n}{n+1}\frac{a-\bar{c}}{b} \\ P &= a-b\left(\frac{n}{n+1}\frac{a-\bar{c}}{b}\right) = \frac{1}{n+1}a+\frac{n}{n+1}\bar{c} \end{aligned}$$

and everything is almost exactly like the symmetric model...except for one thing. Now *high cost* firms will drop out, to be precise $q_i = 0$ if $P \leq c_i$. This means that we are moving *towards* a production efficient outcome. Assuming enough low cost firms enter, as the number who enter gets large the high cost firms will drop out and in the competitive limit *only* the firms with the lowest marginal cost will produce.

3 Some results with general cost functions.

It is surprising how much we can know for general cost functions. Assume the costs of firm i are $c_i(q_i)$ with $c_i(0) = 0$ (no fixed costs) and $mc_i(q_i) \geq 0$. Then:

$$\pi_i(q_i + Q_{-i}) = P(q_i + Q_{-i})\,q_i - c_i(q_i) \tag{14}$$

and the first derivative is:

$$\frac{\partial \pi_i}{\partial q_i} = P + \frac{dP}{dQ}q_i - mc_i\ . \tag{15}$$

If this firm was a perfect competitor their output would not affect the price so their objective function would be:

$$\pi_i^c(q_i) = Pq_i - c_i(q_i) \tag{16}$$

With the first derivative of:

$$\frac{\partial \pi_i^c}{\partial q_i} = P - mc_i \tag{17}$$

Let q_i^c and Q^c be the competitive output of the firm and all firms in the market, and q_i^* be the equilibrium output of firm i, and Q^* be the equilibrium output of all firms. Then we can immediately establish that:

Lemma 6 *$q_i^* < q_i^c$ and thus $Q^* < Q^c$.*
Proof. *For all P we notice that:*

$$\frac{\partial \pi_i^c}{\partial q_i} = P - mc_i > P - mc_i + \frac{dP}{dQ}q_i = \frac{\partial \pi_i}{\partial q_i}$$

thus when $\frac{\partial \pi_i}{\partial q_i} = 0$, $\frac{\partial \pi_i^c}{\partial q_i} > 0$ and it must cross zero at a higher value of q_i. This shows that $q_i^* < q_i^c$ for all i, and thus we must have $Q^* < Q^c$. ■

Now summing the first order condition over all firms such that $P \geq mc_i$ we get:

$$\sum_{i=1}^{n} \frac{\partial \pi_i}{\partial q_i} = nP + \frac{dP}{dQ} Q^* - n\overline{mc} = 0 \tag{18}$$

if we let $\overline{mc} = \frac{1}{n} \sum_{i=1}^{n} mc_i$. Alternatively we can write:

$$Q^* = n \left(P - \overline{mc} \right) / \left| \frac{dP}{dQ} \right| \tag{19}$$

and this will lead to an interesting result when n increases. We need to be clear here, in our analysis a firm is only "counted" if $P \geq mc_i$ for some q_i. Thus n increasing means that the number of firms dropping out (if any) are less than the number of more efficient firms entering. Define Q_0 as $P(Q_0) = 0$—or the quantity that is demanded when the price is zero.

Lemma 7 *If* $Q_0 < \infty$ *and* $\left| \frac{dP}{dQ} \right| \leq \bar{b} < \infty$ *then as* $n \to \infty$ $P \to \overline{mc}$
Proof. *Since* $P \geq mc_i \geq 0$ $Q^* \leq Q_0$. *Thus*

$$\infty > Q_0 \geq Q^* \geq n \left(P - \overline{mc} \right) / \left| \frac{dP}{dQ} \right| \geq n \left(P - \overline{mc} \right) / \bar{b} \text{ or} \tag{20}$$

$$\frac{\bar{b} Q_0}{n} \geq \left(P - \overline{mc} \right), \tag{21}$$

and since the left hand side coverges to zero as $n \to \infty$ *we must have* $\left(P - \overline{mc} \right) \to 0$ *or* $P \to \overline{mc}$. ■

So we know a great deal about the Cournot equilibrium. First, they will always produce less than the efficient amount and thus the price will be too high. Second as the number of firms goes to infinity, the price will fall to marginal cost. In essence the key results from the constant marginal cost analysis generalize.