

The Envelope Theorem.

ECON 203

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The envelope theorem is an extremely simple result. Basically the essence of the result is that the answer is simple. Let me give you the problem, say that the price of one good you buy (CDs) increases by a small amount. How much more money should you ask from your parents so that you won't suffer from this price increase?

Now before I give you the incredibly simple answer let me dwell on why this seems like a difficult question. The first thing to realize is that you are now going to buy fewer CDs, obviously the price of the good goes up the demand goes down. Secondly, notice that increasing one good's price lowers the relative price of every other good, or from the introductory lecture "everything's relative." What this means is that since the relative price of other goods (like DVDs) has gone down you are going to buy more of them instead of buying CDs. You might also put off buying that new CD player you've been considering, because now you are going to buy fewer CDs, and etceteras. Basically this one change is going to set off a long chain of reactions on your part, and so how do we compare your happiness after all of these reactions to your happiness beforehand?

So, if the price of one good you buy increases by a small amount how much should your income increase to exactly balance it out?

Enough so that you can still buy the goods you used to buy.

No, surely that's wrong, I must have cheated somewhere. Heh. I didn't. It's that easy. Let me restate this result using some more technical terminology.

When taking the derivative of an optimized function, it will be the same as the derivative of the unoptimized function.

When taking the derivative of an optimized function all indirect effects will disappear, leaving only the direct effect.

When taking the derivative of an optimized function the cost and benefit of all other endogenous variables will already be balanced out, leaving only the direct effect of the variable on the function.

To understand this more clearly let me illustrate it (and prove it) for an expenditure function. Let us simplify your purchasing decision by assuming that you only buy CDs (denoted C), DVDs (D), food (F), and gasoline for your car (G). Let the price of good x be p_x (x is either C, D, F , or G). Then the expenditure minimization problem your parents solve to determine your allowance is:

$$I(p_c, p_d, p_f, p_g, \bar{u}) = \max_{\lambda} \min_{C, D, F, G} p_c C + p_d D + p_f F + p_g G - \lambda (U(C, D, F, G) - \bar{u})$$

Now let us look at the derivative with respect to p_c :

$$\begin{aligned} \frac{\partial I}{\partial p_c} &= C \\ &+ p_c \frac{\partial C}{\partial p_c} + p_d \frac{\partial D}{\partial p_c} + p_f \frac{\partial F}{\partial p_c} + p_g \frac{\partial G}{\partial p_c} \\ &- \lambda \frac{\partial U}{\partial C} \frac{\partial C}{\partial p_c} - \lambda \frac{\partial U}{\partial D} \frac{\partial D}{\partial p_c} - \lambda \frac{\partial U}{\partial F} \frac{\partial F}{\partial p_c} - \lambda \frac{\partial U}{\partial G} \frac{\partial G}{\partial p_c} \\ &- \frac{\partial \lambda}{\partial p_c} (U(C, D, F, G) - \bar{u}) \end{aligned}$$

Anytime one is faced with a very complicated expression like this one should always collect terms. For example, we have $\frac{\partial C}{\partial p_c}$ twice in the above expression. First it is multiplied by p_c , second it is multiplied by $-\lambda \frac{\partial U}{\partial C}$. So let's collect terms like this:

$$\begin{aligned} \frac{\partial I}{\partial p_c} &= C \\ &+ \left(p_c - \lambda \frac{\partial U}{\partial C} \right) \frac{\partial C}{\partial p_c} \\ &+ \left(p_d - \lambda \frac{\partial U}{\partial D} \right) \frac{\partial D}{\partial p_c} \\ &+ \left(p_f - \lambda \frac{\partial U}{\partial F} \right) \frac{\partial F}{\partial p_c} \\ &+ \left(p_g - \lambda \frac{\partial U}{\partial G} \right) \frac{\partial G}{\partial p_c} \\ &- \frac{\partial \lambda}{\partial p_c} (U(C, D, F, G) - \bar{u}) \end{aligned}$$

Now these expressions should look familiar. Look at the last one, for example, $-\frac{\partial \lambda}{\partial p_c} (U(C, D, F, G) - \bar{u})$. The value \bar{u} represents how happy your parents have decided you should be, after due consideration about their budget and your siblings and etcetera. If $U(C, D, F, G) > \bar{u}$ that means they have goofed up and ended up making you happier than they have already decided you should be. If $U(C, D, F, G) < \bar{u}$ then they have failed at their goal. So we *know* that in any optimal solution $U(C, D, F, G) - \bar{u} = 0$ and there is no impact from $\frac{\partial \lambda}{\partial p_c}$.

In general look at the first order conditions of your parent's optimization

problem.

$$\begin{aligned} \left(p_c - \lambda \frac{\partial U}{\partial C} \right) &= 0 \\ \left(p_d - \lambda \frac{\partial U}{\partial D} \right) &= 0 \\ \left(p_f - \lambda \frac{\partial U}{\partial F} \right) &= 0 \\ \left(p_g - \lambda \frac{\partial U}{\partial G} \right) &= 0 \\ (U(C, D, F, G) - \bar{u}) &= 0 \end{aligned}$$

so this means that the derivative in question is:

$$\begin{aligned} \frac{\partial I}{\partial p_c} &= C + (0) \frac{\partial C}{\partial p_c} + (0) \frac{\partial D}{\partial p_c} + (0) \frac{\partial F}{\partial p_c} + (0) \frac{\partial G}{\partial p_c} - \frac{\partial \lambda}{\partial p_c} (0) \\ &= C \end{aligned}$$

which is only the direct effect. The absolute amount I have to change I in order to keep you happy is then $\partial I = \partial p_c C$, which is exactly what I said it would be above. Increase the income so that you can still afford to by the same number of CDs.

That is a formal proof, or if you phrased it more precisely it would be. But I want to give a slightly more general proof so you can understand how general the result is. To produce this more general proof I will need an arbitrary function— $F(p, q, r, w, X, Y, Z)$ —and I will need an optimized version of this function:

$$H(p, q, r, w) = \max_{X, Y, Z} F(p, q, r, w, X, Y, Z)$$

or:

$$L(p, q, r, w) = \min_{X, Y, Z} F(p, q, r, w, X, Y, Z)$$

or a mixture of the two like:

$$M(p, q, r, w) = \max_X \min_{Y, Z} F(p, q, r, w, X, Y, Z)$$

The the envelope theorem states:

Theorem 1 (The Envelope Theorem) *If a function like $H(p, q, r, w)$, $L(p, q, r, w)$, or $M(p, q, r, w)$ is an optimized version of another function like $F(p, q, r, w, X, Y, Z)$ then the impact of $\{p, q, r, w\}$ on the function $H(\cdot)$, $L(\cdot)$, or $M(\cdot)$ is the direct effect only. I.e. the indirect impact on X, Y and Z can be ignored. Formally $\frac{\partial H}{\partial p} = \frac{\partial L}{\partial p} = \frac{\partial M}{\partial p} = \frac{\partial F}{\partial p}$, $\frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} = \frac{\partial M}{\partial q} = \frac{\partial F}{\partial q}$, $\frac{\partial H}{\partial r} = \frac{\partial L}{\partial r} = \frac{\partial M}{\partial r} = \frac{\partial F}{\partial r}$, and $\frac{\partial H}{\partial w} = \frac{\partial L}{\partial w} = \frac{\partial M}{\partial w} = \frac{\partial F}{\partial w}$.*

The proof of this is quite simple. Let's optimize $F(p, q, r, w, X, Y, Z)$ over X , Y , and Z . The first order conditions are:

$$\begin{aligned}\frac{\partial F}{\partial X} &= 0 \\ \frac{\partial F}{\partial Y} &= 0 \\ \frac{\partial F}{\partial Z} &= 0\end{aligned}$$

whether you are minimizing or maximizing over X , Y , and Z . So we can then explicitly calculate (for example) $\frac{\partial H}{\partial p}$:

$$\frac{\partial H}{\partial p} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial X} \frac{\partial X}{\partial p} + \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial p} + \frac{\partial F}{\partial Z} \frac{\partial Z}{\partial p}$$

but since we optimized $F(\cdot)$ we know that $\frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0$ and this simplifies to $\frac{\partial H}{\partial p} = \frac{\partial F}{\partial p}$. And that was a formal proof. That wasn't so painful was it?

We use this many different times in economics. One thing that you won't use but is used sometimes in more advanced classes is:

$$\begin{aligned}V(p_f, p_c) &= \min_{\lambda} \max_{F, C} U(F, C) - \lambda(p_f F + p_c C - I) \\ \frac{\partial V}{\partial p_f} &= -F\end{aligned}$$

the function on the left hand side is the indirect utility function.

Another way you can use it is to calculate "shadow prices." Say that you want to know the value of an irrigation project to some farmers. In order to do this you need to figure out the price they would put on an extra unit of water, or the marginal benefit of a unit of water. To precisely model this let us explicitly let the production function be affected by water, W , so we have $Q \leq f(L, K, W)$. Then the firm's cost minimization problem is:

$$C(w, r, Q, W) = \max_{\lambda} \min_{L, K} wL + rK - \lambda(f(L, K, W) - Q)$$

and from the envelope theorem we know that $\frac{\partial C}{\partial W} = -\lambda \frac{\partial f}{\partial W}$. We further know that $\frac{\partial C}{\partial Q} = \lambda$ so the marginal benefit of increasing water is: $\frac{\partial C}{\partial W} = -\frac{\partial C}{\partial Q} \frac{\partial f}{\partial W}$. This can be easily measured and tells us how much the farmers would pay for an irrigation system.

But how will we use this? Well one way is the producers version of expenditure minimization:

$$C(w, r, Q) = \max_{\lambda} \min_{L, K} wL + rK - \lambda(f(L, K) - Q)$$

we can use this in several different ways. First of all, $\frac{\partial C}{\partial w} = -L$, or the marginal cost of a rise in the wage is proportional to the number of laborers. Another

use is $\frac{\partial C}{\partial Q} = \lambda = \frac{w}{MP_L}$, this is marginal cost, a very important concept. Another use is in comparing the short run and long run cost function. In the short run the amount of capital you have is fixed:

$$C^{SR}(w, r, K, Q) = \max_{\lambda} \min_L wL + rK - \lambda (f(L, K) - Q)$$

now let's figure out what the marginal cost of capital is:

$$\frac{\partial C^{SR}}{\partial K} = r - \lambda \frac{\partial f}{\partial K}$$

notice I just used the envelope theorem. Now a fundamental problem is that we do not observe long run cost functions, we only observe the short run variety in general. So how do we construct the long run cost function? Well, think about it. There has got to be some K^* such that:

$$\frac{\partial C^{SR}(w, r, K^*, Q)}{\partial K^*} = r - \lambda \frac{\partial f}{\partial K} = 0$$

and lo and behold this is the same first order condition as we have in the long run cost function. Using this we can find the optimal amount of K for any Q and $\{w, r\}$.