On the Core Method for Matrices.

ECON 225
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1 Introduction

Mathematics is, at its most beautiful, nothing more than a descriptive language. It is supposed to describe reality, not impose structure on it. Honestly it is at its best when the descriptive structure is simple. This means we can understand and explain a great many things using something that is both easy to understand and describe.[1]

One such structure that mathematicians have found incredibly useful is matrices. It is important to recognize that there are two separate, apparently unrelated, reasons to be interested in matrices. In fact there are more, but those we will discover later.

1.1 Linear Algebra: a System of Equations

Basis number one is the fairly common problem of having a set of variables and system of linear equations. Like for example:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \ . \end{aligned} \tag{1}$$

Using matrices we can rewrite this as:

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \ , \\ Ax &= b \ . \end{aligned} \tag{2}$$

When analyzed from this direction the questions we are interested in are: "Do we have a solution? Do we have one solution or many?"

1.2 Euclidean Spaces: Vectors and the Inner Product

The other direction from which we have thought about matrices is vectors. A vector does not look unfamiliar. It looks just like a point in n dimensional space. For example a point in the three dimensional reals—$p = (x, y, z) \in \mathbb{R}^3$—could also be written as a vector $v = (x, y, z) \in \mathbb{R}^3$. The difference is in the way I interpret it. The point means "starting at $(0, 0, 0)$ go to the right x, up y, and out z." The vector means "starting at any initial point, go right x, up y, and out z." You get it? When I tell you the point p it is fixed in the real numbers.

[1] Translation: In math simple = beauty = truth. If it's useful that's a bonus. (But of course I wouldn't teach it if it wasn't useful, ... probably.)

You can go to that location from anywhere you might be. When I tell you the vector v it just tells you where to move to. The point you end up at will depend on your starting point. The analysis of vectors requires that we also have the inner product. For $a \in \mathbb{R}^3$, $x \in \mathbb{R}^3$:

$$a \cdot x = (a_{11}, a_{12}, a_{13}) \cdot (x_1, x_2, x_3) = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ . \tag{3}$$

Whoops, was that a mistake? Or just me being lazy? My notation suggests that the first linear equation (equation 1, line 1) is the exact same thing as the inner product (equation 3). That is not, in fact, a mistake. The book establishes the standard that if $v = (x, y, z)$ it is understood that it is also a column vector $v = \begin{bmatrix} x & y & z \end{bmatrix}^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, thus $a \cdot x = a^T x$ by definition.

Vectors can be used to "redescribe" the real numbers. What do I mean? Well the standard reals can be based on the vector $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. I.e. the point "5" is $5e_1$, the point 100 is $100e_1$ and so forth. Why would we want to do this? Well, the Turkish government seems to think its a good idea. Between 31 December 2004 and 1 January 2005 the Lira lost six zeros. What was previously one million TL was now one TL. Or, for example, if we are analyzing the sine we might want our units to be 2π, because $\sin(x2\pi) = \sin(y2\pi)$ if $x = y + k$, where k is an integer. It's a useful thing to do.

So, first of all, it should be clear that *any* vector $v \in \mathbb{R}^n$ describes a line. What about if we have two vectors? Three? Then what we really have is a *matrix* like:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} , \tag{4}$$

(this would be three vectors from $\mathbb{R}^2$.) The question we now want to answer is: "How much of $\mathbb{R}^n$ can k vectors from $\mathbb{R}^m$ describe?"[2]

2 The Fundamental Methodology

Our joy at this point is unbounded, because we have here two very different ways of thinking about the same object, matrices, and we will always use only one methodology in order to answer all of the questions we have. This methodology will be based on *elementary row operations* or *elementary matrices.* The definitions of these are:

Definition 1 (Elementary Row Operations) *For a system of equations $Ax = b$ one can do any of the three following operations and leave the system of equations unchanged:*

1. *Switch the order of rows i and j. Leave the other equations unchanged.*

[2] Note we can have $n \neq k \neq m$.

2. *Multiply row j by a number and add it to row i (and multiply all the other rows by zero). Leave the other equations unchanged.*

3. *Rescale row i by a non-zero number. Leave the other equations unchanged.*

These are equivalent to:

Definition 2 (Elementary Matrices) *These are equivalent to multiplying $\tilde{A} = [A\ b]$ by the three following elementary matrices.*

1. *E_{ij}—interchanging row i and j of the identity matrix. For example*

$$E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

2. *$E_{ij}(r)$—an identity matrix except that $a_{ij} = r$. For example:*

$$E_{23}(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix}.^{3} \tag{6}$$

3. *$E_i(r)$—an identity matrix except that $a_{ii} = r$. For example:*

$$E_2(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}.^{4} \tag{7}$$

Throughout the rest of this handout I will assume you are comfortable with both classes of operations and how to use them. For those of you who are not (which might be most of you) I include the next subsection.

2.0.1 About Elementary Row Operations and their equivalence to Elementary Matrices

Your first task, which I will not help you with, is to be comfortable with elementary row operations. These should be familiar from your high school math class—where you learned them in order to solve a system of equations using elimination of variables. What I will take more time to explain is why these are equivalent to elementary matrices. I will do it for:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix}.$$

[3] At this point I would like to mention how glad I am not the originator of this notation. It is potentially very confusing. There is no relationship between E_{ij} and $E_{ij}(r)$. (So if you find the notation perplexing you are not alone.)

[4] At least here we could write this as $E_{ii}(r)$ so the second and third notations coincide, granting a little laziness in the third notation.

I know that we all (should) know this probably has a continuum of solutions, but I don't want you guys to get comfortable with square matrices—problems where we might find a unique answer. So, first of all, you notice that in my description of the elementary row operations I always ended with "Leave the other equations unchanged." You may have wondered why I bothered. The reason was because then it makes explaining the "uninteresting" rows of the elementary matrices simpler. For example let's consider multiplying $\tilde{A}$ by the row vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. And let's be sure to carefully go through the multiplication step by step.

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} 1*a_{11}+0*a_{21}+0*a_{31} \\ 1*a_{12}+0*a_{22}+0*a_{32} \\ 1*a_{13}+0*a_{23}+0*a_{33} \\ 1*a_{14}+0*a_{24}+0*a_{34} \\ 1*b_1+0*b_2+0*b_3 \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \end{bmatrix}, \tag{8}$$

(sorry about using the transpose there, had to fit it on the page.) Do you see the point? Multiplying a row by the appropriate identity row just returns the same row again. Likewise you should verify that:

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} \tag{10}$$

So this should explain why:

$$E_{23}\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \end{bmatrix}. \tag{11}$$

OK, I admit it, you may not get that yet. So work on it, keep on thinking about it, until the answer is transparent. There is no point to going on if you can't figure that out.

So you understand that, right? Good, because I'm not going to return to it. Now let's consider the third operation. First of all what would happen if I re-scaled a coefficient of the identity matrix by some r? Well it should be clear that the result would just be that every element of that row will be multiplied by that constant, for example:

$$\begin{bmatrix} 0 & \frac{1}{a_{22}} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} \frac{a_{21}}{a_{22}} & 1 & \frac{a_{23}}{a_{22}} & \frac{a_{24}}{a_{22}} & \frac{b_2}{a_{22}} \end{bmatrix} \tag{12}$$

(I choose $1/a_{22}$ because transforming a_{22} to one is something we do.) Now combining this with the rows from equation (8) and equation (10) we get the matrix $E_2\left(\frac{1}{a_{22}}\right)$ and multiplying this by $\tilde{A}$ does indeed simply rescale the second row.

$$E_2\left(\frac{1}{a_{22}}\right)\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ \frac{a_{21}}{a_{22}} & 1 & \frac{a_{23}}{a_{22}} & \frac{a_{24}}{a_{22}} & \frac{b_2}{a_{22}} \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} . \tag{13}$$

And again, please do me the favor of understanding this point before moving on.

Now we're ready for the most complex operation. Multiply row j by a number and add it to row i. Now what we want to do is for $k \in \{1, 2, 3, 4\}$:

$$ra_{kj} + a_{ki} \ . \tag{14}$$

Let's have $j = 2$ and $i = 3$ (so we can build up to $E_{23}(r)$) then we can also write this as for $k \in \{1, 2, 3, 4\}$:

$$0a_{k1} + ra_{k2} + a_{k3} \ . \tag{15}$$

You are probably ready to ask me why in the world I would want to do that. Well, I'm glad you asked. Because then we can realize that this is the same as:

$$\begin{bmatrix} 0 & r & 1 \end{bmatrix} \begin{bmatrix} a_{k1} \\ a_{k2} \\ a_{k3} \end{bmatrix} , \tag{16}$$

and the joy of matrix multiplication is that we can then write this for all k (and the b column, which I forgot) as:

$$\begin{bmatrix} 0 & r & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & b_3 \end{bmatrix} = \begin{bmatrix} a_{31} + ra_{21} \\ a_{32} + ra_{22} \\ a_{33} + ra_{23} \\ a_{34} + ra_{24} \\ b_3 + rb_2 \end{bmatrix}^T \tag{17}$$

Then... we slap on rows equation (8) and equation (9) and we've got an elementary matrix!

$$E_{23}(r) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix}$$

Just to give an example we use, let $r = -\frac{a_{32}}{a_{22}}$:

$$E_{23}\left(-\frac{a_{32}}{a_{22}}\right)\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & b_2 \\ a_{31} - \frac{a_{32}}{a_{22}}a_{21} & 0 & a_{33} - \frac{a_{32}}{a_{22}}a_{23} & a_{34} - \frac{a_{32}}{a_{22}}a_{24} & b_3 - \frac{a_{32}}{a_{22}}b_2 \end{bmatrix} .$$

Of course in practice we would first transform a_{21} and a_{31} so that $a_{21} = a_{31} = 0$ by using $E_{12}\left(-\frac{a_{21}}{a_{11}}\right)$ and $E_{13}\left(-\frac{a_{31}}{a_{11}}\right)$, but that's part of the grand algorithm—to which we now return!

2.1 The Grand Algorithm

As I said before there is one method that you are going to use to answer all of the questions that I might answer on this section. There are three different types of matrices you might want to use it on.

1. A—the coefficient matrix, for questions that can be answered using rank or about the determinant. Let's use equation (4) here, or:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}. \tag{18}$$

2. $\tilde{A} = [A\ b]$—the augmented matrix, for questions about solving for x in $Ax = b$. For example:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \end{bmatrix}.$$

3. $A^+ = [A\ I]$—a matrix which they don't have a term for. This is used if you want to find the inverse of a matrix. For example:

$$A^+ = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 1 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 \end{bmatrix}, \tag{19}$$

   (though in this case we should notice that since the number of rows and columns in A are not the same there can't be an inverse.)

But what you do to the matrix, however you denote it, will always be the same.

1. (Gaussian Elimination or transforming to Row-Echelon Form) Use elementary row operation two ($E_{ij}(r)$) so that $a_{i1} = 0$ for $i > 1$. After that do the same so that $a_{i2} = 0$ for $i > 2$. Continue until each row of the matrix starts with strictly more zero coefficients than the row before. *Note: If A is square and it is full rank the result will be upper-triangular.* Because of this we will denote the row-echelon form of A as U.

   At this point you are done if your question is about rank or the determinant. The rank is the number of non-zero rows, the determinant is the product of the diagonal elements. **Note: If your question is about rank you can use operations one (E_{ij}) and three ($E_i(r)$) as much as you want. However if it is about the determinant this will change it.**

   As an example to get equation (4) into row echelon form all I need to do is change a_{21} to zero. To do this I find out what r is such that:

$$\begin{aligned} ra_{11} + a_{21} &= 0 \\ r &= -\frac{a_{21}}{a_{11}}, \end{aligned} \tag{20}$$

and then multiply by the matrix $E_{21}\left(-\frac{a_{21}}{a_{11}}\right)$:

$$\begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}}a_{12} & a_{23} - \frac{a_{21}}{a_{11}}a_{13} \end{bmatrix}. \tag{21}$$

Of course there will be more steps if you have more than one row.

2. (Reduced Row-Echelon Form) If U is in row-echelon form then transform it so that:

   (a) The first non-zero coefficient in any row is one.

   (b) All coefficients above that first non-zero coefficient are zero.

If the rank of U is the number of rows of U then the result is that the left part of the matrix will be the identity matrix.[5] Let's allow that

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \end{bmatrix}, \tag{22}$$

and work on this matrix. Now the first thing we want to do is rescale so that $a_{11} = 1$, and to do this we multiply by:

$$E_1\left(\frac{1}{a_{11}}\right)U = \begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \end{bmatrix} = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} \end{bmatrix}. \tag{23}$$

Do it again with $E_2\left(\frac{1}{\tilde{a}_{22}}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\tilde{a}_{22}} \end{bmatrix}$:

$$E_2\left(\frac{1}{\tilde{a}_{22}}\right)E_1\left(\frac{1}{a_{11}}\right)U = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ 0 & 1 & \frac{\tilde{a}_{23}}{\tilde{a}_{22}} \end{bmatrix}. \tag{24}$$

Finally we find the r such that $r + \frac{a_{12}}{a_{11}} = 0$, or $r = -\frac{a_{12}}{a_{11}}$.

$$\begin{aligned} E_{12}\left(-\frac{a_{12}}{a_{11}}\right) &= \begin{bmatrix} 1 & -\frac{a_{12}}{a_{11}} \\ 0 & 1 \end{bmatrix}, \qquad (25) \\ E_{12}\left(-\frac{a_{12}}{a_{11}}\right)E_2\left(\frac{1}{\tilde{a}_{22}}\right)E_1\left(\frac{1}{a_{11}}\right)U &= \begin{bmatrix} 1 & 0 & \frac{a_{13}}{a_{11}} - \frac{a_{12}}{a_{11}}\frac{\tilde{a}_{23}}{\tilde{a}_{22}} \\ 0 & 1 & \frac{\tilde{a}_{23}}{\tilde{a}_{22}} \end{bmatrix}. \end{aligned}$$

You need to do this second step when ever you are working on either an augmented matrix ($\tilde{A}$) or in order to find the inverse matrix (A^{+}). For example if this was an augmented matrix we would know that:

$$x_1 = \frac{a_{13}}{a_{11}} - \frac{a_{12}}{a_{11}}\frac{\tilde{a}_{23}}{\tilde{a}_{22}},\ x_2 = \frac{\tilde{a}_{23}}{\tilde{a}_{22}} \tag{26}$$

[5] Remember the rank could be the number of columns or something that is lower than both the number of rows or columns.

3 Examples

Let's go through this for several examples. I will do two that are "nice" or square, and then two that are not.

Example 3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}, b_A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \tag{27}$$

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, A^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

first, if $r = -3$ *then* $r + 3 = 0$*, so we multiply by:*

$$E_{21}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{29}$$

$$E_{21}(-3)\tilde{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \tag{30}$$

$$E_{21}(-3)A^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{31}$$

Next we see the if $r = -1$ *then* $r + 1 = 0$*, so we go to:*

$$E_{31}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{32}$$

$$E_{31}(-1)\ldots\tilde{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 1 \\ 0 & -2 & -3 & 2 \end{bmatrix}, \tag{33}$$

$$E_{31}(-1)\ldots A^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & -2 & -3 & -1 & 0 & 1 \end{bmatrix}. \tag{34}$$

Now what r *makes* $-5r - 2 = 0$*? The answer is* $r = -\frac{2}{5}$*, so we multiply by:*

$$E_{32}\left(-\frac{2}{5}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 1 \end{bmatrix} \tag{35}$$

$$E_{32}\left(-\frac{2}{5}\right)...\tilde{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -5 & -7 & 1 \\ 0 & 0 & -\frac{1}{5} & \frac{8}{5} \end{bmatrix}, \quad (36)$$

$$E_{32}\left(-\frac{2}{5}\right)...A^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -7 & -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{1}{5} & -\frac{2}{5} & 1 \end{bmatrix}. \quad (37)$$

At this point we know two things. First of all, rank $(A) = 3$*, second of all* $\det(A) = 1 * (-5) * \left(-\frac{1}{5}\right) = 1$*. Thus we know it is invertible and has a unique solution for any b. Thus we go on to get it into reduced row-echelon form. To get the second diagonal coefficient to one we multiply that row by* $-1/5$ *or:*

$$E_{2}\left(-\frac{1}{5}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (38)$$

to get the third to one we multiply the last row by -5*, or*

$$E_{3}(-5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}, \quad (39)$$

the result of these operations is:

$$E_{3}(-5)...\tilde{A} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -8 \end{bmatrix}, \quad (40)$$

$$E_{3}(-5)...A^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -1 & 2 & -5 \end{bmatrix}. \quad (41)$$

Now what r *gives* $r + 2 = 0$*? Obviously* $r = -2$*, so we multiply by*

$$E_{12}(-2) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{12}(-2)...\tilde{A} = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{7}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -8 \end{bmatrix}, \quad (42)$$

$$E_{12}(-2)...A^{+} = \begin{bmatrix} 1 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -1 & 2 & -5 \end{bmatrix}. \quad (43)$$

Likewise we can see we need to use:

$$E_{13}\left(-\frac{1}{5}\right) = \begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{23}\left(-\frac{7}{5}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{5} \\ 0 & 0 & 1 \end{bmatrix} \quad (44)$$

$$E_{13}\left(-\frac{1}{5}\right)E_{23}\left(-\frac{7}{5}\right)...\tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -8 \end{bmatrix}, \quad (45)$$

$$E_{13}\left(-\frac{1}{5}\right)E_{23}\left(-\frac{7}{5}\right)...A^{+} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & -3 & 7 \\ 0 & 0 & 1 & -1 & 2 & -5 \end{bmatrix}. \quad (46)$$

and we can immediately see from $\tilde{A}$ that $x_1 = 2$, $x_2 = 11$, $x_3 = -8$. More generally we can see that

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -3 & 7 \\ -1 & 2 & -5 \end{bmatrix}$$

now since there is an inverse we know the general solution is $x = A^{-1}b_A$ so we can check our previous conclusion by seeing that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -3 & 7 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \\ -8 \end{bmatrix}, \quad (47)$$

and the result is the same.

Example 4

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}, b_D = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\tilde{D} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & 1 & 4 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}, D^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (48)$$

First $E_{21}(-3) = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$E_{21}(-3)\tilde{D} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad (49)$$

$$E_{21}(-3)D^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -3 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad (50)$$

now $E_{31}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$,

$$E_{31}(-1)\,E_{21}(-3)\,\tilde{D} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 0 & -2 & -2 & 2 \end{bmatrix}, \tag{51}$$

$$E_{31}(-1)\,E_{21}(-3)\,D^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -3 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{bmatrix}, \tag{52}$$

finally we use $E_{32}\left(-\frac{2}{5}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{2}{5} & 1 \end{bmatrix}$,

$$E_{32}\left(-\frac{2}{5}\right) E_{31}(-1)\,E_{21}(-3)\,\tilde{D} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -5 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{53}$$

$$E_{32}\left(-\frac{2}{5}\right) E_{31}(-1)\,E_{21}(-3)\,D^{+} = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -5 & -5 & -3 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & -\frac{2}{5} & 1 \end{bmatrix} \tag{54}$$

and we have row-echelon form. But now we can see that rank $(D) = 2$ *and* $\det(D) = 1 * (-5) * (0) = 0$, *thus this matrix is not invertible. But notice that we* ***can*** *solve the system of equations, because rank* $\left(\tilde{D}\right) = 2$. *I had to choose* b_D *just right so this would work out. To find this solution we have to continue solving the system of equations until it is in reduced row-echelon form. First we use:*

$$E_{2}\left(-\frac{1}{5}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55}$$

$$E_{2}\left(-\frac{1}{5}\right) E_{32}\left(-\frac{2}{5}\right) E_{31}(-1)\,E_{21}(-3)\,\tilde{D} = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{56}$$

then $E_{12}(-2) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_{12}(-2)\,E_{2}\left(-\frac{1}{5}\right) E_{32}\left(-\frac{2}{5}\right) E_{31}(-1)\,E_{21}(-3)\,\tilde{D} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{57}$$

and to characterize our solution we convert it back into standard equation form:

$$\begin{aligned}\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} &= \begin{bmatrix}1\\-1\\0\end{bmatrix} \\ x_1+x_3 &= 1 \\ x_2+x_3 &= -1\end{aligned} \tag{58}$$

Now I am going to do two non-square examples. With these examples I know I can not find the inverse, however everything else in the general algorithm may work.

Example 5

$$F=\begin{bmatrix}1&2\\4&1\\0&2\end{bmatrix}, b_F=\begin{bmatrix}1\\4\\1\end{bmatrix}, \tilde{F}=\begin{bmatrix}1&2&1\\4&1&4\\0&2&1\end{bmatrix} \tag{59}$$

First

$$E_{21}(-4)=\begin{bmatrix}1&0&0\\-4&1&0\\0&0&1\end{bmatrix}, E_{21}(-4)\tilde{F}=\begin{bmatrix}1&2&1\\0&-7&0\\0&2&1\end{bmatrix} \tag{60}$$

then $E_{32}\left(\frac{2}{7}\right)=\begin{bmatrix}1&0&0\\0&1&0\\0&\frac{2}{7}&1\end{bmatrix}$

$$E_{32}\left(\frac{2}{7}\right)E_{21}(-4)\tilde{F}=\begin{bmatrix}1&2&1\\0&-7&0\\0&0&1\end{bmatrix} \tag{61}$$

and of course we aren't that surprised that rank $(F)=\min(\mathrm{col}(F),\mathrm{row}(F))=2$*, we also know that there is no point in solving this problem because our reduced problem has three equations:*

$$\begin{bmatrix}1&2\\0&-7\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} \tag{62}$$

$$\begin{bmatrix}x_1+2x_2\\-7x_2\\0*x_1+0*x_2\end{bmatrix} = \begin{bmatrix}x_1+2x_2\\-7x_2\\0\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix} \tag{63}$$

and there is no way that we can ever have $0=1$*. So the rank is two and there is no solution to this problem. Furthermore the determinant is zero and there is no inverse because it's not square. On the other hand if you were given these two vectors you could conclude that their span is* $\mathbb{R}^2$*.*

Example 6

$$G=\begin{bmatrix}1&1&3\\1&2&4\end{bmatrix},b_G=\begin{bmatrix}2\\0\end{bmatrix},\tilde{G}=\begin{bmatrix}1&1&3&2\\1&2&4&0\end{bmatrix}. \qquad (64)$$

Like before there is no point in solving for G^{-1}*, the determinant is zero and there is no inverse because it is not square. Thus first:*

$$E_{21}(-2)=\begin{bmatrix}1&0\\-1&1\end{bmatrix},E_{21}(-2)\tilde{G}=\begin{bmatrix}1&1&3&2\\0&1&1&-2\end{bmatrix}. \qquad (65)$$

is the row-echelon form, and the rank is two. We don't have to rescale any of the coefficients so we just multiply it by:

$$E_{12}(-1)=\begin{bmatrix}1&-1\\0&1\end{bmatrix},E_{12}(-1)E_{21}(-2)\tilde{G}=\begin{bmatrix}1&0&2&4\\0&1&1&-2\end{bmatrix} \qquad (66)$$

and the solution is:

$$\begin{aligned}\begin{bmatrix}1&0&2\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}&=\begin{bmatrix}4\\-2\end{bmatrix}\\x_1+2x_3&=4\\x_2+x_3&=-2\end{aligned} \qquad (67)$$

Whew, so now you have four examples basically covering the possible cases. Now how you use this information, and what information you are after, will depend on the question you are faced with. I should also point out that except for the last example I didn't use any of the "cheats" that could make this problem easier for you. If diagonal coefficients end up being one and if there are a lot of zeros in the matrix then it will be easier to analyze, and you can expect assistance like that on quizzes and exams. To be perfectly frank once you've done one $E_{ij}(r)$ and $E_i(r)$ step you've shown you can do it. I might want more than one just to double check, but if I include too many steps (obviously) you won't be able to do the exam. You should be much more prepared for unsolvable problems, problems with a continuum of solutions, and problems with unique solutions—and their vector analysis counterparts.