1 General Equilibrium in an Exchange Economy.

When one studies a partial equilibrium model one assumes that all other prices are not affected. However this is not actually the case, consider—for example—if the price of coffee increases. Tea is a substitute for coffee, and so that means that the demand for tea will increase. What will this do? Why obviously it will increase the price of tea, and since coffee is a substitute for tea the demand for coffee will increase.

Wild! An increase in the price of a good has increased the demand for the good! Hunh? Well, yes, actually that is exactly what has happened. To be precise one needs to understand the path of causality, but this is precisely what could happen. And there's no reason for it to stop! When the demand for coffee increases that will increase the price of coffee, which will increase the demand for tea, which will increase the price of tea, which will increase the demand for coffee, which will increase the price of coffee...

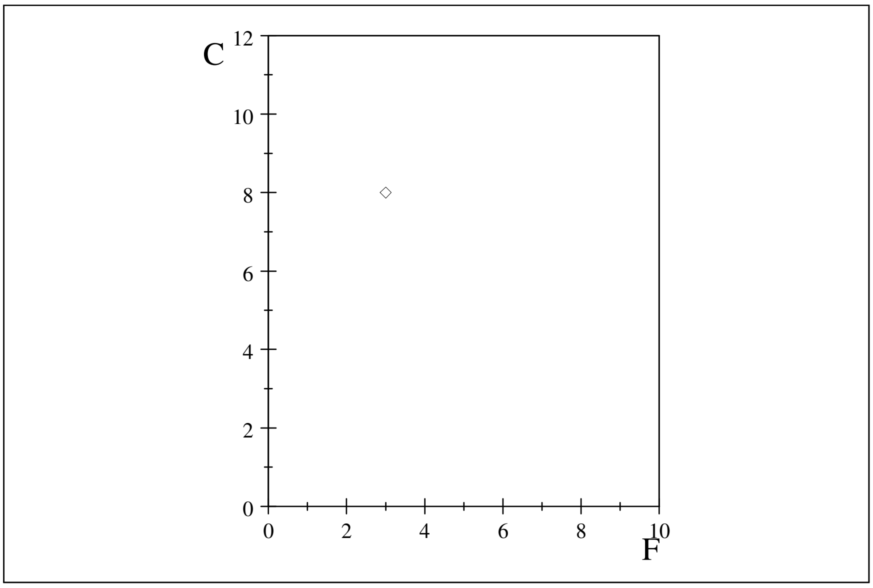
See the point? Without any more general understanding of this economy we could end up in a situation where the price of coffee and tea increases without bound, and the demand for the two goods could increase without bound. WILD!

So what do we do? Well we need to develop a more general framework. This handout is intended to present the simplest self-contained framework—the exchange economy. Much of what is done here will parallel the book, Snyder and Nicholson, but we will be doing it in a little bit more depth and in a slightly different format. The difference is primarily one of preference, and also has to do with the fact that I want to mathematically solve the problem—and I expect you to do so as well.

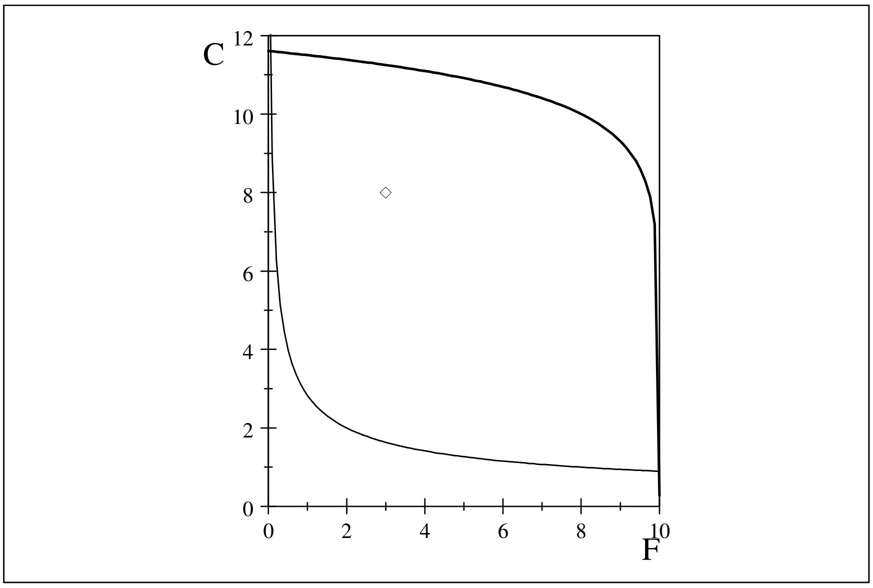
1.1 The Edgeworth Box Exchange Economy

If we are going to get anywhere in our understanding of this problem we need a problem simple enough to graph. So we are going to assume that there are only two people and two goods in the economy. These people are going to do nothing other than trade goods back and forth between each other. Call the two people 1 and 2 and the two goods F and C (for food and clothing.)

Now, how do we graph the consumption of both people on the same graph? It is actually not that difficult, let $\bar{F}$ be the total amount of food in the economy, and $\bar{C}$ be the total amount of clothing. Now let's graph the consumption possibilities of person 1. (Let $\bar{F} = 10$ and $\bar{C} = 12$).



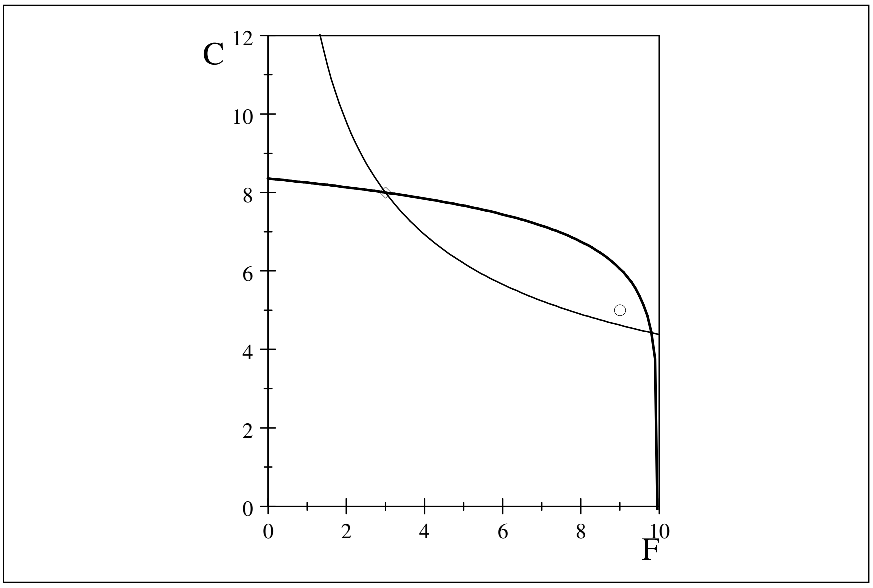
Person 1 can consume any point in that graph, like for example the diamond at $(3, 8)$ in the graph above. Now, what will person 2 have to consume? Well obviously $10 - 3 = 7$ units of F and $12 - 8 = 4$ units of C. But wait a second, if we were to flip this graph upside down then that would be exactly what point would be represented by the diamond. Isn't that convenient? In other words if we think of the origin for person 2 being $\left(\bar{F}, \bar{C}\right)$ then any point in the space becomes a consumption point for both people. Lets assume that person 1 has the utility function $U_1\left(F_1, C_1\right) = F_1 C_1^2$ and that person 2 has the utility function $U_2\left(F_2, C_2\right) = \ln\left(F_2\right) + C_2$, then we can draw an indifference curve for both of them in that graph. The thin curve that is bowed down is an indifference curve for person 1, The thick curve that is bowed up (bowed down towards person 2's origin) is an indifference curve for person 2.



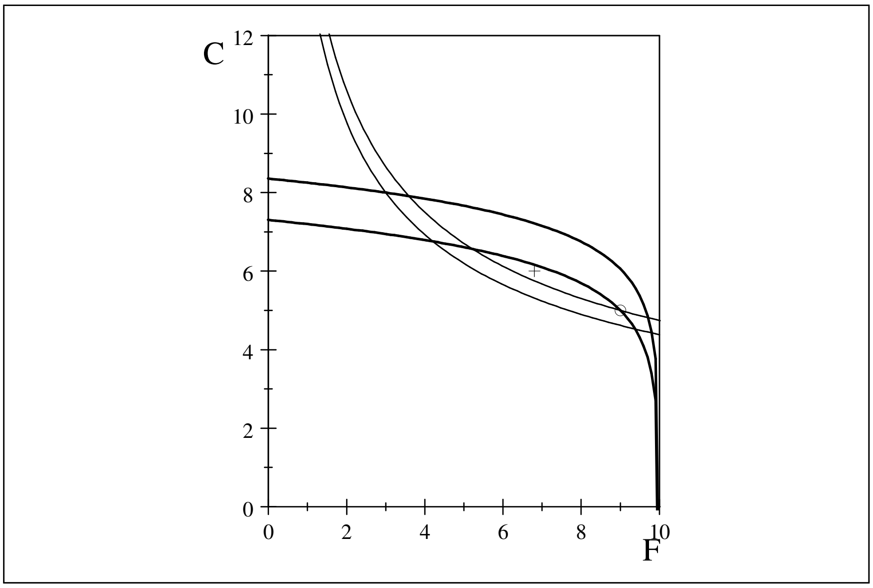
Thus we now have an elegant method for representing both people's consumption and preferences in a nice little graph. Now, what will happen if their initial endowments are $(F_1, C_1) = (3, 8)$ and $(F_2, C_2) = (7, 4)$ (this is the diamond in the graph)?

1.1.1 Efficiency is a Necessary condition with free trade.

So let's now graph the consumption bundle of these two individuals if they consume their initial endowments.



I will always draw person 2's indifference curves as a darker line than person 1's. Now, can this initial endowment be a final stopping point? No, clearly not, because both parties can be made happier by trading to some point like the circle, where $(F, C) = (9, 5)$.



So again, can they agree to stopping at the circle? No, clearly not because the cross is, again, better for both so they will clearly agree to switch to that bundle. And we could go on, and on, and on.

But let's cut to the chase, when will they be able to both agree to trade? If there is an eye created by the indifference curves through the bundle. So when can they not trade? If there is no eye. What will happen at this bundle? Their indifference curves will be tangent, or have the same slope.

Conclusion 1 *At any equilibrium in the exchange economy:*

$$MRS_1 = MRS_2$$

or

$$\frac{MU^1_{F_1}}{MU^1_{C_1}} = \frac{MU^2_{F_2}}{MU^2_{C_2}} \tag{1}$$

and $F_1 + F_2 = \bar{F}$, $C_1 + C_2 = \bar{C}$

Now that is rather interesting, and the most interesting thing about it is that this also indicates that:

Proposition 2 *Every equilibrium in the exchange economy is Pareto Efficient, or there is no way to make all parties better off.*

In fact I got you to agree to this proposition while I was arguing that the points above could not be an equilibrium. So what are the possible equilibria? Well we know that they have to be Pareto Efficient so we can use that fact to pin them down to a great degree.

Definition 3 *The* Contract Curve *is the set of Pareto Efficient outcomes in an exchange economy.*

Finding this is quite simple, in general what we do is solve condition 1 as an equation of C_1 in terms of F_1. In general this is:

$$\frac{MU^1_{F_1}(F_1, C_1)}{MU^1_{C_1}(F_1, C_1)} = \frac{MU^2_{F_2}(F_2, C_2)}{MU^2_{C_2}(F_2, C_2)}$$

so we use the fact that $F_2 = \bar{F} - F_1$ and $C_2 = \bar{C} - C_1$ to solve this for C_1.

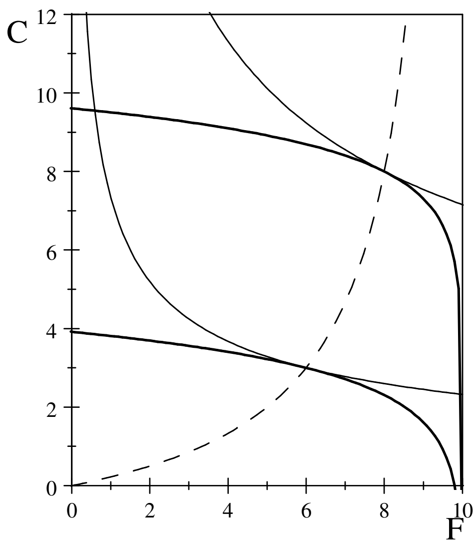
$$\frac{MU^1_{F_1}(F_1, C_1)}{MU^1_{C_1}(F_1, C_1)} = \frac{MU^2_{F_2}\left(\bar{F} - F_1, \bar{C} - C_1\right)}{MU^2_{C_2}\left(\bar{F} - F_1, \bar{C} - C_1\right)}$$

For example with the utility functions above

$$\begin{aligned} MU^1_{F_1}(F_1, C_1) &= \frac{U}{F_1}, MU^1_{C_1}(F_1, C_1) = 2\frac{U}{C_1}, \frac{MU^1_{F_1}(F_1, C_1)}{MU^1_{C_1}(F_1, C_1)} = \frac{\frac{U}{F_1}}{2\frac{U}{C_1}} = \frac{1}{2}\frac{C_1}{F_1} \\ MU^2_{F_2}(F_2, C_2) &= \frac{1}{F_2}, MU^2_{C_2}(F_2, C_2) = 1, \frac{MU^2_{F_2}(F_2, C_2)}{MU^2_{C_2}(F_2, C_2)} = \frac{\frac{1}{F_2}}{1} = \frac{1}{F_2} = \frac{1}{10 - F_1} \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\frac{C_1}{F_1} &= \frac{1}{10 - F_1} \\ C_1 &= \frac{2F_1}{10 - F_1} \end{aligned}$$

and we can draw this curve in the graph, it is the dashed increasing line. Notice that it goes through the origin for person 1, but it doesn't for person 2. This is because if person 2 is poor (they have a very low endowment) then they will do anything to consume F. To be precise when $F_2 \leq 1.4286$ the contract curve is the horizontal line where person 2 consumes only F and person 1 consumes everything else.



I have also drawn the indifference curves when $(F_1, C_1) = (6, 3)$ and $(8, 8)$, you can clearly see that they are tangent and it is not possible for both parties to find a new bundle that they both want to trade to.

1.1.2 Finding the General Equilibrium using Two different Methods.

First of all there is nothing in this problem that is any different from a standard utility maximization problem except that my income is now the prices times my endowment. Let's call the endowment for 1 (F_1^e, C_1^e) and the endowment for 2 (F_2^e, C_2^e) then we can just solve the problem like usual with $I_1 = p_f F_1^e + p_c C_1^e$.

For example for person 1 we have:

$$\begin{aligned}
\frac{MU^1_{F_1}}{p_f} &= \frac{MU^1_{C_1}}{p_c} \\
\frac{\frac{U}{F_1}}{p_f} &= \frac{2\frac{U}{C_1}}{p_c} \\
F_1 &= \frac{1}{2}C_1\frac{p_c}{p_f}
\end{aligned}$$

$$\begin{aligned}
p_f F_1 + p_c C_1 &= I_1 = p_f F_1^e + p_c C_1^e \\
p_f\left(\frac{1}{2}C_1\frac{p_c}{p_f}\right) + p_c C_1 &= p_f F_1^e + p_c C_1^e \\
\frac{3}{2}C_1 p_c &= p_f F_1^e + p_c C_1^e \\
C_1 &= \frac{2}{3}C_1^e + \frac{2}{3}\frac{p_f}{p_c}F_1^e \\
F_1 &= \frac{1}{2}C_1\frac{p_c}{p_f} = \frac{1}{3}\frac{p_c}{p_f}C_1^e + \frac{1}{3}F_1^e
\end{aligned}$$

For person 2 we have (assume he consumes both goods):

$$\begin{aligned}
\frac{MU^2_{F_2}}{p_f} &= \frac{MU^2_{C_2}}{p_c} \\
\frac{\frac{1}{F_2}}{p_f} &= \frac{1}{p_c} \\
F_2 &= \frac{p_c}{p_f}
\end{aligned}$$

$$\begin{aligned}
p_f F_2 + p_c C_2 &= I_2 = p_f F_2^e + p_c C_2^e \\
p_f\left(\frac{p_c}{p_f}\right) + p_c C_2 &= p_f F_2^e + p_c C_2^e \\
p_c\left(C_2 + 1\right) &= p_f F_2^e + p_c C_2^e \\
C_2 &= C_2^e + \frac{p_f}{p_c}F_2^e - 1
\end{aligned}$$

Now we need to solve for the third condition, supply equals demand. Well we have the demand of the two consumers above, so what we need to figure out is what is supply. But that is quite simple! It is nothing more than the sum of the endowments of the two parties. So in general:

$$\begin{aligned}
F_1\left(p_f, p_c\right) + F_2\left(p_f, p_c\right) &= F_1^e + F_2^e \\
C_1\left(p_f, p_c\right) + C_2\left(p_f, p_c\right) &= C_1^e + C_2^e
\end{aligned}$$

We call these the *market clearing equations*, in our example:

$$\begin{aligned}\frac{1}{3}\frac{p_c}{p_f}C_1^e + \frac{1}{3}F_1^e + \frac{p_c}{p_f} &= F_1^e + F_2^e \\ \frac{2}{3}C_1^e + \frac{2}{3}\frac{p_f}{p_c}F_1^e + C_2^e + \frac{p_f}{p_c}F_2^e - 1 &= C_1^e + C_2^e\end{aligned}$$

Now we can solve the first equation for p_c and we get $p_c = \frac{2F_1^e p_f + 3F_2^e p_f}{C_1^e + 3}$, now if we plug that into the second equation we get:

$$\frac{2}{3}C_1^e + \frac{2}{3}\frac{p_f}{\frac{2F_1^e p_f + 3F_2^e p_f}{C_1^e+3}}F_1^e + C_2^e + \frac{p_f}{\frac{2F_1^e p_f + 3F_2^e p_f}{C_1^e+3}}F_2^e - 1 = C_1^e + C_2^e$$

Wait, that can't be right, when I simplify the left hand side I get the right hand side. Why can't we solve for p_f? Well, this goes back to one of the great insights of Microeconomic theory, *everything is relative.* Who cares about the absolute value of p_f or p_c? All we care about is their relative values. You can clearly see this from the budget constraint:

$$p_f F_1 + p_c C_1 = p_f F_1^e + p_c C_1^e$$

if we multiply all prices by $\alpha > 0$ then whatever the value of α the equation still holds. So what does this mean? It means we can set p_f or p_c or $p_f + p_c$ to a constant and proceed. What constant? Well 1 is always good. Which one do we set to one? Whichever you want, in *some* cases it may matter, and generally choosing the right one makes your work easier, but heh. But even if we do this we are not done simplifying our problem yet. Let's set $p_f = 1$, just because. Then our two equations are:

$$\begin{aligned}\frac{1}{3}p_c C_1^e + \frac{1}{3}F_1^e + p_c &= F_1^e + F_2^e \\ \frac{2}{3}C_1^e + \frac{2}{3}\frac{1}{p_c}F_1^e + C_2^e + \frac{1}{p_c}F_2^e - 1 &= C_1^e + C_2^e\end{aligned}$$

Lets simplify the second one,

$$\begin{aligned}\left(\frac{2}{3}C_1^e + \frac{2}{3}\frac{1}{p_c}F_1^e + C_2^e + \frac{1}{p_c}F_2^e - 1\right)p_c &= \left(C_1^e + C_2^e\right)p_c \\ \frac{2}{3}C_1^e p_c - p_c + C_2^e p_c + \frac{2}{3}F_1^e + F_2^e\left(+\frac{1}{3}F_1^e - \frac{1}{3}F_1^e\right) &= C_1^e p_c + C_2^e p_c\end{aligned}$$

$$\begin{aligned}\frac{2}{3}C_1^e p_c - p_c + C_2^e p_c - \frac{1}{3}F_1^e + F_1^e + F_2^e - \left(\frac{2}{3}C_1^e p_c - p_c + C_2^e p_c - \frac{1}{3}F_1^e\right) &= \\ C_1^e p_c + C_2^e p_c - \left(\frac{2}{3}C_1^e p_c - p_c + C_2^e p_c - \frac{1}{3}F_1^e\right)&\end{aligned}$$

$$F_1^e + F_2^e = p_c + \frac{1}{3}C_1^e p_c + \frac{1}{3}F_1^e$$

Whoops, this is the first equation... What? Why? Is this always true? Yes it is, these two facts give us what is called *Walrus's Law.*

Definition 4 (Walrus's Law) *In any general equilibrium problem if there are m goods then you can only solve for $m-1$ prices and only need to solve $m-1$ market clearing condition.*

Now let's get a little more concrete and solve our simplified problem with the endowments $(F_1^e, C_1^e) = (6,3)$ and $(F_2^e, C_2^e) = (4,9)$, then we can solve the market clearing equation for F.

$$\begin{aligned} \frac{1}{3}p_c(3) + \frac{1}{3}(6) + p_c &= 6+4 \\ 2p_c + 2 &= 10 \\ p_c &= 4 \end{aligned}$$

Now that was what I call pretty easy. I won't ask you to solve the general, abstract problem, I will only ask you to solve it for given, fixed, endowments. But is there another way? Yes there is. Is it easier? Well... in general it's one of those personal things. I always find it easier because I don't have to solve for demand curves. But perhaps you like solving for demand curves, like I said, it's personal.

Finding the General Equilibrium using the Contract Curve and Utility Maximization. We can also use the fact that we know the general equilibrium must also be Pareto Efficient. Remember above we said it must be true in equilibrium through the counter factual argument I made before stating the proposition. So we can use that to our advantage, we only need that and solving one person's utility maximization problem to solve the problem, remember that $(F_1^e, C_1^e) = (6,3)$ and the contract curve is:

$$C_1 = \frac{2F_1}{10 - F_1}\ .$$

So the two equations that give us utility maximization are:

$$\begin{aligned} F_1 &= \frac{1}{2}C_1 p_c \\ F_1 + p_c C_1 &= 6 + p_c 3 \end{aligned}$$

We can plug the contract curve into the first one:

$$\begin{aligned}
F_1 &= \frac{1}{2}\left(\frac{2F_1}{10-F_1}\right)p_c \\
F_1 &= -F_1\frac{p_c}{F_1-10} \\
1 &= -\frac{p_c}{F_1-10} \\
F_1 &= -p_c+10 \\
C_1 &= 2\frac{F_1}{p_c} = 2\frac{(-p_c+10)}{p_c} = \frac{20}{p_c}-2
\end{aligned}$$

and plug these two equations into the budget constraint:

$$\begin{aligned}
-p_c+10+p_c\left(2\frac{(-p_c+10)}{p_c}\right) &= 6+p_c3 \\
30-3p_c &= 6+p_c3 \\
30-6 &= 6p_c \\
p_c &= 4
\end{aligned}$$

voila! It actually might have been easier if I had started with the other person, but it isn't, but I will show you that method just for kicks. The hardest bit about this method is that I solved the contract curve in terms of F_1 and C_1, I need to change it so that it is in terms of $C_2 = 12 - C_1$ and $F_2 = 10 - F_1$.

$$\begin{aligned}
F_2 &= p_c \\
F_2+p_cC_2 &= 9p_c+4
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{2F_1}{10-F_1} \\
C_2 &= \frac{14F_2-20}{F_2}
\end{aligned}$$

$$\begin{aligned}
p_c+p_c\left(\frac{14p_c-20}{p_c}\right) &= 9p_c+4 \\
15p_c-20 &= 9p_c+4 \\
p_c &= \frac{4+20}{15-9} = 4
\end{aligned}$$