

The Income and the Substitution Effect.

ECON 203
Kevin Hasker

1 Introduction

I hate Consumer Theory. Really I do. Why? Because of the Income and Substitution effects.

You know demand curves are downward sloping, right? Yea, everyone knows that. If you lower the price of something you sell more of it. Just imagine how ridiculous the opposite would be, you are having a hard time selling fur coats so you raise the price? Nobody but an idiot could think such a thing.

Or a theorist, one named Giffen. I hate the guy, not that he was wrong. He was right, it's just that what he proved is annoying. It is possible for a demand curve to be upward sloping, because the Income effect can outweigh the Substitution effect. In honor of this insane discovery we have named goods with upward sloping demand curves after this guy, or Giffen Goods. The only reason that you should remember them is because they will help you understand the substitution and income effects.

If it wasn't for the Income effect Consumer Theory would be just the same thing as producer theory. It's almost a joke that we can do the exact same theory twice, once we get to call it Consumer theory and once we call it Producer theory. And you guys think you have to learn an entire new theory, it's fun. But there is one difference—only one. The Income effect. I hate it.

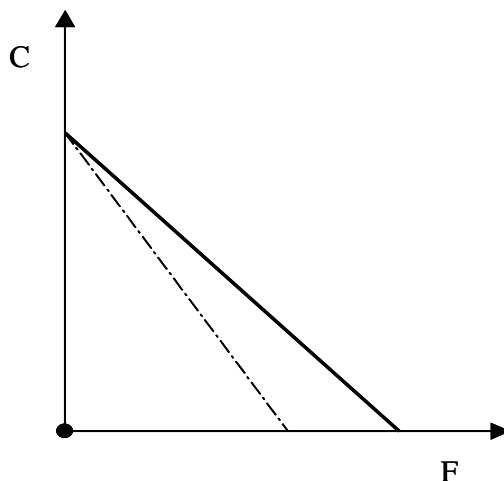
First let's define income properly, and then look at a graphical analysis of what happens when a price changes. Remember how I said on the first day of class that everything's relative? This goes for income as much as anything. If I tell you you're going to make 10,000,000 a year do you know what your first question should be? "Lira or Dollars?" Yep, it makes a difference, as I think you know.

So what do I mean when I want to talk about a income effect? I am talking about the change in *real income*, or the budget set.

$$RI = \{\{F, C\} | p_f F + p_c C \leq I\}$$

or in words my real income is the possible bundles of goods I can buy. If this set increases—for any reason—then I'm happier. For example, if I promised you that if you consumed 5 F and 3 C then I would give you 2 F, this is an increase in real income. Even if you actually don't consumer 5 F and 3 C just this possibility increases your real income. Of course your "income" in the traditional sense hasn't changed, but what do you care? If you choose 5 F and 3 C you can now get more!

OK, now on to the next step, a graphic analysis. Below we have an increase in the price of food from $p_f = p_c$ to $p_f = \frac{3}{2}p_c$. What two changes have occurred?



Well first of all, the slope of the budget set has changed. This slope is $-\frac{p_f}{p_c}$. But there is a second effect, *the new box is smaller!* So the two effects are:

1. Substitution or Relative Price Effect—change in $-\frac{p_f}{p_c}$, in this case it is increasing in absolute value.
2. Income or Real Income Effect—change in RI , and it always falls when a price rises.

Now the first effect is very predictable, the relative price of a good increases the demand for that good falls. That's a fact, we know it. But the second effect is not so predictable. Sometimes when income rises people demand more of a good, sometimes they demand less. It all depends on...

1.1 The Elasticity of Income and *ENGEL CURVES*

First of all the elasticity of income is the percentage change in quantity consumed with respect to the percentage change in income, or:

$$e_F(I) = \frac{\partial F}{\partial I} \frac{I}{F} \approx \frac{\% \Delta F}{\% \Delta I} = \frac{\frac{F_o - F_n}{F_o}}{\frac{I_o - I_n}{I_o}} = \frac{F_o - F_n}{I_o - I_n} \frac{I_o}{F_o}$$

If you don't understand this then I suggest you go read chapter seven. It is—essentially—a unitless way of measuring the change in F when I changes. There are three important ranges for it:

1. Inferior— $e_F(I) \leq 0$ —these goods are things only poor people buy. Like public transportation, as people's income increases they stop using public

transportation as much. Another example is bread, if your poor your dinner will be “bread with a little bit of something” if your rich your dinner will be “something with a little bit of bread.” The quantity of bread you consume decreases as you get richer.

2. Normal— $0 \leq e_F(I) \leq 1$ —these goods you spend more of as your income increases, but the percentage of your income you spend on these goods decreases. These are goods the “normal” or middle class people consume. People who aren’t too worried about their income but still can’t afford only the best. A car, owning your apartment, standard clothes. To show you what I mean about “the share decreases” here’s a precise proof:

$$\begin{aligned}
 s_f &= \frac{p_f F}{I}, \frac{\partial s_f}{\partial I} \leq 0 \Rightarrow \frac{\partial}{\partial I} \left(\frac{p_f F}{I} \right) \leq 0 \\
 \frac{\partial}{\partial I} \left(\frac{p_f F}{I} \right) &= \frac{p_f}{I} \frac{\partial F}{\partial I} - \frac{p_f F}{I^2} \leq 0 \\
 \left(\frac{p_f}{I} \frac{\partial F}{\partial I} - \frac{p_f F}{I^2} \right) \frac{I}{F} &\leq 0 * \frac{I}{F} \\
 \frac{p_f}{I} \frac{\partial F}{\partial I} \frac{F}{I} - \frac{p_f}{I} &= \frac{p_f}{I} e_f(I) - \frac{p_f}{I} = \frac{p_f}{I} (e_f(I) - 1) \leq 0
 \end{aligned}$$

and there it is.

3. Luxury— $e_F(I) \geq 1$ —the rich people’s goods. Mercedes, designer clothes, your very own yacht, that sort of things. Things average people spend money on just because they can, not because they really think they’re that important.

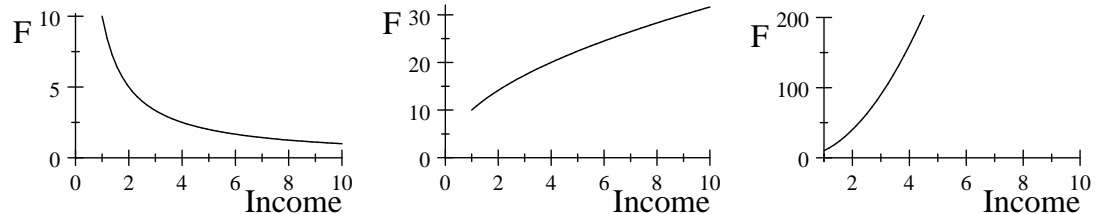
The three type of curves can be illustrated graphically. On the horizontal axis is Income, on the vertical axis is Quantity. The Demand curve that all of these are based on is:

$$F = \frac{\alpha I^\psi}{P^\beta}$$

for this demand curve:

$$e_F(I) = \psi$$

I set $P = 1, \alpha = 100$ for simplicity, and then in the left hand graph $\psi = -1$ and the good is inferior, in the center graph $\psi = \frac{1}{2}$, in the right hand graph $\psi = 2$.



.....Inferior Good.....Normal Good.....Luxury Good

These are called “Engel Curves” after the economist who first studied them. He found an astonishing and well established empirical regularity. Food (in aggregate) is a normal good. Or in other words $\frac{\partial F}{\partial I} \geq 0$ —richer people consume more food, but $\frac{\partial s_f}{\partial I} \leq 0$ —the percentage of their income that they spend on food decreases.

1.2 The Substitution and Income Effect

Now we want to translate what we have seen into a more precise mathematical form. In this section we will start with our basic insight and get it into a graphical representation. In the next section we will transform this representation into a mathematical form.

Now what we have seen is that:

$$\begin{aligned} \text{Total effect of a change in } p_f \text{ on } F &= \text{Change due to change in Relative Price } \left(\frac{p_f}{p_c}\right) \\ &\quad + \text{Change due to decrease in Real Income.} \\ &= \text{Substitution Effect} \\ &\quad + \text{Income Effect} \end{aligned}$$

using mathematical notation we can write this as:

$$\frac{\Delta F}{\Delta P_f} = \frac{\Delta F}{\Delta \left(\frac{p_f}{p_c}\right)} - \frac{\Delta F}{\Delta RI}$$

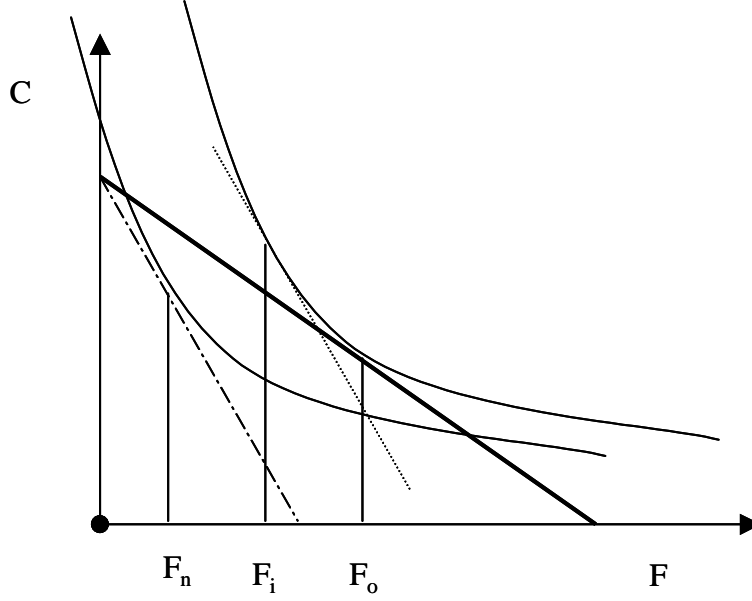
where the minus sign is used to indicate that when p_f increase RI decreases.

Let’s think about $\frac{\Delta F}{\Delta \left(\frac{p_f}{p_c}\right)}$. What is the difference between this and $\frac{\Delta F}{\Delta P_f}$?

The total effect takes into consideration the change in real income, thus when we look at $\frac{\Delta F}{\Delta \left(\frac{p_f}{p_c}\right)}$ we want to analyze the pure effect of the change in relative prices on behavior. Now if our utility changed when we analyzed $\frac{\Delta F}{\Delta \left(\frac{p_f}{p_c}\right)}$ then there would be two effects in our analysis. First the change due to the change in our happiness level and then the change due to prices, so we want to hold this constant.

Graphically this means we want to stay on the same indifference curve, and

this means that graphically our analysis looks something like this:



F_o is the old level of consumption of food, F_n is the new level, and F_i is the level when the change in real income is controlled for. Given this example we can also write the changes as:

$$\begin{aligned}\frac{\Delta F}{\Delta P_f} &= F_n - F_o \\ \frac{\Delta F}{\Delta \left(\frac{p_f}{p_c} \right)} &= F_i - F_o \\ -\frac{\Delta F}{\Delta RI} &= F_n - F_i \\ \frac{\Delta F}{\Delta P_f} &= \frac{\Delta F}{\Delta \left(\frac{p_f}{p_c} \right)} + \left(-\frac{\Delta F}{\Delta RI} \right)\end{aligned}$$

1.2.1 Comparing this to Expenditure/Cost Minimization

What we are actually doing in the second step is minimizing our expenditure while keeping our happiness level at some predetermined level, \bar{U} . Mathematically this is:

$$\begin{aligned}\min & p_f F + p_c C \\ \text{s.t. } & U(F, C) \geq \bar{U}\end{aligned}$$

Intuitively this is something similar to what parent's do with their children's allowance. If the price of—say—candy bars goes up the child goes running to his mother, begging for more money. “After all, I always get to buy one candy bar a day, it's just sooooo important.” Did that work with your mother? Right, not with mine either. But if IMPORTANT expenses went up I could indeed expect a raise in my allowance. My parents were basically trying to keep me at a certain happiness level, \bar{U} , while not spending too much money. Formally the solution to this problem is:

$$I(p_f, p_c, \bar{U}) = \min_{F, C} \max_{\lambda} p_f F + p_c C - \lambda (U(F, C) - \bar{U})$$

and $I(p_f, p_c, \bar{U})$ is the expenditure function. This will be important in a second. What also will be important? One of the most fundamental results in economic theory.

The Envelope Theorem. You are talking to your parents about your younger siblings. And your parents say “the price of candy bars just went up 5%, how much should we raise their allowances?” (Remember that in the fantasy land of this handout your parents reaction is not “zero.”) Your first reaction is that this is a simple problem, how about just 5% times the number of candy bars they consume? Your parents are all in panic however. “But the thing is even if we give them that (5% * # Candy Bars) they still will change how many candy bars they consume. They might buy more baklava or (shudder) even more fruit!” And you think about their answer, and grant them how they are right. People do react to price changes, even if you work to keep them just as happy as before.

But, you say, obviously your folks don't know about the Envelope theorem. Let's assume that it's the price of food that has risen, then formally your parents want to know what:

$$\frac{\partial I(p_f, p_c, \bar{U})}{\partial p_f}$$

is. Well let's take this derivative.

$$\begin{aligned} \frac{\partial I(p_f, p_c, \bar{U})}{\partial p_f} &= F \\ &+ p_f \frac{\partial F}{\partial p_f} + p_c \frac{\partial C}{\partial p_f} \\ &- \lambda \frac{\partial U}{\partial F} \frac{\partial F}{\partial p_f} - \lambda \frac{\partial U}{\partial C} \frac{\partial C}{\partial p_f} \end{aligned}$$

now we can re-arrange this and write it as:

$$\begin{aligned}\frac{\partial I(p_f, p_c, \bar{U})}{\partial p_f} &= F \\ &+ \left(p_f - \lambda \frac{\partial U}{\partial F}\right) \frac{\partial F}{\partial p_f} \\ &+ \left(p_c - \lambda \frac{\partial U}{\partial C}\right) \frac{\partial C}{\partial p_f}\end{aligned}$$

so what we need to figure out is what exactly $(p_f - \lambda \frac{\partial U}{\partial F}) \frac{\partial F}{\partial p_f}$ is equal to. But wait a minute! Let's look at the first order conditions of our expenditure minimization problem.

$$\begin{aligned}p_f - \lambda \frac{\partial U}{\partial F} &= 0 \\ p_c - \lambda \frac{\partial U}{\partial C} &= 0 \\ U(F, C) - \bar{U} &= 0\end{aligned}$$

Woa Nelly! But that means that $(p_f - \lambda \frac{\partial U}{\partial F}) \frac{\partial F}{\partial p_f} = 0$, and $(p_c - \lambda \frac{\partial U}{\partial C}) \frac{\partial C}{\partial p_f} = 0$ and that means:

$$\frac{\partial I(p_f, p_c, \bar{U})}{\partial p_f} = F$$

which is what you told your folks in the first place. If they don't understand this explanation tell them that Dr. Hasker will gladly try to explain it to them if they stop by. Or they could just trust you that you paid attention in ECON 201. They really should be trusting you by now anyway, sheesh.

In general the envelope theorem is that when you take the derivative of an optimized function (like the expenditure function, the cost function, or the profit function) then the only effect is the direct effect. All of the indirect effects (like the change in your consumption of F and C in this question) drop out because the marginal benefit and marginal cost is equalized in any optimum.

Notice how beautiful this result is, it changed what was a nearly impossible problem (keeping your kids happy when prices change) into one of just knowing what your kids consume. Cool eh? I love this one. But why am I telling you this? Stay tuned for the answer to this and other important questions!!!!

1.2.2 The Slutsky Equation

The solution to the expenditure minimization problem above are the *hicksian demand curves*, we denote these

$$\begin{aligned}h_f(p_f, p_c, u) \\ h_c(p_f, p_c, u)\end{aligned}$$

h_f is the quantity of food this person will buy, and h_c is the quantity of clothing. Notice the big difference between these and the “normal” or *marshallian demand curves*:

$$\begin{aligned} F(p_f, p_c, I) \\ C(p_f, p_c, I) \end{aligned}$$

in the Marshallian demand curves your demand is affected by your income, while in the Hicksian demand curves your demand is affected by your utility, or happiness level. One of the deepest results in economic theory is the *duality theorem*. One thing this means that if $I = I(p_f, p_c, u)$ then

$$h_f(p_f, p_c, u) = F(p_f, p_c, I)$$

or more succinctly we know that:

$$h_f(p_f, p_c, u) = F(p_f, p_c, I(p_f, p_c, u))$$

at this point let's take the derivative of both sides with respect to p_f .

$$\frac{\partial h_f}{\partial p_f} = \frac{\partial F}{\partial p_f} + \frac{\partial F}{\partial I} \frac{\partial I}{\partial p_f}$$

And looking back at the envelope theorem, we know that $\frac{\partial I}{\partial p_f} = F$. So this equation can be written as:

$$\begin{aligned} \frac{\partial h_f}{\partial p_f} &= \frac{\partial F}{\partial p_f} + \frac{\partial F}{\partial I} F \\ \frac{\partial F}{\partial p_f} &= \frac{\partial h_f}{\partial p_f} - \frac{\partial F}{\partial I} F \end{aligned}$$

and this is the grand conclusion. This is the big one. In this one little equation everything I've been talking about is included. This is a one line representation of the difference between Consumer Theory and Producer Theory. The Slutsky Equation. Wow, now we can understand this result more precisely, those two effects I was speaking about earlier are summarized in this equation, to be precise:

$$\begin{aligned} \frac{\Delta F}{\Delta P_f} &\approx \frac{\partial F}{\partial p_f} \\ \frac{\Delta F}{\Delta \left(\frac{p_f}{p_c} \right)} &\approx \frac{\partial h_f}{\partial p_f} \\ \frac{\Delta F}{\Delta R I} &\approx \frac{\partial F}{\partial I} F \end{aligned}$$

It also assists our analysis to translate this into elasticity terms:

$$\begin{aligned}\frac{\partial F}{\partial p_f} \frac{p_f}{F} &= \left(\frac{\partial h_f}{\partial p_f} - \frac{\partial F}{\partial I} F \right) \frac{p_f}{F} \\ e_f(p_f) &= \frac{\partial h_f}{\partial p_f} \frac{p_f}{F} - \frac{\partial F}{\partial I} \frac{1}{F} p_f F \frac{I}{I} \\ e_f(p_f) &= e_f^h(p_f) - \frac{\partial F}{\partial I} \frac{I}{F} \frac{p_f F}{I} \\ e_f(p_f) &= e_f^h(p_f) - e_f(I) s_f\end{aligned}$$

where $s_f = \frac{p_f F}{I}$ is the share of income spent on food. Now, let's return to the crazy phenomena that I started this entire discussion with. How can it be possible that demand curves can slope up?

$$\begin{aligned}e_f(p_f) &\geq 0 \\ e_f^h(p_f) - e_f(I) s_f &\geq 0 \\ -e_f(I) s_f &\geq -e_f^h(p_f)\end{aligned}$$

I've written this so the right hand side is positive, we know that the hicksian demand curve is downward sloping, so the RHS of this inequality is always positive. Thus the left hand side must be positive or:

1. F must be an *inferior good*. ($e_f(I) \leq 0$ thus $-e_f(I) s_f \geq 0$)
2. The *income effect* ($e_f(I) s_f$) must outweigh the *substitution effect* ($e_f^h(p_f)$).
In general this requires that:

- (a) $|e_f(I)|$ is "large."
- (b) s_f is "large."

One example that is always given is potatoes during the Irish potato famine. Rich Irish thought potatoes were gross, so it had a large $e_f(I)$ (large and negative of course). As well, since it was the primary food stuff for the poor, s_f was large. But I don't know if I believe this, Giffen goods are primarily a theoretic curiosity. The only common example is the demand for leisure. The price of an hour of leisure is the wage you are giving up by not working, so the question is if someone offered you enough money would you work less? The answer is almost surely yes, but this means that your demand for leisure increases when the price of your leisure increases! Notice that in this case s_l is large because every hour you don't work you spend on leisure.

But Giffen goods aren't important. Their only real import is to help you understand the Income and Substitution effect. Which I hope you do now.

$$\begin{aligned}
 \text{Total effect of a change in } p_f \text{ on } F &= \text{Change due to change in Relative Price } \left(\frac{p_f}{p_c}\right) \\
 &+ \text{Change due to decrease in Real Income.} \\
 &= \text{Substitution Effect} \\
 &+ \text{Income Effect} \\
 \frac{\Delta F}{\Delta P_f} &= \frac{\Delta F}{\Delta \left(\frac{p_f}{p_c}\right)} - \frac{\Delta F}{\Delta RI} \\
 \frac{\partial F}{\partial p_f} &= \frac{\partial h_f}{\partial p_f} - \frac{\partial F}{\partial I} F
 \end{aligned}$$

1.3 A Mathematical Example.

Consider the utility function

$$U(F, C) = FC$$

I expect you to be able to derive that:

$$\begin{aligned}
 F(p_f, p_c, I) &= \frac{1}{2} \frac{I}{p_f} \\
 C(p_f, p_c, I) &= \frac{1}{2} \frac{I}{p_c}
 \end{aligned}$$

and let $I = 80$, $p_f = 10$ and $p_c^0 = 2$, $p_c^n = 8$. Thus

$$\begin{aligned}
 C_o &= 20 \\
 C_n &= 5
 \end{aligned}$$

Now how do we find the intermediate level of C ? Well what we want to do is keep the level of happiness at the old level, which is $U_o = F_o C_o = \frac{1}{2} \frac{I}{p_f} * 20 = 4 * 20 = 80$. And we want to minimize our expenditure. You can check that this means you want to equalize your "buck for the bang" which is just the inverse of the "bang for the buck."

$$\begin{aligned}
 \frac{p_f}{MU_f} &= \frac{p_c}{MU_c} \\
 \frac{p_f}{\frac{U}{F}} &= \frac{p_c}{\frac{U}{C}} \\
 F_I &= C_I \frac{p_c}{p_f}
 \end{aligned}$$

and then we plug this into

$$\begin{aligned} U_o &= F_I C_I \\ 80 &= \left(C_I \frac{p_c}{p_f} \right) C_I \\ C_I &= \sqrt{80 \frac{p_f}{p_c}} \end{aligned}$$

and we use the new prices, or $\frac{p_f}{p_c} = \frac{10}{8}$ $C_I = 10$. Thus:

$$\begin{aligned} \frac{\Delta C}{\Delta P_c} &= C_n - C_o = 5 - 20 = -15 \\ \frac{\Delta C}{\Delta \left(\frac{p_f}{p_c} \right)} &= C_i - C_o = 10 - 20 = -10 \\ -\frac{\Delta C}{\Delta R_I} &= C_n - C_i = 5 - 10 = -5 \end{aligned}$$

and this is the solution for this example.