Handout on
Inverting Matrices using Elementary Matrices

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For Econ 225

1 Introduction

In this handout I am going to go through how to use elementary matrices to invert matrices. Beyond a methodology handout this handout also seeks to point out:

1. As a general methodology, elementary row operations are the best method for solving a system of linear equations.
2. Elementary matrices are simply another way of writing elementary row operations.
3. The amount of work required to solve for the inverse matrix is only slightly more than to solve a given system of equations.
4. If you solve for the inverse matrix you will have solved the problem for any right hand side variables.

I will take it as given that you know how to use the elementary row operations, which are:

Definition 1 *The* ***elementary row operations*** *are:*

1. *Interchanging rows of the matrix.*
2. *Adding a multiple of one row to another row.*
3. *Multiplying a row by a non-zero constant.*

The general problem we will be looking at can be written as:

$$Ax = b \ , \tag{1}$$

where $A = [a_{ij}]_{i=1...n}^{j=1...n}$ is an $n \times n$ matrix, $x = [x_1 \ x_2 \ x_3 \ ...x_n]^T$ and $b = [b_1 \ b_2 \ b_3 \ ...b_n]^T$. When A is non-singular and A^{-1} is it's inverse the general solution to this is

$$x = A^{-1}b \ . \tag{2}$$

Finding A^{-1} is only a little harder than solving the augmented matrix: $\tilde{A} = [A|b]$

2 The Elementary Matrices, and how they are equivalent.

So let I be the $n \times n$ identity matrix, or $a_{ii} = 1$ for $i \in \{1, 2, 3, ..., n\}$ and $a_{ij} = 0$ if $i \neq j$. Then the elementary matrices are:

1. E_{ij}—switch row i and j in I. (equivalent to elementary row operation 1)
2. $E_{ij}(r)$—take the identity matrix and change it so that $a_{ij} = r$ (equivalent to operation 2, to be precise we are multiplying row j by a constant and adding it to row i).
3. $E_i(r) = E_{ii}(r)$—in the identity matrix have $a_{ii} = r$ (operation 3, multiply row i by r).

Now to see how these operations work let's consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \tag{3}$$

Now, let's first switch rows one and three. This is supposed to be done by multiplying E_{13} times this:

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{4}$$

First of all, does it work? Yes, notice that

$$E_{13}A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \tag{5}$$

Second of all, why does it work? There is a way to understand it by thinking about each row at a time. The first row is basically:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \tag{6}$$

and that first matrix ($\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$) tells us to pick out the third element of each column. Of course that third element is what we wanted in the first place. Hopefully thinking about that example for a while makes it clear why this works.

Now onto the mysterious one, $E_{ij}(r)$. In order to explain this one let us convert A (equation 3) into row-echelon form. Now the first thing we would want to do is multiply the first row by -1 and add it to the second row.

$$(-1)\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}. \tag{7}$$

And let's realize that we could, quite easily, also add zero times the third row into this, then we would have the sum

$$(-1)\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} + (1)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + (0)\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}, \tag{8}$$

: but who would be stupid enough to do that? Well, me for one. Because once I do this I see that I want the second row to be the result of multiplying $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$ times my matrix:

$$\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}.$$

Cool, I like that. Now I want the first and the third rows to be the same as before, so I use the identity row like before. For the third one (for example) I would use:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}. \tag{9}$$

So this would give me the matrix:

$$E_{21}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$E_{21}(-1)\,A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$$

OK, and we need to do the same thing again to get the 2 in the last row to be zero. Or in other words we want to do:

$$(-2)\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} + (0)\begin{bmatrix} 0 & -1 & -1 \end{bmatrix} + (1)\begin{bmatrix} 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -3 \end{bmatrix}. \tag{11}$$

and hopefully you realize that this means we actually want to use

$$E_{31}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \tag{12}$$

$$E_{31}(-2)\,E_{21}(-1)\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{bmatrix}. \tag{14}$$

Now, a comment is called for at this point. One can do both of these steps at once, no reason not to. This matrix is not one of the elementary matrices, but it does the exact same job.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} M_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad (15) \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \\ \begin{bmatrix} -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \end{bmatrix} \end{aligned}$$

$$M_1 A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{bmatrix} \qquad (16)$$

The second expansion shows you how you will actually calculate each row, and you can see that you can easily change rows 2 and 3 at the same time.

Now hopefully you can see how to continue this. In our new matrix we need to multiply row 2 by -2 and add it to row 3. You should be able to go through the logic of why this would be the same as multiplying by:

$$\begin{aligned} E_{32}(-2) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad (17) \\ E_{32}(-2) M_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \end{aligned}$$

and we have achieved row-echelon form. Let me continue to reduced row echelon form, which will essentially turn our square matrix into the identity matrix (unless it is singular, or not invertible). The next step is to rescale each row, or in other words we want to multiply the second and the third row by -1. Which is clearly equivalent to:

$$\begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \qquad (18)$$

and we can do this in either two steps or one.

$$E_2(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{19}$$

and of course I'm lazy so I'd do it as

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{20}$$

either way the result is:

$$\begin{aligned} E_2(-1)E_{32}(-2)M_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \\ E_3(-1)E_2(-1)E_{32}(-2)M_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ M_2E_{32}(-2)M_1A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Now let's get rid of the 2 in row 1 column 3, to do this we want to multiply the matrix by:

$$\begin{bmatrix} 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \tag{21}$$

and to get rid of the 1 in row 2 column 3 we want to multiply the third row by negative one, or the matrix by:

$$\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{22}$$

So this is a combination of matrices $E_{13}(-2)$ and $E_{23}(-1)$ or

$$\begin{aligned} M_3 &= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (23) \\ M_3M_2E_{32}(-2)M_1A &= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Now we just need to multiply it by $E_{12}(-2)$ and we are done.

$$E_{12}(-2)M_3M_2E_{32}(-2)M_1A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

So, let me just remind you of the sequence of elementary matrices we just used. They were: $E_{21}(-1)$, $E_{31}(-2)$, $E_{32}(-2)$, $E_2(-1)$, $E_3(-1)$, $E_{13}(-2)$, $E_{23}(-1)$, and $E_{12}(-2)$. A lot of steps, but of course in reality you can combine some. I hope you can now see the equivalence to the elementary row operations that you have used before, and you can see that these steps were not harder, though perhaps more abstract and therefore harder to understand.

3 Inverting a Matrix

Assume that you have a system of equations

$$Ax = b$$

Now the general solution method for this system is:

1. Construct the augmented matrix $\tilde{A} = [A|b]$.
2. Convert the A in this augmented matrix to row-echelon form.
3. Convert the A in this augmented matrix to reduced row-echelon form.
4. Read off the answers.

The steps to find an inverse are:

1. Construct the matrix $A^+ = [A|I]$
2. Convert the A in this matrix to row-echelon form.
3. Convert the A in this matrix to reduced row-echelon form.
4. The second block will now be A^{-1}.

Hopefully you can see the similarity. In fact the steps required are exactly the same. To see this let me invert the example, making sure to use the same steps for each one.

$$\tilde{A} = \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 1 & 1 & 1 & b_2 \\ 2 & 2 & 1 & b_3 \end{bmatrix}, A^+ = \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

$$\begin{aligned} E_{21}(-1)\,E_{31}(-2)\,\tilde{A} &= \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & -1 & -1 & b_2 - b_1 \\ 0 & -2 & -3 & b_3 - 2b_1 \end{bmatrix} \\ E_{21}(-1)\,E_{31}(-2)\,A^+ &= \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{bmatrix} \end{aligned} \tag{26}$$

$$
\begin{aligned}
E_{32}(-2)\,E_{21}(-1)\,E_{31}(-2)\,\tilde{A} &= \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & -1 & -1 & b_2-b_1 \\ 0 & 0 & -1 & b_3-2b_2 \end{bmatrix} \\
E_{32}(-2)\,E_{21}(-1)\,E_{31}(-2)\,A^{+} &= \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{bmatrix}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
E_{3}(-1)\,E_{2}(-1)\ldots\tilde{A} &= \begin{bmatrix} 1 & 2 & 2 & b_1 \\ 0 & 1 & 1 & b_1-b_2 \\ 0 & 0 & 1 & 2b_2-b_3 \end{bmatrix} \\
E_{3}(-1)\,E_{2}(-1)\ldots A^{+} &= \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
E_{13}(-2)\,E_{23}(-1)\ldots\tilde{A} &= \begin{bmatrix} 1 & 2 & 0 & b_1-4b_2+2b_3 \\ 0 & 1 & 0 & b_1-3b_2+b_3 \\ 0 & 0 & 1 & 2b_2-b_3 \end{bmatrix} \\
E_{13}(-2)\,E_{23}(-1)\ldots A^{+} &= \begin{bmatrix} 1 & 2 & 0 & 1 & -4 & 2 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
E_{12}(-2)\ldots\tilde{A} &= \begin{bmatrix} 1 & 0 & 0 & 2b_2-b_1 \\ 0 & 1 & 0 & b_1-3b_2+b_3 \\ 0 & 0 & 1 & 2b_2-b_3 \end{bmatrix} \\
E_{12}(-2)\ldots A^{+} &= \begin{bmatrix} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{bmatrix}
\end{aligned} \tag{30}
$$

Whew, quite a few steps. Now you may wonder why I did both steps in parallel. The reason is because I want you to realize *finding the inverse matrix takes no more steps than solving a given problem.* This does not mean they are equally difficult, because there are more computations in each step (we have six columns to calculate instead of four). But at the cost of only a little more difficulty we have the inverse matrix. So you doubt the solution are the same? Well we know that using the first method we found that:

$$
x = \begin{bmatrix} 2b_2-b_1 \\ b_1-3b_2+b_3 \\ 2b_2-b_3 \end{bmatrix}. \tag{31}
$$

the second method would say that:

$$
x = A^{-1}b = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -3 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_2-b_1 \\ b_1-3b_2+b_3 \\ 2b_2-b_3 \end{bmatrix} \tag{32}
$$

So there you have it, you can find the inverse matrix almost as easily as solving a given problem. Of course, as you may have noticed, you could also just solve it for abstract coefficients, $\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$, but the inverse of a matrix is a powerful and useful tool. Knowing how to find one can help you in cases where finding the abstract coefficients might be very hard.

4 The general solution for 2×2 Matrices.

As another example let me solve an arbitrary 2×2 matrix. This won't seem that simple but it should help you see how to use the techniques.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^+ = \begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}.$$

Row-echelon form is found in one step:

$$E_{21}\left(-\frac{c}{a}\right) A^+ = \begin{bmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 1 & 0 \\ 0 & \frac{1}{a}(ad-bc) & -\frac{1}{a}c & 1 \end{bmatrix}$$

Next we rescale the pivot coefficients.

$$E_1\left(\frac{1}{a}\right) E_{21}\left(-\frac{c}{a}\right) A^+ \tag{33}$$

$$= \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & 1 & 0 \\ 0 & \frac{1}{a}(ad-bc) & -\frac{1}{a}c & 1 \end{bmatrix} \tag{34}$$

$$= \begin{bmatrix} 1 & \frac{1}{a}b & \frac{1}{a} & 0 \\ 0 & \frac{1}{a}(ad-bc) & -\frac{1}{a}c & 1 \end{bmatrix} \tag{35}$$

$$E_2\left(\frac{a}{ad-bc}\right) E_1\left(\frac{1}{a}\right) E_{21}\left(-\frac{c}{a}\right) A^+ \tag{36}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{a}b & \frac{1}{a} & 0 \\ 0 & \frac{1}{a}(ad-bc) & -\frac{1}{a}c & 1 \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} 1 & \frac{1}{a}b & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \tag{38}$$

Finally we change the last problematic cell to zero:

$$E_{12}\left(-\frac{b}{a}\right) E_2\left(\frac{a}{ad-bc}\right) E_1\left(\frac{1}{a}\right) E_{21}\left(-\frac{c}{a}\right) A^+ \tag{39}$$

$$= \begin{bmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{a}b & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} \tag{40}$$

$$= \begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

And so:

$$A^{-1} = \frac{1}{ad-bc}\left[\begin{array}{cc} d & -b \\ -c & a \end{array}\right]. \tag{41}$$

At some point you will memorize this formula, but of course that won't help you on quizzes and exams. You must show your work.