Handout on Non-Degenerate Contracts

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It is uncommon that equilibria can be characterized as eloquently as contracts under asymmetric information. Without knowing anything about the demand side of the market we can characterize the set of feasible contracts quite generally. They are—without loss of generality—monotonic, and we also know that only the nearby incentive compatibility constraints need to be considered. This result applies to signalling games, second degree price discrimination, principle agent problems, and more.

1 A General Model of Asymmetric Information.

There are two types of agents, workers and firms. Workers have information that firms do not. Workers have a type $\theta \in \Theta \subseteq R^{+}$, and an action $e \in E \subseteq R^{+}$ with $0 \in E$, the type is never observable and the action might be. Workers have a revenue impact of $\pi(\theta, e)$ and an outside option $r(\theta)$. If e is observable it can be used to signal, if e is not then this is a model of moral hazard and (possibly) adverse selection, if $E = 0$ then this is a pure model of adverse selection. If it is a model of moral hazard then $\pi(\theta, e)$ must be stochastic, otherwise it is deterministic for simplicity. In this handout we will always assume that e is observable and $E \backslash 0 \neq \emptyset$.

Worker's utility function must be the form:

$$u(w, e, \theta) = \begin{cases} v(w - c(e, \theta)) & \text{if the worker chooses } \{w, e\} \\ v(r(\theta)) & \text{if the worker chooses the outside option.} \end{cases}$$

where $v(\cdot)$ is strictly increasing and concave, and we assume that $c(0, \theta) = 0$, $c_e > 0$, $sign(c_\theta) = sign(c_{e\theta})$ and $c_\theta \neq 0$, $c_{e\theta} \neq 0$. Since all of these functions are take as primitives assume they are smooth, or infinitely differentiable.

We will generally assume $c_{e\theta} < 0$—this means high θ is a better worker. The sign of c_θ affects the value of $c(\cdot, \cdot)$, the sign of $c_{e\theta}$ affects the slope of $c(\cdot, \cdot)$. Notice that as long as $c_{e\theta}$ has constant sign so does c_θ because $c_\theta = \int_0^e e_{e\theta} + c_\theta(0, \theta) = \int_0^e e_{e\theta}$ since $c(0, \theta) = 0$ for all θ. If $c_{e\theta}$ is not constant then e can not be used as a signal. For many cases it is intuitively acceptable. For example the paradigmatic example is where e is education. It is completely reasonable to assume that what firms are looking for is a combination of education and the ability to learn. It is obvious that high θ (more intelligent workers) will have both $c_\theta < 0$ and $c_{e\theta} < 0$.

The fact that utility is essentially quasi-linear is harder to defend. However notice that the word *essentially* is important here. It means that Worker's welfare is:

$$W(\alpha, w, e) = \sum_{j=1}^{I} \alpha_j v_j (\max\{w_j - c(e_j, \theta_j), r(\theta)\})$$

and that we can not assume for all j, $\alpha_j = 1$. Thus the definition of a Pareto Improvement and Pareto Efficiency are the standard one, we are not maximizing the sum of individuals' utilities. We can let $v_j(\cdot) = \tau_j v_{j'}(\cdot) + \beta_j$ for $\tau_j > 0$ and $\beta_j \in R$, but we can not allow $c(\cdot, \cdot)$ to vary with individual without a great deal more work.

2 Contracts with observable actions (e).

A *contract* is a wage/education schedule $\{w(e), e\}$ where the interpretation is that if you choose education level e you will get $w(e)$.

Definition 1 *A contract is* non-degenerate *if for every e in the contract there is a type θ which chooses e.*

Notice that by this definition a contract can not cover all education levels unless θ is a subset of R, but we can extend our contract to cover those states. The reason for focusing on non-degenerate contracts is because obviously we can not say anything about $\{w(e), e\}$ which no type will choose.

Notice how complex a contract could be. For example we could have a (degenerate) contract where their are many different $\{w(e), e\}$ but everyone chooses the same contract. This essentially would allow for the possibility of "lying" as an equilibrium strategy. If we could not overcome this problem our space of contracts would be humongous, and we've already restricted the contracts to be a function only of e. However we can focus only on contracts of the form $\{w(\theta), e(\theta)\}$, where type θ chooses $\{w(\theta), e(\theta)\}$ in equilibrium. This important result is the *revelation principle* that we will not prove here.

Definition 2 (The Revelation Principle) *Assume there exists an equilibrium contract of the form $\{w(e), e\}$, then there exists an equilibrium contract of the form $\{w(\theta), e(\theta)\}$ where:*

1. *Worker's are asked to report their type, $\rho_j \in \Theta$.*
2. *Workers receive $\{w(\rho_j), e(\rho_j)\}$.*
3. *It is a (weakly) dominant strategy that $\rho_j = \theta_j$*

This simplifies the space of viable contracts to those that satisfy two types of constraints:
Individual Rationality:

$$u(w(\theta), e(\theta), \theta) \geq u(r(\theta))$$

Incentive compatibility: $\forall \theta' \in \Theta$

$$u(w(\theta), e(\theta), \theta) \geq u(w(\theta'), e(\theta'), \theta)$$

While this is an incredible step forward compared to the original problem there are still too many of these constraints. Combining the two if there I types there are I^2 constraints. Without simplification clearly this problem is still not solvable. We will use our assumptions about $c(\cdot,\cdot)$ to get rid of most of these constraints.

So what do we know about such contracts? The results can be stated in four lemmas:

Lemma 1 $\frac{\Delta w}{\Delta e} > 0$.

P roof. *This proof is based on $c_e > 0$.*

By maximization we know that if

$$\frac{\Delta w}{\Delta e} - \frac{\Delta c(e,\theta)}{\Delta e} < 0$$

then no one will be willing to increase their e, but since $\frac{\Delta c(e,\theta)}{\Delta e} > 0$ it must be that $\frac{\Delta w}{\Delta e} > 0$. ■

Notice that we are allowing for all types of changes in education. This is a global result. We also know:

Lemma 2 $\hat{e}(\theta)$ *is weakly monotonic increasing.*

P roof. *This proof is based on $c_{e\theta} < 0$.*

Assume $e(\theta)$ is not monotonic increasing, specifically that there is a $\theta > \theta'$ such that $e(\theta) < e(\theta')$ then we know that $w(\theta') \geq w(\theta)$ (we already proved $w(e)$ was mono-increasing) but then if:

$$\begin{aligned} w(\theta) - c(e(\theta), \theta) &\geq w(\theta') - c(e(\theta'), \theta) \\ w(\theta') - w(\theta) &\geq c(e(\theta), \theta) - c(e(\theta'), \theta) \end{aligned}$$

and since $c_{e\theta} < 0$ for all e and θ we know that:

$$\begin{aligned} \frac{\Delta c}{\Delta e}\Delta e &= c(e(\theta), \theta') - c(e(\theta'), \theta') < 0 \\ \frac{\Delta^2 c}{\Delta e \Delta\theta}\Delta e\Delta\theta &= (c(e(\theta), \theta) - c(e(\theta'), \theta)) - (c(e(\theta), \theta') - c(e(\theta'), \theta')) > 0 \end{aligned}$$

since $\frac{\Delta^2 c}{\Delta e\Delta\theta} < 0$, $\Delta e = e(\theta) - e(\theta') < 0$ and $\Delta\theta = \theta - \theta' > 0$, thus

$$c(e(\theta), \theta) - c(e(\theta'), \theta) > c(e(\theta), \theta') - c(e(\theta'), \theta')$$

but this means

$$\begin{aligned} w(\theta) - w(\theta') &> c(e(\theta), \theta') - c(e(\theta'), \theta') \\ w(\theta) - c(e(\theta), \theta') &> w(\theta') - c(e(\theta'), \theta') \end{aligned}$$

and the lemma is proven. ■

So we can now restrict ourselves to analyzing monotonic increasing $\{w(\theta), e(\theta)\}$. Now we want to simplify the incentive compatibility constraints.

Lemma 3 *For $\theta > \theta'$ if*

$$u(w(\theta), e(\theta), \theta) \geq u(w(\theta'), e(\theta'), \theta)$$

then this is also true for $\tilde{\theta} \geq \theta$.

P roof. *This proof is based on $c_{e\theta} < 0$.*

By assumption:

$$\begin{aligned} w(\theta) - c(e(\theta), \theta) &\geq w(\theta') - c(e(\theta'), \theta) \\ w(\theta) - w(\theta') &\geq c(e(\theta), \theta) - c(e(\theta'), \theta) \end{aligned}$$

$$\begin{aligned} \frac{\Delta c}{\Delta e}\Delta e &= c(e(\theta), \theta) - c(e(\theta'), \theta) \\ \frac{\Delta^2 c}{\Delta e\Delta\theta}\Delta e\Delta\theta &= c\left(e(\theta), \tilde{\theta}\right) - c\left(e(\theta'), \tilde{\theta}\right) - (c(e(\theta), \theta) - c(e(\theta'), \theta)) \leq 0 \end{aligned}$$

Since $\Delta e = e(\theta) - e(\theta') \geq 0$ and $\Delta\theta = \tilde{\theta} - \theta \geq 0$.

$$c\left(e(\theta), \tilde{\theta}\right) - c\left(e(\theta'), \tilde{\theta}\right) \leq (c(e(\theta), \theta) - c(e(\theta'), \theta)) \ .$$

but then

$$\begin{aligned} w(\theta) - w(\theta') &\geq c\left(e(\theta), \tilde{\theta}\right) - c\left(e(\theta'), \tilde{\theta}\right) \\ w(\theta) - c\left(e(\theta), \tilde{\theta}\right) &\geq w(\theta') - c\left(e(\theta'), \tilde{\theta}\right) \end{aligned}$$

and we are done. ■

We are going to use exactly the same proof going the other direction.

Lemma 4 *For $\theta > \theta'$ if*

$$u\left(w\left(\theta'\right), e\left(\theta'\right), \theta'\right) \geq u\left(w\left(\theta\right), e\left(\theta\right), \theta'\right)$$

then this is also true for $\tilde{\theta} \leq \theta'$.

P roof. *This proof is based on $c_{e\theta} < 0$.*

Like before we have:

$$\begin{aligned} w\left(\theta'\right) - c\left(e\left(\theta'\right), \theta'\right) &\geq w\left(\theta\right) - c\left(e\left(\theta\right), \theta'\right) \\ w\left(\theta\right) - w\left(\theta'\right) &\geq c\left(e\left(\theta'\right), \theta'\right) - c\left(e\left(\theta\right), \theta'\right) \end{aligned}$$

$$\begin{aligned} \frac{\Delta c}{\Delta e}\Delta e &= c\left(e\left(\theta'\right), \theta'\right) - c\left(e\left(\theta\right), \theta'\right) \\ \frac{\Delta c}{\Delta e}\Delta e \Delta\theta &= c\left(e\left(\theta'\right), \theta'\right) - c\left(e\left(\theta\right), \theta'\right) - \left(c\left(e\left(\theta'\right), \tilde{\theta}\right) - c\left(e\left(\theta\right), \tilde{\theta}\right)\right) \geq 0 \end{aligned}$$

Since $\Delta e = e\left(\theta'\right) - e\left(\theta\right) \leq 0$ and $\Delta\theta = \theta' - \tilde{\theta} \geq 0$ thus:

$$c\left(e\left(\theta'\right), \theta'\right) - c\left(e\left(\theta\right), \theta'\right) \geq \left(c\left(e\left(\theta'\right), \tilde{\theta}\right) - c\left(e\left(\theta\right), \tilde{\theta}\right)\right)$$

$$w\left(\theta\right) - w\left(\theta\right) \geq c\left(e\left(\theta'\right), \tilde{\theta}\right) - c\left(e\left(\theta\right), \tilde{\theta}\right)$$

■

For the highest and the lowest type this leaves only one constraint to check, for everyone else this leaves only two. Thus if there are I types the total number of incentive compatibility constraints is $2(I-2)+2$. There are I individual rationality constraints so the total number of constraints is $3I-2$. Since this is a finite constant times the number of types we can allow I to converge to infinity, or even a continuum, and the problem can still be solved.

What we essentially have is three bounding function. The upper bound is given by the function where all constraints in Lemma 4 are satisfied. The first lower bound is given by the function where all constraints in Lemma 3 are satisfied, and the second lower bound is given by the individual rationality constraints.

I will prove one final lemma. This lemma will not simplify the space of contracts without further assumptions but does show an important characteristic of the revelation principle.

Lemma 5 *For $\theta > \theta'$ $u\left(w\left(\theta\right), e\left(\theta\right), \theta\right) > u\left(w\left(\theta'\right), e\left(\theta'\right), \theta\right)$.*

P roof. We know that

$$w\left(\theta\right) - c\left(e\left(\theta\right), \theta\right) \geq w\left(\theta'\right) - c\left(e\left(\theta'\right), \theta\right)$$

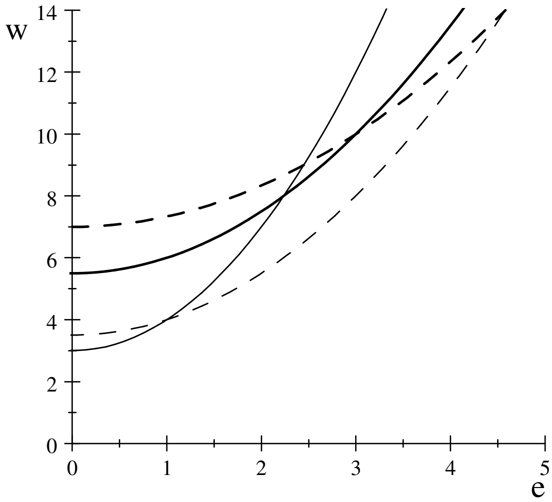
and since $c_\theta < 0$ when $\Delta\theta = \theta - \theta'$

$$\frac{\Delta c}{\Delta\theta}\Delta\theta = c\left(e\left(\theta'\right), \theta\right) - c\left(e\left(\theta'\right), \theta'\right) < 0$$

thus we are done. ■

How does the revelation principle work? By "paying" people to announce they are a higher type. Since high θ is better higher θ must get a strictly better payoff. This is essential to how the mechanism works.

Now assume that $r(\theta) = r$, or it is constant, given the last lemma this means we can ignore all of the individual rationality constraints except for the lowest type. Graphically we can look on this problem as choosing a $\{w_1, e_1\}$, given this choice we have an upper and lower bound for $\{w_2, e_2\}$ given by the indifference curves of type 1 and 2 at $\{w_1, e_1\}$. Given these constraints we can choose a $\{w_2, e_2\}$, and then we have two constraints for $\{w_3, e_3\}$ and so on. In the graph below, $u = w + \frac{e^2}{\theta}$,$\theta_1 = 1$, $\theta_2 = 2$, $\theta_3 = 3$ and $\{w_1, e_1\} = \{4, 1\}$, $\{w_2, e_2\} = \{10, 3\}$. $\{w_2, e_2\}$ had to be between the two thin lines ($u_1 = u_1(4, 1)$ and $u_2 = u_2(4, 1)$). $\{w_3, e_3\}$ has to be between the two thick lines ($u_2 = u_2(10, 3)$ and $u_2 = u_2(10, 3)$).



For completeness we can actually solve for these functions in the discrete case. The lower bound is:

$$\begin{aligned}
w_1 &= c(e_1, \theta_1) + r \\
w_2 - c(e_2, \theta_2) &= w_1 - c(e_1, \theta_2) \\
w_2 &= c(e_2, \theta_2) + (w_1 - c(e_1, \theta_2)) = c(e_2, \theta_2) + (c(e_1, \theta_1) - c(e_1, \theta_2)) + r \\
w_{i+1} &= c(e_{i+1}, \theta_{i+1}) + \sum_{j=2}^{i+1} (c(e_{j-1}, \theta_{j-1}) - c(e_{j-1}, \theta_j)) + r
\end{aligned}$$

Notice we still need to solve for the I levels of e, but whatever levels we choose we will satisfy these constraints.

The upper bound is almost this solution reversed. For the lower bound we have the starting point $v(u_1) = v(r)$, but for the upper bound we have no such condition for u_I. So we have one unknown wage:

$$\begin{aligned}
w_{I-1} - c(e_{I-1}, \theta_{I-1}) &= w_I - c(e_I, \theta_{I-1}) \\
w_{I-1} &= c(e_{I-1}, \theta_{I-1}) + (w_I - c(e_I, \theta_{I-1})) \\
w_{I-2} &= c(e_{I-2}, \theta_{I-2}) + (w_I - c(e_I, \theta_{I-1})) + (c(e_{I-1}, \theta_{I-1}) - c(e_{I-1}, \theta_{I-2})) \\
w_i &= c(e_i, \theta_i) + (w_I - c(e_I, \theta_{I-1})) + \sum_{j=i}^{I-2} (c(e_{i+1}, \theta_{i+1}) - c(e_{i+1}, \theta_i))
\end{aligned}$$

, leaving us with $I + 1$ unknowns.

If $r(\theta)$ is not constant then it is clear that either the upper or the lower bound could be confused by this binding constraint. In this case we have either:

$$\begin{aligned} w_i - c(e_i, \theta_i) &= \max\{w_{i-1} - c(e_{i-1}, \theta_i), r_i\} \\ w_i &= \max\{w_{i-1} - c(e_{i-1}, \theta_i), r_i\} + c(e_i, \theta_i) \\ w_i - c(e_i, \theta_i) &= \max\{w_{i+1} - c(e_{i+1}, \theta_i), r_i\} \\ w_i &= \max\{w_{i+1} - c(e_{i+1}, \theta_i), r_i\} + c(e_i, \theta_i) \ . \end{aligned}$$

Notice that if $r(\theta)$ is not constant this simplifies our set of contracts, every time r_i is binding e_i can be chosen without reference to either e_{i+1} or e_{i-1}.