

On Price Discrimination

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1 Monopoly and Price Discrimination:

A very good text on this subject is Jean Tirole, “The Theory of Industrial Organization” Chp 3. While it might not be necessary to cover price discrimination in the first year it is—without a doubt—the most commonly observed evidence of market power. I have also found many students have been perplexed by price discrimination—it’s been a paradox more than once—thus I want to be sure you understand the subject.

In this analysis we are—of course—going to get away from price that is linear in quantity. This is what price discrimination means. I.e. we will not assume that if you buy quantity q then your fee is $F = pq$. We will consider other types of fees, like the two part tariff $F = A + pq$. This is a common type of fee, for example telephone companies usually charge a connection fee and then a per-minute usage fee, taxis charge a pickup fee and a per kilometer fee. We will be allowing more general types of fees, $F(q)$, and we can describe an equilibrium as a $\{q, F\}$ offer curve, where q is the quantity and F is the fee charged. We will always assume that this offer curve is *non-degenerate* or that there is no offer that is not chosen by some θ . For simplicity we will also assume quasilinear utility or

$$v(q, F, \theta) = \phi(q, \theta) - F.$$

Notice we are ignoring p (the price vector) and the wealth level, in all other ways this is equivalent to a formal indirect utility function. θ is the type of this consumer, $\theta \in \Theta \subseteq R$. Furthermore we will assume that $v_q > 0$ and $v_{qq} < 0$.

There are three types of price discrimination we will look at:

1. First Degree or Perfect Price Discrimination. $\{q(\theta), F(\theta)\}$ —type θ can only buy $\{q(\theta), F(\theta)\}$
2. Second Degree or Quantity Based Price Discrimination. $\{q, F\}$ —type θ can choose any $\{q, F\}$.
3. Third Degree or Characteristic Based Price Discrimination. $\{q(\rho), F(\rho)\}$ —where the set of ρ forms a partition of Θ , and if $\theta \in \rho$ then θ can only choose $\{q(\rho), F(\rho)\}$.

Notice that in the third case quantity is *only* a function of type. As well the set of ρ are assumed to be *completely* observable. In most applications one or the other of these assumptions (and often both) is violated, in this case you are looking at a blend of second and third degree price discrimination. For example airlines often charge lower prices if you stay over a Saturday night. This is a type of third degree price discrimination, but anyone can choose to stay over thus it’s really a type of second degree... and on it goes.

The first type of price discrimination is uncommon. It is our “ideal” model. The second type is very common, many goods are sold with volume discounts. Everything from soap to breakfast cereal and “value menus”—available at most fast food restaurants—are another example. The third type is also extremely common. It is used at amusement parks, museums, or airlines. Usually it is called a “discount” for some group (kids or retired people) but it is entirely consistent with profit maximization.

1.0.1 First Degree Price Discrimination.

Lets first analyze the case where there is only one consumer of known type θ

$$\max_{q, F} F - c(q) \text{ s.t. } \phi(q, \theta) - F \geq 0$$

And we notice that $\phi(q, \theta) - F > 0$ can not be optimal because $\hat{F} = \phi(q, \theta)$ will give higher revenue without changing anything.

$$\Rightarrow F = \phi(q, \theta)$$

$$\max_{q, F} \phi(q, \theta) - c(q)$$

$$\phi_q - c'(q) = 0$$

or the *Pareto Efficient level of output is provided*. But the monopolist gets all of the surplus. Good? Bad? By the Pareto Ranking it is Pareto Optimal. But do you think so? Notice that all we need for this to be true is $\frac{\partial F(q^*)}{\partial q} = \phi_q(q^*, \theta)$ which does not require that the Monopolist get all of the revenue.

Notice as well that this can easily be decomposed into a two part tariff, with $p = c'(q^*)$ and $A = \phi(q^*, \theta) - c'(q^*)q^*$. The solution to the general problem is pretty much exactly the same:

By the way you might be disturbed by the fact that the net utility of buying the good and not buying it are the same. If this is true, how can we assume that everyone will buy? Surely a fee of $\phi(q, \theta) - \varepsilon$ for some $\varepsilon > 0$ would be more sensible? Then they would *have* to buy. Sure, that's right, but which ε ? For any $\varepsilon \frac{\varepsilon}{2}$ would give the monopolist higher profits... If we don't assume people will buy when indifferent we have an "open set" problem. This is just an artifact of working in the continuum and should be ignored. If F could only be in liras then $F = \lfloor \phi(q, \theta) \rfloor$, or the highest lira less than $\phi(q, \theta)$ and the problem would no longer occur. Don't worry about it, just assume that when indifferent people will do what you want.

Now for the general solution. Assume that the distribution of θ is given by a density function $g(\theta)$. This allows for both a discrete number of people and a continuum. For example if there 9 people, each of a unique type, then

$$g(\theta) = \begin{cases} 1 & \theta \in \{1, 2, 3, \dots, 9\} \\ 0 & \text{else} \end{cases}$$

if there are three types, 4 of type 1, 2 of type 2 and 6 of type 3 then:

$$g(\theta) = \begin{cases} 4 & \theta = 1 \\ 2 & \theta = 2 \\ 6 & \theta = 3 \\ 0 & \text{else} \end{cases}$$

and so on. It really doesn't matter. Then the general problem is:

$$\max_{q(\theta), F(\theta)} \int F(\theta) g(\theta) d\theta - c \left(\int q(\theta) g(\theta) d\theta \right) \text{ s.t. } \phi(q(\theta), \theta) - F(\theta) \geq 0$$

and like before $F(\theta) = \phi(q(\theta), \theta)$ is optimal.¹

$$\max_{q(\theta)} \int \phi(q(\theta), \theta) g(\theta) d\theta - c \left(\int q(\theta) g(\theta) d\theta \right)$$

and we get:

$$\phi_q(q(\theta), \theta) = c' = \phi_q(q(\theta'), \theta') .$$

1.0.2 Third Degree Price Discrimination.

It might seem odd to solve for this one before second degree, but it is easier.

$$\max_{q(\rho), F(\rho)} \int F(\rho(\theta)) 1_{\theta \in \rho} g(\theta) d\theta - c \left(\int q(\rho(\theta)) 1_{\theta \in \rho} g(\theta) d\theta \right) \text{ s.t. } \\ 1_{\theta \in \rho} = 1 \text{ if } \theta \in \rho \text{ and } \phi(q(\rho), \theta) - F(\rho) \geq 0$$

Define

$$D_{\rho}(F_{\rho}) = \int_{\theta \in \rho} 1_{\theta} g(\theta) d\theta$$

¹Notice that if there is a continuum of types then for any set which has probability 0 $F(\theta) \neq \phi(q(\theta), \theta)$ is also optimal. But this is stupid, ignore such solutions.

then the above can be simplified to:

$$\max_{q(\rho), F(\rho)} \Sigma_{\rho} F_{\rho} D_{\rho}(F_{\rho}) - c(\Sigma_{\rho} q_{\rho} D_{\rho}(F_{\rho}))$$

and if we let $Q_{-\rho} = \Sigma_{\rho' \neq \rho} q_{\rho'} D_{\rho'}(F_{\rho'})$ then we can analyze the problem for group ρ in isolation:

$$\max_{q, F} F D_{\rho}(F) - c(Q_{-\rho} + q D_{\rho}(F))$$

If we assume constant marginal cost then we can maximize:

$$\max_{q, F} (F - cq) D(F)$$

Now clearly, again, it can not be optimal for $\phi(q, \underline{\theta}) - F > 0$ for all θ such that $1_{\theta} = 1$. Thus $F = \phi(q, \underline{\theta})$ where $\underline{\theta}$ is defined as $\phi(q, \underline{\theta}) - F = 0$, and our function is:

$$\max_{q, F} (\phi(q, \underline{\theta}) - cq) D(F)$$

let $D' = \frac{\partial D}{\partial F}$ then:

$$\begin{aligned} (\phi_q(q, \underline{\theta}) - c) D(F) + \phi_{\theta}(q, \underline{\theta}) \frac{\partial \theta}{\partial q} D(F) + (\phi(q, \underline{\theta}) - cq) \frac{\partial D}{\partial \theta} \frac{\partial \theta}{\partial q} &= 0 \\ \phi_{\theta}(q, \underline{\theta}) \frac{\partial \theta}{\partial F} D(F) + (F - cq) D' &= 0 \end{aligned}$$

Now to simplify this we need to find $\frac{\partial \theta}{\partial F}$ and $\frac{\partial \theta}{\partial q}$

$$\begin{aligned} \phi(q, \underline{\theta}) - F &= 0 \\ \phi_{\theta}(q, \underline{\theta}) \frac{\partial \theta}{\partial F} - 1 &= 0 \\ \frac{\partial \theta}{\partial F} &= \frac{1}{\phi_{\theta}} \end{aligned}$$

$$\begin{aligned} \phi_q(q, \underline{\theta}) + \phi_{\theta}(q, \underline{\theta}) \frac{\partial \theta}{\partial q} &= 0 \\ \frac{\partial \theta}{\partial q} &= -\frac{\phi_q}{\phi_{\theta}} \end{aligned}$$

$$\begin{aligned} (\phi_q - c) D(F) - \phi_q \left(D(F) + (\phi - cq) \frac{\partial D}{\partial \theta} \frac{1}{\phi_{\theta}} \right) &= 0 \\ D(F) + (F - cq) D' &= 0 \end{aligned}$$

Now the final step in simplifying this is to note that $D' = \frac{\partial D}{\partial \theta} \frac{\partial \theta}{\partial F} = \frac{\partial D}{\partial \theta} \frac{1}{\phi_{\theta}}$ thus $(D(F) + (\phi - cq) \frac{\partial D}{\partial \theta} \frac{1}{\phi_{\theta}}) = 0$ and the first order conditions are:

$$\begin{aligned} \phi_q(q, \underline{\theta}) - c &= 0 \\ D(F) + (F - cq) D' &= 0 \end{aligned}$$

Notice that for the type $\underline{\theta}$ the solution is the same as under first degree price discrimination. This type gets the optimal quantity but has to give up all of their surplus. We can also compare this with the first order conditions of a normal monopoly profit function:

$$\max_p (p - c) D(p)$$

and we have exactly the same first order condition.

$$D(p) + (p - c) D' = 0$$

So this problem is really just the standard monopoly problem for each class, ρ . There can be other versions of this model, where they can charge a different per unit price, where within a type they have quantity based discrimination, etcetera. But this model contains the basic insight. Now assume that $q = 1$ for all groups, when will a firm not price discriminate? To find this condition first note that:

$$\frac{D_\rho(F_\rho)}{D'_\rho} + F_\rho = c$$

so in equilibrium:

$$\frac{D_\rho(F_\rho)}{D'_\rho} + F_\rho = \frac{D_{\rho'}(F_{\rho'})}{D'_{\rho'}} + F_{\rho'}$$

for all groups. If we are going to have $F_\rho = F_{\rho'}$ then $\frac{D_\rho(F)}{D'_\rho} = \frac{D_{\rho'}(F)}{D'_{\rho'}}$, or $e_{\rho'}(F) = e_\rho(F)$ —the two demand curves have the same elasticity.

Notice that third degree price discrimination does not necessarily decrease consumer surplus. If one part of the population is excluded from a service without price discrimination it often increases consumer surplus. The government also sometimes cares about one group of consumers more than another. For example at Turkish historic sites foreigners have to pay a higher price. This is sensible because the government really only cares about Turk's consumer surplus, and charging a higher price to foreigners means they can charge a lower price to Turk's. So if there is some reason you prefer one group over another third degree price discrimination is not bad.

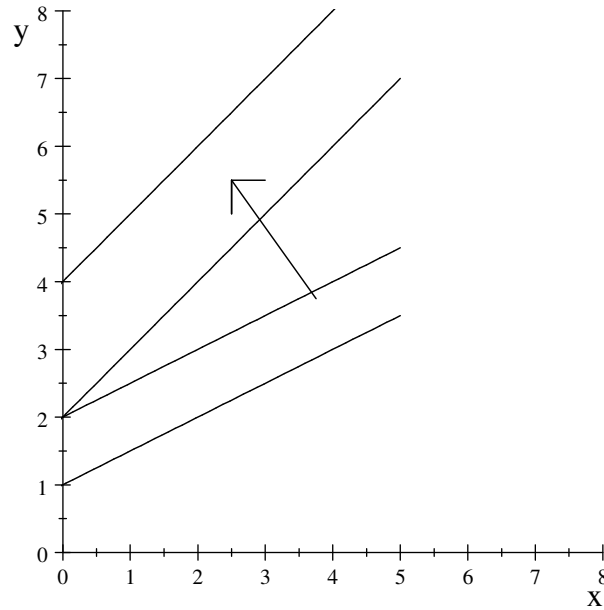
1.0.3 Second Degree Price Discrimination.

Now we are going to need to make some more structural assumptions about $v(q, F, \theta)$. Specifically we will assume $v_\theta > 0$ and $v_{q\theta} > 0$. I will explain this model by going through a simple example. In this example we will have two types, $\theta_l < \theta_h$, and the utility functions will be linear given θ :

$$v(q, F, \theta) = \theta q - F$$

We can now depict this situation graphically.

$$\begin{aligned} u &= \theta q - F \\ q &= \frac{u + F}{\theta} \\ u &\in \{2, 4\}, \theta_l = 1, \theta_h = 2 \end{aligned}$$



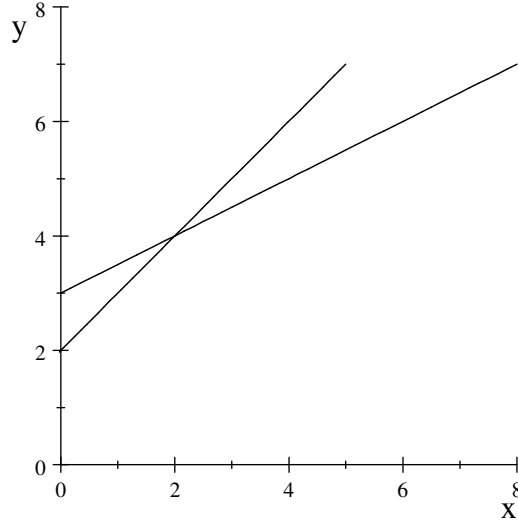
The arrow is the direction of increasing utility. q is on the vertical axis and F is on the horizontal. Notice that these indifference curves have the *single crossing property*.

Definition 1 A family of utility functions $\{v(q, F, \theta)\}$ has the single crossing property if for any u, u' the set $\{q, F\} | v(q, F, \theta) = u\}$ and the set $\{q, F\} | v(q, F, \theta') = u'\}$ have a unique intersection unless $\theta = \theta'$ and $u = u'$.

A sufficient condition for this is $v_{q\theta} < 0$ or $v_{q\theta} > 0$. We will generally assume the latter, which implies *high types are more desirable*.

Now assume that one offer the monopolist makes is $\{q, F\} = \{4, 2\}$ then the indifference curves our two types will be on are:

$$\begin{aligned} v_l &= 4 - 2 = 2 \\ v_h &= 2 * 4 - 2 = 6 \end{aligned}$$



Now remember that the pair of offers must be *non-degenerate*, so if the low type chooses this offer the high type must choose the other one or vice versa. This tells us the other offer can be in either the lower triangle $\{\{0, 3\}, \{4, 2\}, \{0, 2\}\}^2$ or the upper triangle $\{\{10, 8\}, \{4, 2\}, \{7, 8\}\}$ (if we restrict the firm to making offers with charges below 8). If he makes an offer in the lower triangle the l type will take it but the high type will refuse it. If he makes an offer in the upper triangle the h type will take it and the low type will refuse it. These points solve the following *incentive compatibility constraints*:

$$\begin{aligned} v(q_h, F_h, \theta_h) &\geq v(q_l, F_l, \theta_h) \\ v(q_l, F_l, \theta_l) &\geq v(q_h, F_h, \theta_l) \end{aligned}$$

and we also have to satisfy the *individual rationality constraints*:

$$\begin{aligned} v(q_h, F_h, \theta_h) &\geq v(0, 0, \theta_h) = 0 \\ v(q_l, F_l, \theta_l) &\geq v(0, 0, \theta_l) = 0 \end{aligned}$$

which graphically means that the indifference curves cut the axes on the vertical axis.

A second insight we can gain from profit maximizing is that the monopolist will want to charge the highest F given q , or he will only make offers in the line segments $\{\{4, 2\}, \{0, 2\}\}$ and $\{\{4, 2\}, \{6, 8\}\}$. This means that either $v(q_h, F_h, \theta_h) = v(q_l, F_l, \theta_h)$ or $v(q_l, F_l, \theta_l) = v(q_h, F_h, \theta_l)$. (By the single crossing property if both are satisfied then $q_l = q_h$.)

The third insight we derived before, does it make sense for everyone to have a positive level of utility? No. Now notice that if $v(q_h, F_h, \theta_h) = 0$ and $F_h > 0$ then $v(q_l, F_l, \theta_l) < 0$.

Lemma 1 *Since $v_\theta > 0$ if $v(q_h, F_h, \theta_h) = 0$ and $F_l \neq F_h > 0$ then $v(q_l, F_l, \theta_l) < 0$.*

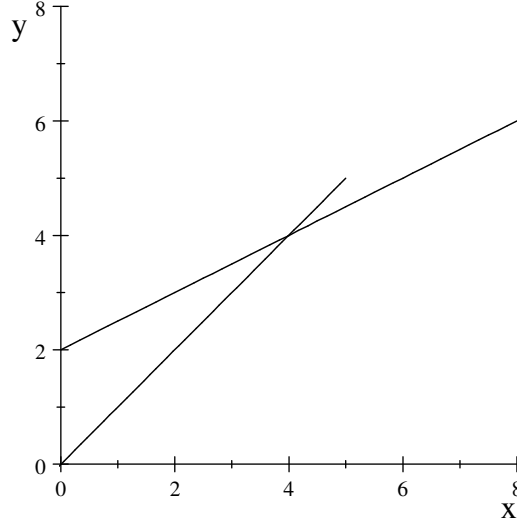
P roof. *Assume that $v(q_h, F_h, \theta_h) = 0$. Then it must either be that $v(q_h, F_h, \theta_h) = v(q_l, F_l, \theta_h)$ or $v(q_l, F_l, \theta_l) = v(q_h, F_h, \theta_l)$. If it is the former then since $v_\theta > 0$ $v(q_l, F_l, \theta_h) > v(q_l, F_l, \theta_l)$ and type l will choose $\{0, 0\}$ instead of $\{q_l, F_l\}$. If it is the latter then $v(q_h, F_h, \theta_l) < v(q_h, F_h, \theta_h) = 0$ and type l will choose $\{0, 0\}$ instead of $\{q_l, F_l\}$. ■*

So,

$$F_l = \theta_l q_l$$

²Notice that I am consistently labeling points $\{y, x\}$ where $y = q$ and $x = F$. I would apologize for this but Marshall did it first (put the dependent variable on the horizontal axis) so I don't need to apologize for continuing a mistake. :-)

Therefore if $q_l = 4$ then $F_l = 4$, and the graph looks like this:



and from the above logic we want to offer the high types a contract somewhere in the line segment $\{(4, 4), (6, 8)\}$.

Now we return to the second insight. We know that the profit maximizer will want to make the high type exactly indifferent between the low offer and the high.

$$\begin{aligned} v(q_h, F_h, \theta_h) &= v(q_l, F_l, \theta_h) \\ \theta_h q_h - F_h &= \theta_h q_l - F_l \\ \theta_h q_h - F_h &= \theta_h q_l - \theta_l q_l \\ F_h &= \theta_h q_h - (\theta_h - \theta_l) q_l \end{aligned}$$

Now if there are λ high types and $(1 - \lambda)$ low types then our profit function is:

$$\begin{aligned} \max_{q_h, q_l} & (\lambda (\theta_h q_h - (\theta_h - \theta_l) q_l) + (1 - \lambda) \theta_l q_l) - c(\lambda q_h + (1 - \lambda) q_l) \\ \max_{q_h, q_l} & (\lambda \theta_h q_h + (\theta_l - \lambda \theta_h) q_l) - c(q_h + q_l) \end{aligned}$$

assume that $\theta_l > \lambda \theta_h$ then the solution is:

$$\begin{aligned} \lambda \theta_h &= \lambda c' \\ \frac{\theta_l - \lambda \theta_h}{1 - \lambda} &= c' \end{aligned}$$

Notice that the high type consumes exactly the optimal amount of good (given the low type's quantity). The low type's quantity is distorted downward ($\frac{\theta_l - \lambda \theta_h}{1 - \lambda} < \theta_l$) because the high type is willing to pay more for the marginal unit. Increasing the quantity that the low type consumes requires that you decrease the fee of the high type by $(\theta_h - \theta_l)$, thus it is better to give the low type to little and the high type the right amount.

If you notice all of the logic above holds for any $v(q, F, \theta) = \phi(q, \theta) - F$ as long as $\phi_\theta > 0$ and $\phi_{\theta q} > 0$.

$$\begin{aligned} F_l &= \phi(q_l, \theta_l) \\ F_h &= \phi(q_h, \theta_h) - \phi(q_l, \theta_h) + \phi(q_l, \theta_l) \end{aligned}$$

$$\begin{aligned} \max_{q_h, q_l} & (\lambda (\phi(q_h, \theta_h) - \phi(q_l, \theta_h) + \phi(q_l, \theta_l)) + (1 - \lambda) \phi(q_l, \theta_l)) - c(\lambda q_h + (1 - \lambda) q_l) \\ \max_{q_h, q_l} & (\lambda \phi(q_h, \theta_h) + (\phi(q_l, \theta_l) - \lambda \phi(q_l, \theta_h))) - c(\lambda q_h + (1 - \lambda) q_l) \end{aligned}$$

$$\begin{aligned}\phi_q(q_h, \theta_h) &= c' \\ \frac{\phi_q(q_l, \theta_l) - \lambda\phi_q(q_l, \theta_h)}{1 - \lambda} &= c'\end{aligned}$$

We generally will just assume that $\phi_q(q_l, \theta_l) - \lambda\phi_q(q_l, \theta_h) > 0$.

We will be analyzing this problem when there is more than two types, but we will put that off for now. That material (and the above) is in 14.c in MCWG.