Handout on the
Properties of the Cost Function

ECON 203

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There are some fundamental properties of the cost function that are always true. These properties might not seem all that exciting in and of themselves but they can be used to prove exciting things.

Let's have a whole bunch of inputs, K to be precise. We can represent these inputs as:

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_K \end{bmatrix}$$

very simple, ehh? Vector notation often makes our analysis easier. We also will have a vector of input prices,

$$W = \begin{bmatrix} w_1 & w_2 & w_3 & ... & w_K \end{bmatrix}$$

in our standard analysis

$$X = \begin{bmatrix} L \\ K \end{bmatrix}, W = \begin{bmatrix} w & r \end{bmatrix}$$

The joy of this notation is that then we can write our cost function as:

$$\begin{aligned} C(W,q) &= \min_X \max_\lambda WX + \lambda (q - f(X)) \\ &= \min_X \max_\lambda \Sigma_{k=1}^K w_k x_k + \lambda (q - f(X)) \\ &= \min_X \max_\lambda w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 ... + w_Kx_K + \lambda (q - f(x_1, x_2, x_3, ..., x_K)) \end{aligned}$$

and isn't the first way of writing it much simpler than the last? That's the point of good notation, you can write rather complicated expressions (like the last one) much more elegantly (like the first one). We can also write the cost function as $C(W,q) = \min_X WX$ such that $q \leq f(X)$. Since all of the properties will be about input prices (W) we will just ignore "such that $q \leq f(X)$" and save ourselves a lot of writing.

The fundamental properties are:

1. $C(W,q)$ is non-decreasing in W. Or if $w_k \geq \tilde{w}_k$ for all $k \in \{1,2,3,...K\}$ then $C(W,q) \geq C\left(\tilde{W}, q\right)$. In simple English "if prices increase so will costs."

2. $C(W,q)$ is homogenous of degree one in W. Or for any $t>0$ $C(tW,q) = tC(W,q)$. In simple English "if all input prices go up by the same amount my costs will go up by that amount." Intuitively "if you increase all prices I will not change what I do, and thus my costs ill increase by that amount."

3. $C(W,q)$ is concave in W. Or for $\alpha \in [0,1]$

$$C\left(\alpha W + (1-\alpha)\tilde{W}, q\right) \geq \alpha C(W,q) + (1-\alpha) C\left(\tilde{W}, q\right)$$

   It's hard to translate this condition into simple English, but what it means is that my costs will always be lower at the extremes (W and $\tilde{W}$) than they will at any point in between.

4. If $X(W,q)$ are my input demands, then they are weakly decreasing in W.

The first three properties are necessary and sufficient for a cost function, or "every cost function has these properties" and "if a function has these properties then it is a cost function. I will not prove the latter result, that they are sufficient, only the former. But notice one fun thing about this fact. It means that the fourth property must be able to be proved only based on the first three. I won't make you do that, but isn't it fun?

So now, let's prove the first one. All of these proofs run along similar lines. "Let X be cost minimizing at W, then it must be that "Z" is true, and then the cost minimizing bundle must cost even less."

For example, let X^* be cost minimizing at W, then we know that since $w_k \geq \tilde{w}_k$ then

$$C(W,q) = WX \geq \tilde{W}X^* \geq \min_{X \text{ s.t. } q \leq f(X)} \tilde{W}X = C\left(\tilde{W}, q\right)$$

seems like I cheated somewhere doesn't it? Too easy. But that's all there is to it, so why do we care? Well it's kind of nice to know that $\frac{\partial C}{\partial w_k} \geq 0$, for example the envelope theorem gives us that $\frac{\partial C}{\partial w_k} = X_k$ and it's nice to know that firms will *always* want to use a positive amount of each input. And we didn't even make any assumptions.

The second one? Well this one just follows from the definition of minimum. Like before let X^* be cost minimizing at W then for every X such that $f(X) \geq q$ it must be that:

$$WX^* \leq WX$$

but this means that for $t>0$ that

$$tWX^* \leq tWX$$

or X^* is cost minimizing at tW but this is exactly what we want:

$$tC(W,q) = tWX^* = C(tW,q)$$

or voila! Now why do we care about this property? Well one benefit of this is that we can differentiate both sides with regards to t.

$$\begin{aligned}\frac{\partial}{\partial t} tC(W,q) &= C(W,q) \\ \frac{\partial}{\partial t} C(tW,q) &= \Sigma_k \frac{\partial C}{\partial w_k} w_k\end{aligned}$$

and since this is an identity we know that this means:

$$C(W,q) = \Sigma_k w_k \frac{\partial C}{\partial w_k}$$

using the envelope theorem we have $\frac{\partial C}{\partial w_k} = x_k^*$ and this means

$$\begin{aligned}C(W,q) &= \Sigma_k w_k x_k^* \\ &= WX^*\end{aligned}$$

or we have re-constructed the cost function from it's derivatives.

The third one? Well now, this is a little trickier (ha). Let $X(\alpha)$ be cost minimizing at $\alpha W + (1-\alpha)\tilde{W}$. Then:

$$\begin{aligned}C\left(\alpha W + (1-\alpha)\tilde{W}, q\right) &= \left[\alpha W + (1-\alpha)\tilde{W}\right] X(\alpha) \\ &= \alpha W X(\alpha) + (1-\alpha)\tilde{W} X(\alpha) \\ &\geq \alpha \min_X WX + (1-\alpha)\tilde{W} X(\alpha) \\ &= \alpha C(W,q) + (1-\alpha)\tilde{W} X(\alpha) \\ &\geq \alpha C(W,q) + (1-\alpha)\min_X \tilde{W} X \\ &= \alpha C(W,q) + (1-\alpha) C\left(\tilde{W}, q\right)\end{aligned}$$

and this is exactly what we want to prove. Notice that the "tricky part" is in lines 3 and 5, this is where I point out that $WX(\alpha) \geq \min_X WX$.

Now for the fourth claim. This one is the most tricky of the lot, and also the most intuitively important. The benefit of this property can be seen immediately, it is very nice to know that based purely on rationality we can conclude that input demand curves are downward sloping, and that's all this proof requires.

By rationality if $C(W,q) = WX$ and $\tilde{W}\tilde{X} = C\left(\tilde{W}, q\right)$ then we know that:

$$\begin{aligned}WX &\leq W\tilde{X} \\ \tilde{W}X &\geq \tilde{W}\tilde{X}\end{aligned}$$

I know that this might seem mysterious, but the first statement is "at the price vector W X has a lower cost than $\tilde{X}$." The second line is "at the price vector $\tilde{W}$ X has a higher cost than $\tilde{X}$." Do you see it? That's exactly the definition of cost minimization, nothing more. But how can this double statement help us out? Well let's first line up the greater than or equal signs.

$$\begin{aligned} WX^* &\leq W\tilde{X} \\ -\tilde{W}X^* &\leq -\tilde{W}\tilde{X} \end{aligned}$$

then we can add these together:

$$\begin{aligned} WX^* - \tilde{W}X^* &\leq W\tilde{X} - \tilde{W}\tilde{X} \\ \left(W - \tilde{W}\right)X^* &\leq \left(W - \tilde{W}\right)\tilde{X} \end{aligned}$$

and if we write

$$\Delta W = W - \tilde{W}$$

then we can write this as:

$$\Delta W X^* \leq \Delta W \tilde{X}$$

and we can move everything to the right hand side and we have:

$$\begin{aligned} \Delta W X^* - \Delta W \tilde{X} &\leq 0 \\ \Delta W \left(X^* - \tilde{X}\right) &\leq 0 \\ \Delta W \Delta X &\leq 0 \end{aligned}$$

Notice that ΔX and ΔW have to have the same direction. If we go from $\tilde{X}$ to X^* then we have to go from $\tilde{W}$ to W, or vice-a-versa.

Now, what do we have? Well it's the "delta" version of downward sloping demand, derivatives are defined at a point—or very small changes in W—this is true for all changes of W.

If we analyze this where $\Delta w_j = 0$ except for Δw_k we instantly get downward sloping demand:

$$\begin{aligned} \Delta W \Delta X &= \Sigma_{j=1}^{K} \Delta w_j \Delta x_j = 0 + 0 + ... + \Delta w_k \Delta x_k + 0 + 0 + 0 \\ &= \Delta w_k \Delta x_k \leq 0 \end{aligned}$$

voila!