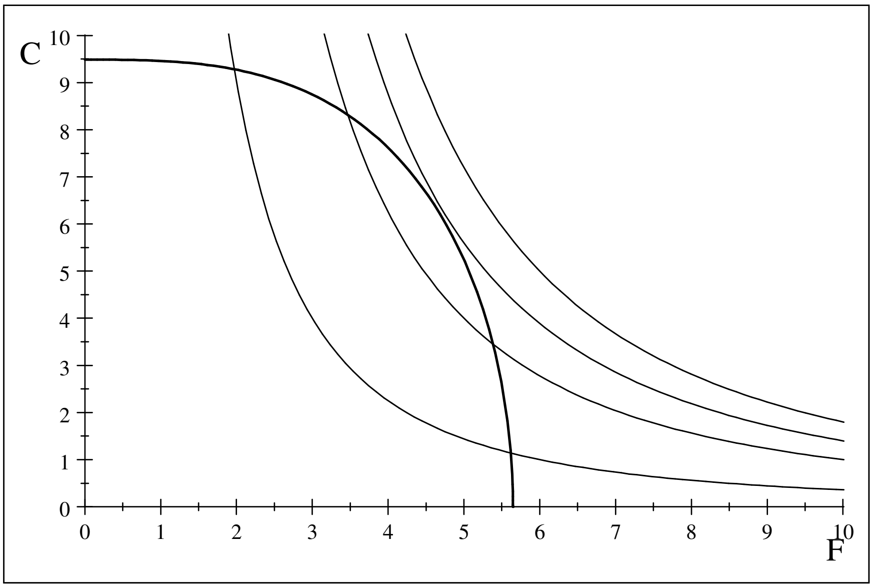
1 General Equilibrium in the Robinson Crusoe Economy.

Robinson Crusoe is a fictional shipwrecked person from a novel by Daniel Defoe.[1] In the novel he lived alone on an Island for 28 years, producing everything he consumed by himself.[2] This term has been adopted by economists to describe a very simple economy, where there is one consumer and one production possibilities frontier (PPF). One can assume that the PPF is created by the various ways the one consumer can divide his labor and other resources. In this handout I will assume preferences are Cobb-Douglass $U(F,C) = F^{\alpha}C^{\beta}$ and that the production possibilities frontier has the following simple form: $a_f F^{\gamma_f} + a_c C^{\gamma_c} = T$, where $a_f, a_c, T > 0$ and $\gamma_f, \gamma_c > 1$.[3] The following graph shows us such an economy.



The dark line is the production possibilities frontier, and the light lines are various indifference curves. Now obviously maximizing the utility over this set will be very similar to maximizing it over a budget set, and so the solution will be at a tangency. The slope of the PPF we call the Marginal Rate of Transformation, or the MRT. We find both the MRS and the MRT by taking

[1] http://en.wikipedia.org/wiki/Robinson_Crusoe

[2] He did have the assistance of "Friday," a local man. But his man Friday was a native American in a novel written by a European, so he didn't count.

[3] In most applications $\gamma_f = \gamma_c = \gamma$.

derivatives of an indifference curve and the PPF respectively.

$$\begin{aligned}
U(F,C) &= \bar{U} \\
MU_f + MU_c \frac{\partial C}{\partial F} &= 0 \\
MRS &= -\frac{\partial C}{\partial F} = \frac{MU_f}{MU_c} \\
MU_f &= \frac{\alpha U}{F}, MU_c = \frac{\beta U}{C}, MRS = \frac{\frac{\alpha U}{F}}{\frac{\beta U}{C}} = \frac{C}{F}\frac{\alpha}{\beta}
\end{aligned}$$

likewise we can find the MRT:

$$\begin{aligned}
a_f \gamma_f F^{\gamma_f - 1} + a_c \gamma_c C^{\gamma_c - 1} \frac{\partial C}{\partial F} &= 0 \\
MRT &= -\frac{\partial C}{\partial F} = \frac{a_f \gamma_f F^{\gamma_f - 1}}{a_c \gamma_c C^{\gamma_c - 1}}
\end{aligned}$$

and the tangency condition is:

$$\begin{aligned}
\frac{\alpha C}{\beta F} &= \frac{a_f \gamma_f F^{\gamma_f - 1}}{a_c \gamma_c C^{\gamma_c - 1}} \\
C^{\gamma_c} \alpha \gamma_c a_c &= F^{\gamma_f} \beta \gamma_f a_f \\
C &= \left(\frac{\beta \gamma_f a_f}{\alpha \gamma_c a_c} F^{\gamma_f} \right)^{\frac{1}{\gamma_c}} .
\end{aligned}$$

To find out how much clothing this consumer should consume we plug this back into the PPF:

$$\begin{aligned}
a_f F^{\gamma_f} + a_c C^{\gamma_c} &= T \\
a_f F^{\gamma_f} + a_c \left(\left(\frac{\beta \gamma_f a_f}{\alpha \gamma_c a_c} F^{\gamma_f} \right)^{\frac{1}{\gamma_c}} \right)^{\gamma_c} &= T \\
a_f F^{\gamma_f} + a_c \frac{\beta \gamma_f a_f}{\alpha \gamma_c a_c} F^{\gamma_f} &= T \\
\left(1 + \frac{\beta \gamma_f}{\alpha \gamma_c} \right) a_f F^{\gamma_f} &= T \\
F &= \left(\frac{\alpha \gamma_c}{\alpha \gamma_c + \beta \gamma_f} \frac{T}{a_f} \right)^{\frac{1}{\gamma_f}}
\end{aligned}$$

you should recognize the close similarity between this solution and the standard Cobb-Douglass demands, In the standard demand $F = \frac{\beta}{\alpha+\beta}\frac{I}{p_f}$ here $\alpha\gamma_c$ replaces β, $\beta\gamma_f$ replaces α and the entire thing is raised to a power. Overall it is not really that different. I will let you solve for the general value for C on your own.

The important thing about this equilibrium—to an economist—is that Robinson's decision process can be separated into Utility maximizing and Revenue maximizing. The revenue maximizer will solve the following problem:

$$R = \max_{F,C} p_f F + p_c C - \lambda \left(a_f F^{\gamma_f} + a_c C^{\gamma_c} - T \right)$$

the utility maximizer will solve the problem

$$\max_{F,C} F^{\alpha} C^{\beta} - \lambda \left(p_f F + p_c C - I \right)$$

and the model will be closed by having $I = R$, or the revenue the firm makes. I should note that in order to solve such a problem we really need $\gamma_f = \gamma_c$, we also need this if the utility function is not a Cobb-Douglass. In general we will assume this.

To give an example assume that $U(F, C) = FC^4$ and $F^2 + C^2 = 125$. Then the Pareto efficient outcome is:

$$\max_{F,C} \min_{\lambda} FC^4 - \lambda \left(C^2 + F^2 - 125 \right)$$

$$\begin{aligned} C^4 - 2F\lambda &= 0 \\ 4C^3 F - 2C\lambda &= 0 \\ C^2 + F^2 - 125 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{1}{2}\frac{C^4}{F} = 2C^2 F \\ F &= \frac{1}{2}C \\ C^2 + \left(\frac{1}{2}C\right)^2 &= 125 \\ \frac{5}{4}C^2 &= 125 \\ C &= 10 \\ F &= \frac{1}{2}(10) = 5 \end{aligned}$$

and we can decentralize this by finding the appropriate prices.

$$MRS = \frac{MU_f}{MU_c} = \frac{C^4}{4FC^3} = \frac{1}{4}\frac{C}{F} = \frac{1}{2} = \frac{p_f}{p_c}$$

let us just let $p_c = 1$ and $p_f = \frac{1}{2}$ for the fun of it. Then we can solve the revenue maximization problem first:

$$\max_{F,C} \min_{\lambda} \frac{1}{2}F + C - \lambda \left(C^2 + F^2 - 125 \right)$$

$$
\begin{aligned}
\frac{1}{2} - \lambda 2F &= 0 \\
1 - \lambda 2C &= 0 \\
\left(C^2 + F^2 - 125\right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= \frac{1}{4F} = \frac{1}{2C} \\
F &= \frac{1}{2}C
\end{aligned}
$$

and at this point from above you can easily see that the solution is $C = 10$ and $F = 5$. This means that Robinson's Income is $I = \frac{1}{2}(5) + 10 = \frac{25}{2}$, and his utility maximization problem is:

$$
\max_{F,C} \min_{\lambda} FC^4 - \lambda \left(pF + C - \frac{25}{2} \right)
$$

$$
\begin{aligned}
C^4 - \lambda p &= 0 \\
4FC^3 - \lambda &= 0 \\
\frac{1}{2}F + C - \frac{25}{2} &= 0
\end{aligned}
$$

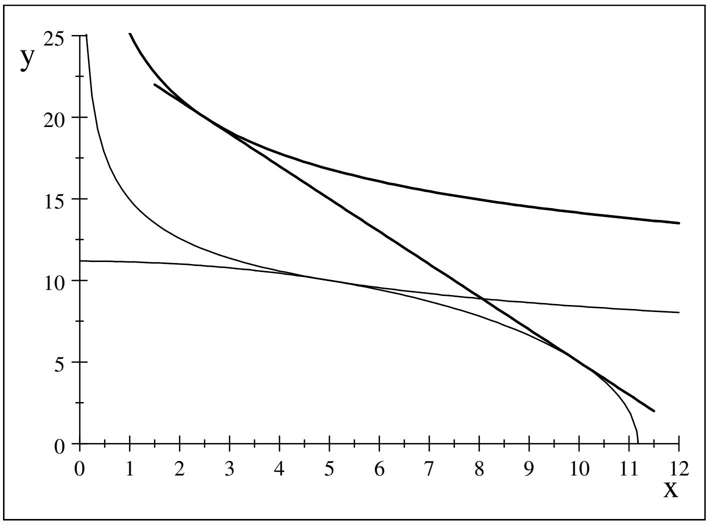
$$
\begin{aligned}
\lambda &= 2C^4 = 4FC^3 \\
F &= \frac{1}{2}C \\
\frac{1}{2}\left(\frac{1}{2}C\right) + C - \frac{25}{2} &= 0 \\
C &= 10
\end{aligned}
$$

and as you can see the solution is the same.

1.1 Robinson Crusoe and International Trade.

Now what would happen if Robinson Crusoe could trade at a different price vector from $p = \frac{1}{2}$ as above? What would happen if he could trade at $p = 2$? Obviously now he would choose a different bundle to produce (as a revenue maximizer) and consume (as a utility maximizer.) In the example above

$(U(F,C) = FC^4$ and $F^2 + C^2 = 125)$ this would result in the following graph:



The revenue maximizing point is $(F,C) = (10,5)$ the utility maximizing point is $(F,C) = \left(\frac{5}{2}, 20\right)$ so obviously this will not work. There is no way that Robinson can get the extra 15 units of C that he needs and nothing to do with the $\frac{15}{2}$ units of leftover food. Or is there? What if he could trade with other islands? This is what happens in a model of international trade.

Now say that Robinson has just discovered that the next island over has a very productive economy run by highly civilized Native Americans that do **not** run around half naked killing every European they come across.[4] Should Robinson open up trade? Should the Native Americans? After all they can produce everything Robinson can, and do it more efficiently. The answer is yes, and there is a very simple way to figure out if both economies will benefit from trade. Are the (implicit) price vectors different in the two economies? (Up to a scaling factor of course, we still can only identify $m-1$ prices in a General Equilibrium model.) If yes, then they will strictly benefit from trade. Pretty amazing isn't it?

What could cause these different price vectors? Well many different things, these are a few in the order of relative importance.

1. Difference in Resource Endowments
2. Differences in Technology
3. Differences in Demand
4. Economies of Scale

[4] Though, come to think of it, it probably would have been better for all Native Americans if they had been a little more... blood thirsty... when meeting the first Europeans.

I should point out that the last one could mean that the PPF was not concave, if this happens it could cause problems—but it generally doesn't. Notice as well that the terms of trade will change, almost surely input prices will shift when trade is opened up. This means that some people will gain and some will loose. When I say that they should open up to trade I mean that the winners could pay off the losers. Unfortunately in general this does not happen—this is a serious problem but not one I want to discuss here.

1.2 Robinson Crusoe and the Dead Weight Loss from Taxes.

Now I want to explore the idea of dead weight loss caused by taxes. To be precise what we are going to do is look at a Robinson Crusoe economy where first the government taxes each unit of labor a fixed amount t and then returns the funds to him as a transfer $\tau = tL^*$, where L^* is the optimal amount of labor Robinson supplies. Since we are trying to mimic an economy where there are millions of laborers we will assume that Robinson does not think about the fact that τ will change if L^* does, or we will assume Robinson acts as if he is a competitive worker. Notice as well that this government is a waste of time. If they had any brains they wouldn't collect a tax just to give it back, in general there are much better things to do with the money—we will discuss this later in the semester.

But to continue with this problem, assume the production function for food and clothing is $F = \sqrt{L_f}$ and $C = \sqrt{L_c}$ and Robinson decides how much labor to supply by maximizing his utility function: $U(F, C, L) = FC(24 - L)$ subject to the budget constraint $p_f F + p_c C \leq wL$.

$$\max FC(24 - L) - \lambda(p_f F + p_c C - wL)$$

$$\begin{aligned}
\frac{U}{F} - \lambda p_f &= 0 \\
\frac{U}{C} - \lambda p_c &= 0 \\
-\frac{U}{24 - L} + \lambda w &= 0 \\
p_f F - p_c C - wL &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\frac{U}{F}}{\frac{U}{C}} &= \frac{\lambda p_f}{\lambda p_c} \\
F &= C\frac{p_c}{p_f} \\
p_f\left(C\frac{p_c}{p_f}\right) + p_c C - wL &= 0
\end{aligned}$$

$$
\begin{aligned}
\frac{\frac{U}{24-L}}{\frac{U}{C}} &= \frac{\lambda w}{\lambda p_c} \\
L &= -\frac{C}{w}p_c + 24
\end{aligned}
$$

$$
\begin{aligned}
p_f\left(C\frac{p_c}{p_f}\right) + p_cC - w\left(-\frac{C}{w}p_c + 24\right) &= 0 \\
C &= 8\frac{w}{p_c} \\
F &= C\frac{p_c}{p_f} = 8\frac{w}{p_c}\frac{p_c}{p_f} = 8\frac{w}{p_f} \\
L &= -\frac{C}{w}p_c + 24 = -\frac{\left(8\frac{w}{p_c}\right)}{w}p_c + 24 = 16
\end{aligned}
$$

Notice the rather shocking fact that the quantity of Labor supplied is independent of all prices. To understand this we should think about him consuming *leisure* instead of selling Labor. Leisure is the time this guy uses to do anything besides work (watch TV, sleep, have dinner, etceteras) so $l = 24 - L$. If we substitute for L in the budget constraint we get:

$$
\begin{aligned}
p_fF + p_cC &\leq w(24 - l) \\
p_fF + p_cC + wl &\leq w24
\end{aligned}
$$

so you can see that his total *potential* income is $24w$ and the price of a unit of leisure is also w, since in Cobb-Douglass demands the general form is $\frac{\alpha_m}{\Sigma_{m=1}^{N}\alpha_n}\frac{I}{p_m}$ and in this case $p_m = w$, $I = 24w$ we can see why the labor supply is constant. Now we want to solve for the supply of food and clothing, this is easiest to do if we write the profit function in terms of F, in order to do this since $F = L_f^{\frac{1}{2}}$ we can see that $L_f = F^2$

$$
\begin{aligned}
&\max_{L_f} p_fF - wF^2 \\
p_f - 2wF &= 0 \\
F_s &= \frac{1}{2w}p_f \\
F_s &= \frac{1}{2w}p_f = 8\frac{w}{p_f} = F_d \\
p_f &= 4w, F_s = F_d = \frac{1}{2w}(4w) = 2
\end{aligned}
$$

and obviously by symmetry $p_c = 4w$, and at this point let us normalize the wage rate to one. So our equilibrium is$F = C = 2, L = 16$ and $p_f = p_c = 4$.

Now with the tax our new problem is:

$$
\max FC(24 - L) - \lambda(p_fF + p_cC - L - \tau)
$$

obviously the only difference is that the income of the consumer has increased by τ, so the solution will still be

$$F = C\frac{p_c}{p_f}, L = -Cp_c + 24$$

when we put this into the budget constraint:

$$\begin{aligned}
p_f F + p_c C - L - \tau &= 0 \\
p_f\left(C\frac{p_c}{p_f}\right) + p_c C - (-Cp_c + 24) - \tau &= 0 \\
C &= \frac{8}{p_c} + \frac{1}{3}\frac{\tau}{p_c} \\
F &= \frac{8}{p_f} + \frac{1}{3}\frac{\tau}{p_c} \\
L &= -\left(\frac{8}{p_c} + \frac{1}{3}\frac{\tau}{p_c}\right)p_c + 24 \\
&= -\frac{1}{3}\tau - 8 + 24 \\
&= 16 - \frac{1}{3}\tau
\end{aligned}$$

but now let's plug back in the fact that $\tau = tL$, and we get:

$$\begin{aligned}
L &= 16 - \frac{1}{3}tL \\
L &= \frac{48}{t+3} = \frac{(48)\frac{1}{3}}{(t+3)\frac{1}{3}} = 16\frac{1}{1+\frac{t}{3}}
\end{aligned}$$

and obviously I like this solution because if $t = 0$, $\frac{1}{1+\frac{t}{3}} = 1$. Plugging this in all over the place we get:

$$\begin{aligned}
C &= \frac{8}{p_c} + \frac{1}{3}\frac{\tau}{p_c} = \frac{1}{3p_c}\left(t16\frac{1}{1+\frac{t}{3}} + 24\right) = \frac{1}{3p_c}\left(\frac{16t + 24\left(1+\frac{t}{3}\right)}{1+\frac{t}{3}}\right) = \frac{8}{p_c}\left(\frac{1+t}{1+\frac{t}{3}}\right) \\
F &= \frac{8}{p_f}\left(\frac{1+t}{1+\frac{t}{3}}\right)
\end{aligned}$$

and to solve the other side of the problem:

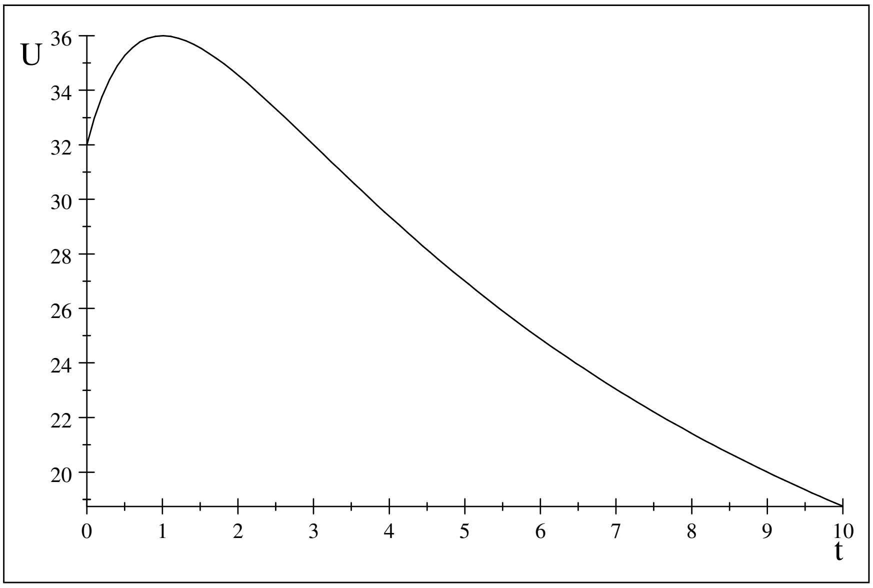
$$\begin{aligned}
&\max_{L_f} p_f F - (1+t)F^2 \\
p_f - 2(1+t)F &= 0 \\
F_s &= \frac{p_f}{2(1+t)}
\end{aligned}$$

$$
\begin{aligned}
F_s &= \frac{p_f}{2(1+t)} = \frac{8}{p_f}\left(\frac{1+t}{1+\frac{t}{3}}\right) = F_d \\
p_f^2 &= 16(1+t)^2 \frac{1}{1+\frac{t}{3}} \\
p_f &= 4(1+t)\sqrt{\frac{1}{1+\frac{t}{3}}} \\
F &= \frac{p_f}{2(1+t)} = \frac{1}{2(1+t)} 4(1+t)\sqrt{\frac{1}{1+\frac{t}{3}}} = 2\sqrt{\frac{1}{1+\frac{t}{3}}}
\end{aligned}
$$

Not elegant, I agree, but heh. The final solution is $F = C = 2\sqrt{\frac{1}{1+\frac{t}{3}}}, L = 16\frac{1}{1+\frac{t}{3}}$ and $p_f = p_c = 4(1+t)\sqrt{\frac{1}{1+\frac{t}{3}}}$. So how do we compare the two situations? If we wanted to be precise we should use compensating variation (how much more income would we have to give the consumer to make him indifferent) but I am not going to require that this year. An easy rule of thumb is to just look at the two utilities. The utility after the tax is:

$$
\begin{aligned}
U(F(t), C(t), L(t)) &= FC(24-L) \\
&= \left(2\sqrt{\frac{1}{1+\frac{t}{3}}}\right)\left(2\sqrt{\frac{1}{1+\frac{t}{3}}}\right)\left(24 - 16\frac{1}{1+\frac{t}{3}}\right) \\
&= \left(\frac{4}{1+\frac{t}{3}}\right)\left(\frac{24\left(1+\frac{t}{3}\right) - 16}{1+\frac{t}{3}}\right) \\
&= \left(\frac{4}{1+\frac{t}{3}}\right)\left(\frac{8t+8}{1+\frac{t}{3}}\right) \\
&= 32\frac{t+1}{\left(1+\frac{t}{3}\right)^2}
\end{aligned}
$$

The case without a tax is $t = 0$, and utility is equal to 32 at this level. Now theroetically this should be monotonically decreasing in t, but it is not, it only decreases after $t \geq 1$, the maximum is actually at $t = 1$ as you can see in the graph.



I find this rather peculiar, and don't really understand what is going on. The best I can understand is that increasing the tax in this general equilibrium model has two effects. First there is the increase in all prices, second is the increase in income. Apparently in this model the second effect outweighs the first for low levels of taxes.