On the equilibria of two signalling games.

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April 26, 2017

This handout addresses how to find the equilibria in two classic signalling games, the *beer-quiche* game from economics and the *Sir Phillip Sidney* game from biology. A *signalling game* is one where people can take an action that either reveals or hides some underlying characteristic.

Throughout we will use the following standard methodology:

1. Solve the game as if it were sequential, i.e. at every decision everyone knows what has happened before.
2. Rewrite the strategy just found as a strategy of the game as given, and check if it can be a weak sequential equilibrium.
3. Roll over the strategies of the uniformed agent.

This methodology may not be successful. Often the equilibrium found in the first step will not be a weak sequential equilibrium. For example consider transforming a standard normal form game into a sequential one. First Mover's advantage tells us that at best we will pick out one pure strategy Nash equilibrium of the normal form game. However this is not really why we use this technique. The primary reason we use this technique is to generate information. It is very quick and simple to do and we will learn a lot about the basic incentives of the game.

Likewise if one strategy set is very large it is often better to look at the best responses of the informed, and derive information from that. This will happen in a lot of signalling games that we look at further on.

1 The Beer-Quiche Game:

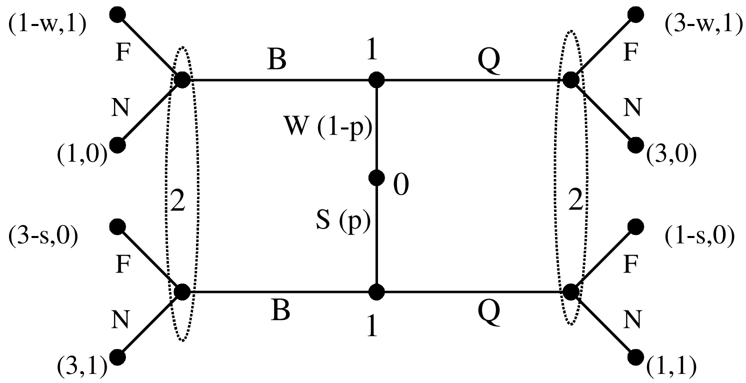
1.1 The story:

There is a bully waiting outside of a restaurant. This bully wants to beat up weak people (F), but does not want to fight the strong (N). The bully can not see if a person is weak (W) or strong (S), but instead only see what they eat for breakfast. The strong want to drink beer for breakfast (B)[1], the weak want to eat quiche (Q).[2]

[1] Thus if you have never drank beer for breakfast you must be weak.

[2] Quiche is a French dish which is, essentially, an omlette baked in a pie shell. Actually, my description is innacurate:
https://en.wikipedia.org/wiki/Quiche

1.2 The extensive form game:



The play in this game starts with decision by player 0 (in the center). Nature will make the player weak with probability $1-p$ and strong with probability p. Nature's action is known by player 1, who then decides whether to have beer or quiche. The bully (player 2) only observes whether the customer chooses beer or quiche, and must decide whether to fight them or not. The parameters w and s are the cost of getting into a fight to the weak and strong, respectively.

1.3 Analysis:

1. I will write down the subgame perfect best responses of the game because adding them to the game is difficult in this format.

   $BR_2(W,Q) = BR_2(W,B) = F$, $BR_2(S,Q) = BR_2(S,B) = N \Rightarrow$

   $BR_1(W) = Q$, $BR_1(S) = B$

   And thus the strategy is:

$$\begin{aligned} S_1^* &= (Q(W), B(S)) \\ S_2^* &= (F(W,Q), F(W,B), N(S,B), N(S,Q)) \end{aligned}$$

2. Rewriting this strategy as a strategy of the game as given obviously $(Q(W), B(S))$ survives unchanged, thus it must be that $S_2 = (F(Q), N(B))$. Is this a weak sequential equilibrium? We don't need to worry about beliefs because every information set is reached with positive probability. We don't need to worry if nature chooses S, because this type of player 1 is getting their maximum payoff in the game. Likewise player 2 is no concern because she is also always achieving her maximum payoff. Thus we only need to check when nature makes player 1 weak. In this case we have to check that:

$$\begin{aligned} U_1(W,Q,F) &\geq U_1(W,B,N) \\ 3-w &\geq 1,\ w \leq 2 \end{aligned}$$

Thus this $(Q(W), B(S))$, $(F(Q), N(B))$ is a weak sequential equilibrium if $w \leq 2$.

3. Roll over the remaining strategies.

   (a) $(F(B), F(Q))$—this means that $P1$ will get in a fight no matter what, so they might as well enjoy their breakfast and $BR_1(F(B), F(Q)) = (Q(W), B(S))$, but $BR_2(Q(W), B(S)) = (F(Q), N(B))$ and thus this can not be an equilibrium.

   (b) $(N(B), N(Q))$—like above $BR_1(N(B), N(Q)) = (Q(W), B(S))$ and this can not be an equilibrium.

   (c) $(F(B), N(Q))$—clearly $BR_1(W) = Q$ because it is his maximum payoff, so we need to check what type S will do.

   If type S chooses B, then again we're at the strategy $(Q(W), B(S))$ and this will result in the bully not wanting to use this strategy. Thus we must have type S choosing Q:

$$\begin{aligned} U_2(S, Q, N) &\geq U_2(S, B, F) \\ 1 &\geq 3 - s,\ s \geq 2 \end{aligned}$$

   If this is true then $(Q(W), Q(S))$ might be part of an equilibrium.

   However we have two more things to do. First we need to check whether the bully's strategy is optimal on the equilibrium path. I.e. does it make sense for the bully not to fight when they eat quiche. In this possible equilibrium:

$$\begin{aligned} EU_2(N, Q) &= \Pr(W|Q)\, U_2(N, Q, W) + \Pr(S|Q)\, U_2(N, Q, S) \\ &= (1-p)\, U_2(N, Q, W) + pU_2(N, Q, S) \\ &= p \\ EU_2(F, Q) &= (1-p)\, U_2(F, Q, W) + pU_2(F, Q, S) \\ &= (1-p) \end{aligned}$$

$$EU_2(N, Q) \geq EU_2(F, Q) \ \text{ if } p \geq \frac{1}{2}$$

   Now likewise we must be sure the bully does want to fight if they observe B. Let the beliefs of player 2 be $\Pr(S|B) = \beta$ then we must be sure that:

$$\begin{aligned} EU_2(N, B) &= \Pr(W|B)\, U_2(N, B, W) + \Pr(S|B)\, U_2(N, B, S) \\ &= \beta \\ EU_2(F, B) &= (1-\beta)\, U_2(F, B, W) + \beta U_2(F, B, S) \\ &= (1-\beta) \end{aligned}$$

$$EU_2(F, B) \geq EU_2(N, B) \ \text{ if } \beta \leq \frac{1}{2}\ .$$

Notice an *a-priori* contradiction here. The strong like beer and the bully does not want to fight the strong, but when the bully sees someone drink beer then she concludes this person is weak. There is no universally accepted refinement of equilibrium (on the lines of subgame perfection or weak sequentiality) that rules out this equilibrium, and I would like to argue that your perception of a "contradiction" here is the problem.

This is a world where everyone eats quiche for breakfast. There are strong social taboos against having beer for breakfast[3]. So what do you conclude when you see someone who orders beer? They're crazy. Why should their underlying type matter if they are insane?

Regardless of whether it is "reasonable" another equilibrium is $(Q(W), Q(S))$, $(F(B), N(Q))$. This requires $s \geq 2$, $\Pr(S) \geq \frac{1}{2}$, $\Pr(S|B) \leq \frac{1}{2}$— where the last is a matter of beliefs.

(d) $(F(Q), N(B))$, $w \geq 2$. We have to return to this strategy because we only analyzed the case where w was low. What if it is high? then $BR_1(F(Q), N(B)) = (B(W), B(S))$, and likewise on path we must have $\Pr(S) = p \geq \frac{1}{2}$ so that the bully does not want to fight someone who drinks beer, and $\Pr(S|Q) = \beta \leq \frac{1}{2}$ so that when someone eats quiche the bully does want to fight them.

1.4 The pure strategy equilibria:

For clarity I will list all of the possible equilibria in one table. The first thing I want to point out is that I am only specifying the pure strategy equilibria. While mixed strategy equilibria are still of interest there are too many tricks that can be played in these types games to want to analyze them. Unless you can not find a pure strategy equilibrium you do not need to worry about mixed strategy equilibria. The table allows us to distinguish the two types of equilibria in this game. They are:

Definition 1 *In a* separating *equilibrium a player's actions reveal their type.*

Definition 2 *In a* pooling *equilibrium all types of players do the same thing.*

Obviously in a model with more than two types we can have some combination of these equilibria. If a parameter does not matter for an equilibrium in the table below I write NA. When I write $-S_1^*$ this is the beliefs when player 1 does not use his optimal strategy. This will only matter in pooling equilibria, when $-S_1^*$ is simply the other strategy.

Strategy	$\Pr(S)$	$\Pr(S\mid -S_1^*)$	w	s	Type
$(Q(W), B(S))$, $(F(Q), N(B))$	NA	NA	≤ 2	NA	Separating
$(Q(W), Q(S))$, $(F(B), N(Q))$	$\geq \frac{1}{2}$	$\leq \frac{1}{2}$	NA	≥ 2	Pooling
$(B(W), B(S))$, $(F(Q), N(B))$	$\geq \frac{1}{2}$	$\leq \frac{1}{2}$	≥ 2	NA	Pooling

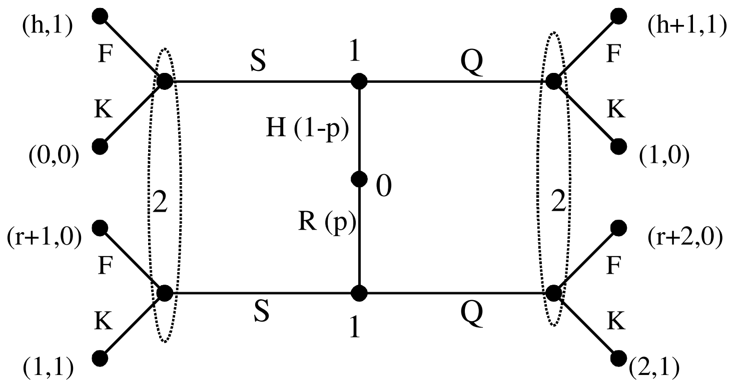
[3] Remind you of any world you know?

2 The Sir Phillip Sydney Game:

2.1 The story:

Biologists frequently observe large colonies of birds where adults return to the colony and feed the baby birds (chicks). What they frequently observe is that some chicks will squawk (S) or make noise and others will keep quiet (Q). The adults will then feed (F) some baby bird or keep (K) the food. Most commonly they will feed a baby bird that is squawking. Now obviously what the adults are doing is realizing that some of the chicks will be hungry (H), while others are full (R, for *replete*[4]). The adults want to feed the hungry and keep the food if the chick is full. Biologists are further interested in the degree relationship between the birds[5], etceteras, making the game in the book overly complex. I will strip it down to the essentials.

2.1.1 The game:



The parameters h and r indicate the marginal benefit of eating for the full and hungry, naturally $h > r > 0$. The baseline utility for the full is higher than that for the hungry, and the cost of squawking has been normalized to one.

2.2 Analysis:

1. The subgame perfect best responses are:

   $BR_2(H,Q) = BR_2(H,S) = F$, $BR_2(R,Q) = BR_2(R,S) = K \Rightarrow$

   $BR_1(H) = Q$, $BR_1(R) = Q$

   And thus the strategy is:

$$
\begin{aligned}
S_1^* &= (Q(H), Q(R)) \\
S_2^* &= (F(H,Q), F(H,S), K(R,S), K(R,Q))
\end{aligned}
$$

[4] https://www.merriam-webster.com/dictionary/replete

[5] As in: "Oh, that's my baby sister, I'll feed her. But that one is only a cousin, nahhh."

2. Rewriting this strategy as a strategy of the game as given obviously $(Q(H), Q(R))$ survives unchanged, but now we notice that there is no way for the adults to feed only the hungry. This can not be a weak sequential equilibrium. This analysis has allowed us to observe that in this game all player's incentives are aligned. This is actually a common model, in fact:

   Definition 3 *A* signal *is an action for which the direct marginal benefit is outweighed by the direct marginal cost. This action is taken to reveal or hide information about some underlying trait of the person.*

   A pure signal is then an action in which the direct marginal benefit is zero, like this one.

3. Roll over the uniformed player's strategies.

   (a) $(G(S), G(Q))$—From our previous analysis we recognize that $BR_1(G(S), G(Q)) = (Q(H), Q(R))$, and since this is a pooling equilibrium we have to make sure that player 2 is using an optimal strategy or:

   $$\begin{aligned} EU_2(K,Q) &= \Pr(H|Q)\,U_2(K,Q,H) + \Pr(R|Q)\,U_2(K,Q,S) \\ &= (1-p)\,U_2(K,Q,H) + pU_2(K,Q,R) \\ &= p \\ EU_2(G,Q) &= \Pr(H|Q)\,U_2(G,Q,H) + \Pr(R|Q)\,U_2(G,Q,S) \\ &= (1-p) \end{aligned}$$

   $$EU_2(G,Q) \geq EU_2(K,Q) \text{ if } p \leq \frac{1}{2}\,.$$

   and we also need to check off path beliefs because no baby squawks. Let $\Pr(R|S) = \beta$ then we need

   $$\begin{aligned} EU_2(K,S) &= \Pr(H|S)\,U_2(K,S,H) + \Pr(R|S)\,U_2(K,S,S) \\ &= (1-\beta)\,U_2(K,S,H) + \beta U_2(K,S,R) \\ &= \beta \\ EU_2(G,S) &= \Pr(H|S)\,U_2(G,S,H) + \Pr(R|S)\,U_2(G,S,S) \\ &= (1-\beta) \end{aligned}$$

   $$EU_2(G,S) \geq EU_2(K,S) \text{ if } \beta \leq \frac{1}{2}\,.$$

   (b) $(K(S), K(Q))$—like above $BR_1(K(S), K(Q)) = (Q(H), Q(R))$ and the incentives of the adult need to be reversed, so we need $\Pr(R) = p \geq \frac{1}{2}$, $\Pr(R|S) = \beta \geq \frac{1}{2}$.

   (c) $(K(S), G(Q))$—obviously $BR_1(K(S), G(Q)) = (Q(H), Q(R))$, and in order for them to feed the quite we must have $\Pr(R) = p \leq \frac{1}{2}$, but now in order to keep it if the baby squawks we must have $\Pr(R|S) = \beta \geq \frac{1}{2}$.

(d) $(G(S), K(Q))$—there are going to be a whole bunch of cases here. This is the strategy that biologists find most interesting because they observe it frequently, but the responses of the babies will depend on h and r.

i. $(S(H), S(R))$ in order for this to be true we must have:

$$\begin{aligned} U_1(H, S, G) &\geq U_1(H, Q, K) \\ h &\geq 1 \end{aligned}$$

also:

$$\begin{aligned} U_1(R, S, G) &\geq U_1(R, Q, K) \\ r + 1 &\geq 2, r \geq 1 \end{aligned}$$

this is a pooling equilibrium, so we need $\Pr(R) = p \leq \frac{1}{2}$ and since they aren't giving to the quiet birds $\Pr(R|Q) = \beta \geq \frac{1}{2}$.

ii. $(Q(H), Q(R))$ from above we need $1 \geq h$ and $1 \geq r$, and since they aren't feeding the babies we must have $\Pr(R) = p \geq \frac{1}{2}$, since they will feed anyone who squawks we must have $\Pr(R|S) = \beta \leq \frac{1}{2}$.

iii. $(S(H), Q(R))$ finally a separating equilibrium! Also the one of greatest interest. In this case we need $h \geq 1 \geq r$ and we don't need to worry about the adult's incentives.

iv. $(Q(H), S(R))$ this can never be a best response because $h > r$ and in order for this to be a best response we need $r \geq 1 \geq h$.

2.3 The pure strategy equilibria:

What a lot of equilibria! I am going to organize them first by how noisy the colony is and then by the probability the baby is already full.

Strategy	$\Pr(R)$	$\Pr(R\| - S_1^*)$	h	r	Type
$(Q(H), Q(R))$, $(G(S), G(Q))$	$\leq \frac{1}{2}$	$\leq \frac{1}{2}$	NA	NA	Pooling
$(Q(H), Q(R))$, $(K(S), G(Q))$	$\leq \frac{1}{2}$	$\geq \frac{1}{2}$	NA	NA	Pooling
$(Q(H), Q(R))$, $(K(S), K(Q))$	$\geq \frac{1}{2}$	$\geq \frac{1}{2}$	NA	NA	Pooling
$(Q(H), Q(R))$, $(G(S), K(Q))$	$\geq \frac{1}{2}$	$\leq \frac{1}{2}$	≤ 1	≤ 1	Pooling
$(S(H), Q(R))$, $(G(S), K(Q))$	NA	NA	≥ 1	≤ 1	Separating
$(S(H), S(R))$, $(G(S), K(Q))$	$\leq \frac{1}{2}$	$\geq \frac{1}{2}$	≥ 1	≥ 1	Pooling

Biologists frequently observe $(G(S), K(Q))$. Notice that this is always part of an equilibrium no matter what $\Pr(R) = p$ is and in fact creates natural separation. The marginal benefit of the food (h and r) obviously will depend on how hungry the chick is, and basically the rule is that if the chick is hungry enough it squawks. Recognize that a more holistic analysis would include that the adult needs the food as well. This strategy guarantees that both adults and chicks survive.

The strategies $(G(S), G(Q))$, $(K(S), G(Q))$ are ones where the adult just feeds any or no chick, and obviously these will be bad for the survival of the entire colony. Either the chicks starve or the adults starve. The strategy $(K(S), G(Q))$ gives the wrong incentives and results in the same outcome as $(G(S), G(Q))$.

From an evolutionary perspective it is clear that adults who follow the $(G(S), K(Q))$ strategy will maximize the survival chances of the adults and the chicks—or the colony. Further analysis would be required to make this argument formal.