Handout on the Axioms of Choice

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This handout should help explain the axioms of choice more clearly, and make the link between the mathematics and economics clearer.

Notation 1 *In this handout:*

1. A, B, A' *are all market bundles, or a complete list of all goods a person consumes. These should be thought of as vectors. For example if I consume food* (F)*, clothing* (C)*, and housing* (H) *then my consumption bundle would be:*

$$A = \begin{bmatrix} F \\ C \\ H \end{bmatrix}$$

where $\{F, C, H\}$ *are all quantities of food, clothing and housing.*

2. $\geq$ *is the standard greater than or equal to ordering. For example,* $5 \geq 3$*,* $-10.000 \geq -10.001$*, and* $4 \geq 4$*.*

3. $\succeq$ *is an abstract ordering representing a person's preferences. For example if you like (Association) football and hate American football I would express that as: football* $\succ$ *American football. If you prefer three bananas to two bananas I would write: 3 bananas* $\succ$ *2 bananas. Notice I am using* $\succ$ *like I use* $>$*, if you want to say that you like something at least as much as something else you use* $\succeq$*. For example: I like a vacation in the mountains as much as a vacation on the beach. So I would write: vacation in the mountains* $\succeq$ *vacation on the beach.*

4. $d(\cdot, \cdot)$ *is a distance metric. The standard one in mathematics is* $d(x, y) = |x - y|$ *like* $d(10, 5) = |10 - 5| = 5$ *or* $d(-333, 267) = |(-333) - 267| = 600$*. However one can define other distance metrics, and I'm just using an "arbitrary" one. I do this because sometimes what we are talking about is not as simple as the real line. What is the appropriate distance between a flower and a motorcycle?*

5. ε*—epsilon, any time someone uses this notation you should think of it as "a very small number."*

Now I will use this terminology to define our axioms of choice. As I like to say, "math is meaningless but it is the language of economics." There is a reason for this, we can be much more precise when we use mathematical terminology than we can using ordinary language. Thus I will define all of the axioms in both mathematical and ordinary language. Most of the time understanding the ordinary language definition will be sufficient, but sometimes the mathematical definition is more precise, and I expect you to understand this definition.

0.1 The Axioms

1. Reflexivity—$A \succeq A$

   In words? We always assume that a bundle is at least as good as itself. Does that make sense? Yes, but it doesn't seem to mean anything. It tells us two things. First, $\succeq$ is like $\geq$, or it is a weak inequality. Second, we don't need to refer to a third object to tell about the ordering between A and A.

   As a simple (but incorrect) counter example if you have traditional Turkish manners even if you smoke you don't do it in front of your parents. In this case the relationship between "smoking" and "not smoking" depends on the presence of a third object, your parents. Such an ordering seems not to satisfy this assumption. However it should because your choices are actually "smoking in front of parents" or "not smoking in front of your parents." That is what I meant when I said a market bundle should include *all* goods you consume. It doesn't make sense to consider smoking without the secondary assessment of where you are. For example do you want a slice of pizza? Well that clearly depends on whether you are hungry. Preferences over every good usually depends on things other than the good alone.

   A good counter example is the "second largest slice of cake" preferences—due to Amartya Sen. Say that a potential mother-in-law invites you over for dinner, after dinner she brings out slices of cake with no two slices the same size. She asks you which slice you want, how do you respond? If you ask for too small of a slice you are suggesting she can not cook. If you ask for too big of a slice then you are greedy. Thus the optimal strategy is to ask for a large slice but not too large. For example you might ask for the second largest slice. These preferences don't satisfy reflexivity because they depend on a third item. For example if the choice set is $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$ then $\frac{1}{3}$ is best and strictly better than $\frac{1}{4}$, but if the choices are$\left(\frac{1}{3}, \frac{1}{4}\right)$ then $\frac{1}{4}$ is best and strictly better than $\frac{1}{3}$.[1]

2. Transitivity—$A \succeq B$ and $B \succeq C$ means that $A \succeq C$

   The best way to understand this axiom is to think about how you would violate it. If your preferences didn't satisfy this axiom then you could get stuck in a cycle comparing various choices and never get out of it. We all go through such cycles when making a decision. For example I and my ex-wife went through one this week, deciding which TV to buy. However, at the end we did reach a decision and bought a TV. If we violated transitivity then we would never be able to make a decision. "TV B is better than TV A because it has more features, and the price isn't that much higher. Well TV C isn't that much more than TV B, and it's got even more features. But TV A is so much cheaper than TV C. Gosh, are

[1] And yes, I know $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$. She's an evil woman, so she prepared two cakse just to make it hard on you.

those features worth it?" Sound familiar? I expect so, but I also hope that sooner or later you came to a decision, and therefore your preferences did satisfy transitivity.

3. Completeness—Either $A \succeq B$ or $B \succeq A$, or everything can be compared.

   For this definition the word definition is exactly as precise as the mathematical one. A simple counter-example for this is choices that are so abstract that you can't make them. For example would you rather live on a colony on Mars or on the moon? Since I can't even imagine life in either situation I can't establish my preferences over these options.

   These three axioms are the Normative definition of Rationality. Someone is Normatively Rational if and only if they satisfy Reflexivity, Transitivity, and Completeness.

4. Continuity—If $A \succ B$ then there is some $\varepsilon > 0$ such that if $d(A, A') < \varepsilon$ then $A' \succ B$, or if one thing is better than a second then things that are "near enough" to the first one are also better than the second.

   This is one where the mathematical notation is important. Intuitively it's fairly easy to understand, but the mathematical language is more precise. Say you're at a restaurant, and the waiter says, "we have big slices of baklava or ice cream for desert." It's a chilly night, so you ask for the baklava. Then the waiter comes back and says, "well the slices of baklava aren't quite as big as I said, do you still want it?" You're not going to decide that on such a chilly night you now want ice cream, you'll take the baklava and maybe tip the waiter less. That's one example, and intuitively clear, but what else is "near enough." Is lokma like baklava or ice cream? That is why you need a precise idea of what is near and what is not, and that's given in the mathematical definition.

   These four axioms give us a Utility function.

   And these are the four axioms that we always will assume, without question. The next one is convenient and I will assume it, even though it is not necessary.

5. Non-satiation—For every A and any $\varepsilon > 0$ there is an A' such that $d(A, A') < \varepsilon$ and $A' \succ A$, or there is never an end to your needs. You can never be satisfied, and it doesn't take big changes to satisfy you, just a little more or less of something will make you happier.

   The main reason that we need the math here is that this is for *every* $\varepsilon > 0$. In other words there is *always* a way to make you happier *no matter how small the changes that you consider.*

   This axiom is very weak. Violating this merely means that someone can be **perfectly** happy, completely happy, blissful in fact. We call such a violation a "bliss point," and it's reasonable to assume that people can't be like that. I once had a colleague who claimed he was perfectly happy, that he was at a bliss point (and yes he did understand the terminology) and so I said, "so you'll give me all of your savings, ehh?" I will let you imagine his answer, savings are, of course, part of your consumption bundle.

This assumption is reasonable, but the next one is absolutely ridiculous. Say I invite you over to my house, and I serve you manti. You like it very much, and ask for a second plate. I give that to you, then insist that you have a third, and a fourth, and a ... At some point you say, "STOP, no more manti please!!!" And I say—what, aren't your preferences monotonic?

6. Monotonicity—If A has at least as much of every good as B and at least one good in strictly greater quantity, $A \succ B$. In words "more is better."

   Notice how much less precise the everyday language is. Of course I could strive to make it more precise but then I would be basically restating the mathematical definition. "More is better" could mean "$A \succsim B$", or "if A has strictly more of every good than B then $A \succ B$" all of these would mean different things. However "more is better" is a good way to put it.

   A simple mathematical condition that is equivalent to this is having a strictly positive marginal utility for all goods. Or in other words if I have a utility function $U(F, C)$ then this is equivalent to $\frac{\partial U}{\partial C} > 0$ and $\frac{\partial U}{\partial F} > 0$. Among the important utility functions we will look at there is only one that does not satisfy this condition. It would be a good exam question to ask you to tell me which one it is.

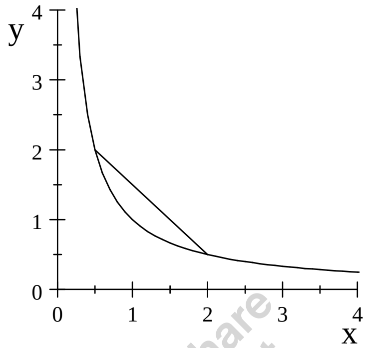
7. Convexity—If $A \succeq C$ and $B \succeq C$ then for $\lambda \in [0, 1]$ $\lambda A + (1 - \lambda) B \succeq C$ or "a little bit of everything is more than all of one."

   This is an assumption we love to make in economics. I'm sure you understand the intuitive definition better than the mathematical one, but the connection between the two is probably not clear in your mind. Here the mathematical statement is precise, so is the intuitive one (kind of) but the connection is unclear.

   Let me give you a mathematical example so you can understand this axiom. For this example, $A = \begin{bmatrix} 10 \\ 3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 11 \end{bmatrix}, C = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$. Then

$$\begin{aligned} \lambda A + (1 - \lambda) B &= \begin{bmatrix} 10 * \lambda + (1 - \lambda) * 2 \\ 3 * \lambda + (1 - \lambda) * 11 \end{bmatrix} \\ &= \begin{bmatrix} 8\lambda + 2 \\ 11 - 8\lambda \end{bmatrix} \end{aligned}$$

   and convexity requires that for every $\lambda \in [0, 1]$ this is better than $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$. For example $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ is better than $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$, $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ is also better than $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$, so is $\begin{bmatrix} 8 \\ 5 \end{bmatrix}$, etcetera. Now, how do we make the transition from the mathematical definition to the intuitive one? Well to understand this first assume that $A \approx B$, or that A and B are equivalent (notice I am using $\approx$ like I would use $=$), then the definition requires that $\lambda A + (1 - \lambda) B \succeq A$ ($\approx B$). Do you see it now? Everything between A and B must be at least as good as the extremes—A and B, or "a little bit of everything ($\lambda A + (1 - \lambda) B$) is better than all of one (A or B)." This condition can also be represented graphically, let's look at an indifference curve.



This indifference curve is convex because everything on the line between $\{2, \frac{1}{2}\}$ and $\{\frac{1}{2}, 2\}$ has a higher utility than those two points, or the indifference curve is bowed towards the origin, or the marginal rate of substitution is decreasing. I won't always make this assumption (look at the homework for example) but I usually will.

So there it is, the axioms that we will sometimes or often assume in our analysis. I hope you look on this as an exercise in understanding the importance of math to economics. Look at the axioms, look at the intuition, try to understand the similarities and differences, and develop your understanding of why economists—who study a social science—use mathematics for our discussions.