1 Utility Functions that will be used for General Equilibrium.

For General Equilibrium we need to use some utility functions, but we don't need to get too technical. Thus we are going to use three basic utility functions: the Cobb-Douglass, the Leontief (Perfect Compliments) and the Linear (Perfect Substitutes).

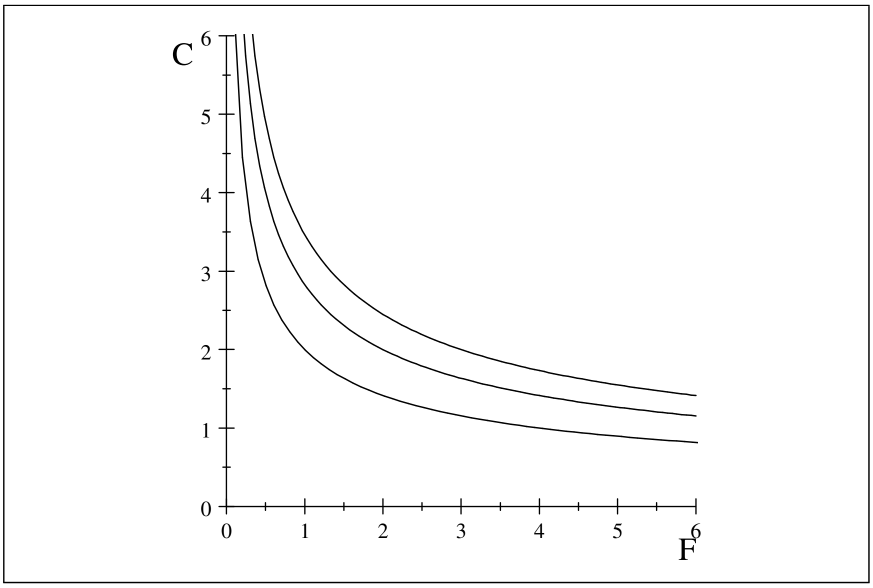
The Cobb-Douglass will be our workhorse, the one we use most commonly. The other two are only included because they make the analysis simpler. If I include one of them it is because I am trying to make the problem easier for you.

1.1 The Cobb-Douglass Utility Function

The function is

$$U(F, C) = F^{\alpha} C^{\beta}$$

where $\alpha \geq 0$ and $\beta \geq 0$. To understand what this utility function it is useful to look at a graph, these are the indifference curves for this function when $\alpha = 1$, $\beta = 2$.



One of the cool things about the Cobb-Douglass is how simple the derivatives are:

$$\frac{\partial U}{\partial C} = \beta F^{\alpha} C^{\beta-1} = \beta \frac{F^{\alpha} C^{\beta}}{C} = \beta \frac{U}{C}$$

where the second step is because $x^{\gamma-1} = \frac{x^\gamma}{x}$ and the third is because $F^\alpha C^\beta = U(F, C)$ by definition. You should be able to establish on your own that:

$$\frac{\partial U}{\partial F} = \alpha \frac{U}{F}$$

given this it is also relatively easy to find the slope of an indifference curve, or the Marginal Rate of Substitution. Now just so you understand where this comes from I will actually derive it from the primitives. An indifference curve can be written as $C(F)$, or how much C you need given the amount of F so that you achieve some fixed level of utility, or $\bar{U}$. Now let me write down the definition mathematically:

$$U(F, C(F)) = \bar{U}$$

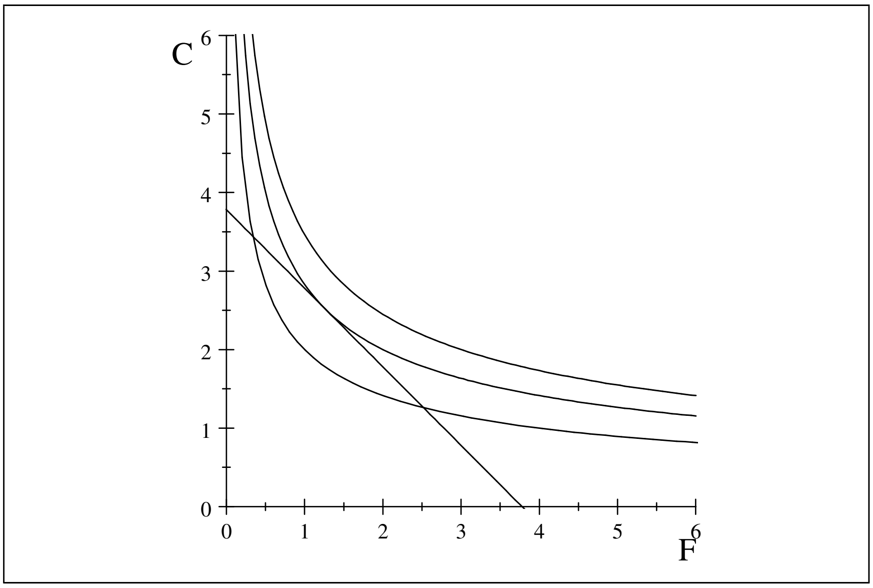
to the marginal rate of substitution is $-\frac{\partial C}{\partial F}$, and we find it by differentiating the above:

$$\begin{aligned} \frac{\partial U}{\partial F} + \frac{\partial U}{\partial C}\frac{\partial C}{\partial F} &= 0 \\ -\frac{\partial C}{\partial F} &= \frac{\frac{\partial U}{\partial F}}{\frac{\partial U}{\partial C}} \end{aligned}$$

So for the Cobb-Douglass this is:

$$MRS = \frac{\frac{\partial U}{\partial F}}{\frac{\partial U}{\partial C}} = \frac{\alpha \frac{U}{F}}{\beta \frac{U}{C}} = \frac{\alpha C}{\beta F} \ .$$

Now a coinvent way to understand utility maximization is to have the marginal rate of substitution equal to the ratio of the prices: $MRS = \frac{p_f}{p_c}$. To understand why look at the graph below:



we are trying to achieve the highest possible indifference curve (maximizing utility) such that we are within our budget constraint. Assuming preferences are convex (and they always will be in this class) this means that we want a tangency between the indifference curve and the budget constraint. This means:

$$MRS = \frac{p_f}{p_c} \ .$$

Now let's derive the demand curves for this consumer. Since we are looking at General Equilibrium we will usually write income as an endowment of food (F_0) and clothing (C_0) so

$$I = p_f F_0 + p_c C_0 \ .$$

We have two conditions for maximization:

$$\begin{aligned} \frac{\alpha C}{\beta F} &= \frac{p_f}{p_c} \\ p_f F + p_c C &= p_f F_0 + p_c C_0 \end{aligned}$$

from the first we get $C = \frac{F}{\alpha}\frac{\beta}{p_c} p_f$ and when we plug this into the second equation we get:

$$\begin{aligned} p_f F + p_c \frac{F}{\alpha}\frac{\beta}{p_c} p_f &= p_f F_0 + p_c C_0 \\ \frac{F}{\alpha} p_f (\alpha + \beta) &= p_f F_0 + p_c C_0 \\ F &= \frac{\alpha}{\alpha+\beta}\frac{C_0 p_c + F_0 p_f}{p_f} \\ &= \frac{\alpha}{\alpha+\beta}\left(C_0 \frac{p_c}{p_f} + F_0\right) \ . \end{aligned}$$

Likewise

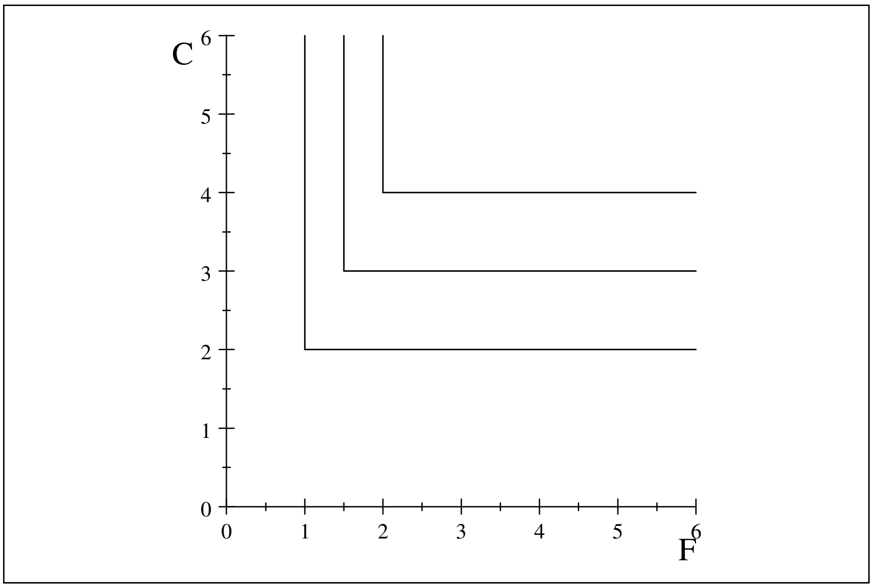
$$\begin{aligned} C &= \frac{F}{\alpha}\frac{\beta}{p_c} p_f \\ &= \frac{1}{\alpha}\frac{\beta}{p_c} p_f \left(\frac{\alpha}{\alpha+\beta}\frac{C_0 p_c + F_0 p_f}{p_f}\right) \\ &= \frac{\beta}{\alpha+\beta}\frac{C_0 p_c + F_0 p_f}{p_c} \\ &= \frac{\beta}{\alpha+\beta}\left(C_0 + F_0 \frac{p_f}{p_c}\right) \ . \end{aligned}$$

1.2 Leontief or Perfect Compliments Utility function:

This function is

$$U(F, C) = \min\{\alpha F, \beta C\}$$

where again $\alpha \geq 0$ and $\beta \geq 0$. For this utility function the indifference curves are quite strange. If $\alpha = 2$ and $\beta = 1$ then look like this:



Calculus isn't going to help us much here. To be frank the Marginal Rate of Substitution is not a function, it is undefined when $\alpha F = \beta C$. So let's use logic. Assume $\alpha = \beta = 1$, then this is the utility function for left shoes and right shoes. Quick, if you lived in a world where left and right shoes were sold separately, in what ratio would you purchase left shoes and right shoes? Or say I give you 10 left shoes, how many right shoes would you want to buy?

The answer is that the ratio of left to right shoes should be one, or for the second question that you would want to buy 10 right shoes.

So now lets make it a little more difficult. Say you have 3 legs, 2 right legs and 1 left leg. In what ratio would you purchase right and left shoes? You would buy 2 right for every 1 left. If right is C and left is F then every left shoe is worth twice as much as a right shoe. Thus this is the utility function above, where $\alpha = 2$ and $\beta = 1$. Now you should be able to generalize this idea and realize that you will obviously purchase so that:

$$\alpha F = \beta C \ .$$

Given this and our budget constraint we can now figure out what you will purchase, clearly $F = \frac{\beta}{\alpha}C$, so:

$$\begin{aligned}
p_f\left(\frac{\beta}{\alpha}C\right)+p_cC &= p_fF_0+p_cC_0 \\
\frac{C}{\alpha}\left(\alpha p_c+\beta p_f\right) &= p_fF_0+p_cC_0 \\
C &= \alpha\frac{p_fF_0+p_cC_0}{\beta p_f+\alpha p_c} \\
F &= \beta\frac{p_fF_0+p_cC_0}{\beta p_f+\alpha p_c}\ .
\end{aligned}$$

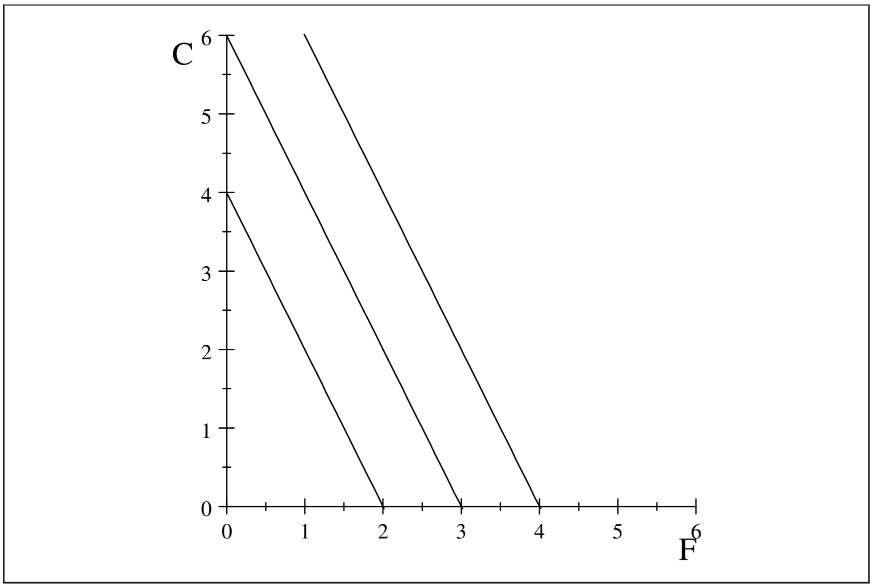
Now why would I use such a ridiculous utility function? Well simply because it makes it much easier to find Pareto Efficient outcomes. In any Pareto Efficient outcome (unless both parties have Leontief utility functions) we must have $\alpha F = \beta C$. This makes it much easier to find the contract curve. I promise I won't give you problems where multiple people have Leontief preferences. That's just a pain in the neck.

1.3 Linear or Perfect Substitutes Utility function:

This function is

$$U(F,C) = \alpha F + \beta C$$

where again $\alpha \geq 0$ and $\beta \geq 0$. For this utility function the indifference curves look fairly simple, but are awfully difficult to deal with. If $\alpha = 2$ and $\beta = 1$ then they look like this:



Now this is quite simple, also the Marginal Rate of substitution is easy to find:

$$MRS = \frac{\frac{\partial U}{\partial F}}{\frac{\partial U}{\partial C}} = \frac{\alpha}{\beta} \ .$$

So what's the problem? Well let's think about an example. Say that—like many people—you really can't tell the difference between Pepsi Cola and Coca Cola. Which do you buy? Whichever's cheaper! But usually if both are available they cost the same. So what do you buy? Whichever one you run across. So in other words you have no idea what your demand is, it depends on what's available when you purchase.

In General Equilibrium we use this to our advantage, having someone with linear preferences exactly pins down the ratio of the prices in equilibrium:

$$\frac{\alpha}{\beta} = \frac{p_f}{p_c} \ .$$

and if this is the ratio of the prices then the person with linear preferences will buy anything we want them to. Again the contract curve is completely determined by the other person's utility function (and the fact that the marginal rate of substitution must be $\frac{\alpha}{\beta}$.)

But what do the demand curves look like in general? Well I will write them down without explaining them too carefully. First you can rewrite the first order condition as:

$$\frac{\alpha}{p_f} \gtreqless \frac{\beta}{p_c} \ .$$

and if $\frac{\alpha}{p_f} > \frac{\beta}{p_c}$ you buy what is (relatively) cheaper, F. If $\frac{\alpha}{p_f} < \frac{\beta}{p_c}$ then you buy C. So the demand curves are:

$$F = \begin{cases} \frac{1}{p_f}(p_f F_0 + p_c C_0) & \text{if } \frac{\alpha}{p_f} > \frac{\beta}{p_c} \\ \tilde{F} & \text{if } \frac{\alpha}{p_f} = \frac{\beta}{p_c} \\ 0 & \text{if } \frac{\alpha}{p_f} < \frac{\beta}{p_c} \end{cases}, C = \begin{cases} 0 & \text{if } \frac{\alpha}{p_f} > \frac{\beta}{p_c} \\ \tilde{C} & \text{if } \frac{\alpha}{p_f} = \frac{\beta}{p_c} \\ \frac{1}{p_c}(p_f F_0 + p_c C_0) & \text{if } \frac{\alpha}{p_f} < \frac{\beta}{p_c} \end{cases}$$

where $p_f \tilde{F} + p_c \tilde{C} = p_f F_0 + p_c C_0$ (notice that's just a long winded way of saying you don't care what you buy.) But like I said we don't really care about the demand curves for this person, we will have $\frac{\alpha}{\beta} = \frac{p_f}{p_c}$ and just give this person whatever the other doesn't want. Notice that like with the Leontief if both people have Linear preferences then you are going to have a real mess. I promise not to do that.