

OPTIMAL STOPPING OF LINEAR DIFFUSIONS WITH RANDOM DISCOUNTING

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ABSTRACT. We propose a new solution method for optimal stopping problems with random discounting for linear diffusions whose state space has a combination of natural, absorbing, or reflecting boundaries. The method uses a concave characterization of excessive functions for linear diffusions killed at a rate determined by a Markov additive functional and reduces the original problem to an undiscounted optimal stopping problem for a standard Brownian motion. The latter can be solved essentially by inspection. The necessary and sufficient conditions for the existence of an optimal stopping rule are proved when the reward function is continuous. The results are illustrated on examples.

1. INTRODUCTION

Optimal stopping problems often arise in economics, finance and statistics. Finding the best time or the best decision rule to exercise American-type financial options, to enter investment contracts or to abandon certain projects, to alert the controller for an abrupt change in a regulated process are important examples; see, e.g., Dixit and Pindyck [11], Karatzas [20], Peskir and Shiryaev [31], Shiryaev [35]. When the underlying stochastic process is governed by a stochastic differential equation, the optimal stopping problem is typically formulated as a free-boundary problem by means of *variational* arguments; see, e.g., Guo and Shepp [18], Karatzas and Ocone [21], Karatzas and Wang [26]. The correct formulation of the free-boundary problem sometimes requires considerable imagination. This is indeed an artful task in that the optimal continuation and stopping regions have to be guessed a priori. The free-boundary problem may then be solved with the help of the smooth-fit principle. The optimality of the candidate continuation and stopping regions are typically proved by direct verification; see, e.g., Bensoussan and Lions [3], Brekke and Øksendal [6], Friedman [16], Grigelionis and Shiryaev [17], Øksendal [29], Shiryaev [34, Section 3.8].

The variational methods become challenging when the form of the reward function and/or the dynamics of the diffusion obscure the shape of the optimal continuation region. If the latter is a disconnected subset of the state space, we may end up with several solutions of the free-boundary problem. Finding the right candidate for the optimal solution may become nontrivial. Let us also mention that there are cases where the smooth-fit principle does not apply; see, e.g., Øksendal and Reikvam [30], Salminen [33, page 98, Example (iii)].

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If the terminal reward upon stopping is *discounted at a constant rate* (i.e., $A_t = \beta t$, $t \geq 0$ for some constant $\beta \geq 0$ in (1.2) below), and the boundaries of the state-space of X are either *absorbing or natural*, a new *direct* solution method was proposed by Dayanik and Karatzas [9]. The method is direct in the sense that it does not require any a-priori guess of the optimal stopping region; therefore, it avoids the difficulties of free-boundary formulation. Instead, it relies on a concave characterization of excessive functions for general linear diffusions. Then the well-known excessive characterization (see, for example, Dynkin [12], Shiryaev [34, Theorem 1, p. 124], Øksendal [29, Theorem 10.1.9]) of the value function has been fully used to solve optimal stopping problems. The new method reduces every optimal stopping problem to a special one which is essentially solved by inspection.

In this paper, we further develop this methodology in two nontrivial directions; namely, when the *discounting is random*, and the underlying diffusion may have *reflecting boundaries*.

Random discounting is important in financial applications and control theory. In finance, new exotic options on stocks are designed to have payoffs discounted by a functional of the underlying stock process. For example, step options are proposed as an alternative to popular barrier options to alleviate the risk management problems inherent to the latter; see, e.g., Linetsky [28]. Pricing and hedging a perpetual American-type “down-and-out” step option with a barrier at B requires the solution of an optimal stopping problem as in (1.2) where the discount rate is the occupation time $A_t = \text{meas}\{u \in [0, t] : S_u \leq b\}$ of the stock price S in $(-\infty, B]$ until time t . In control theory, certain singular stochastic control problems are known to be also equivalent to optimal stopping problems whose payoffs are discounted by some additive functional of the underlying diffusions; see, e.g., Boetius and Kohlmann [4], Karatzas and Shreve [22; 23].

Reflecting boundaries are also common in financial economics. In a competitive market, which is open to entries and exits of price-taking companies, the price process of a commodity is typically assumed to have an upper reflecting barrier. For example, effects of price ceilings on the partially irreversible entry and exit decisions made by companies under uncertainty are studied by Dixit [10]. In the equivalent optimal stopping problem of certain singular control problems mentioned above, the underlying diffusion has sometimes reflecting boundaries as well.

Let us now introduce the mathematical framework. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a Brownian motion $B = \{B_t; t \geq 0\}$ and a diffusion process $X = \{X_t; t \geq 0\}$ on some state-space $\mathcal{I} \subseteq \mathbb{R}$ with dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0 \tag{1.1}$$

for some Borel functions $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto (0, \infty)$. We assume that \mathcal{I} is an interval with endpoints $-\infty \leq a < b \leq +\infty$, and that (1.1) has weak solution with unique probability law, which is guaranteed, see Karatzas and Shreve [24, pp. 329-353] for example, if

$$\forall x \in \text{int}(\mathcal{I}) \quad \exists \varepsilon > 0 \quad \text{such that} \quad \int_{(x-\varepsilon, x+\varepsilon)} \frac{1 + |\mu(y)|}{\sigma^2(y)} dy < \infty.$$

We also assume that X is regular; i.e., X reaches y with positive probability starting at x for every $x \in (a, b)$ and $y \in \mathcal{I}$. We shall denote the natural filtration of X by $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

Let $\{A_t : t \geq 0\}$ be a continuous additive functional of X on the same probability space; we shall use A_t and $A(t)$ interchangeably. Namely, $A(\cdot)$ is an \mathbb{F} -adapted process that is almost surely nonnegative, continuous, vanishing at zero, and has the additivity property

$$A_{s+t} = A_s + A_t \circ \theta_s, \quad s, t \geq 0 \quad \text{a.s.},$$

where θ_s is the usual shift operator: $X_t \circ \theta_s = X_{t+s}$. Let $h(\cdot)$ be a Borel function such that $\mathbb{E}_x[e^{-A_\tau} h(X_\tau) 1_{\{\tau < \infty\}}]$ exists for every \mathbb{F} -stopping time τ and $x \in \mathcal{I}$. Denote by

$$V(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}_x [e^{-A_\tau} h(X_\tau) 1_{\{\tau < \infty\}}], \quad x \in \mathcal{I} \quad (1.2)$$

the *value function* of the optimal stopping problem with terminal *reward function* $h(\cdot)$ and random *discount rate* $A(\cdot)$; the supremum in (1.2) is taken over the set \mathcal{S} of all \mathbb{F} -stopping times. The objective is to find (i) the value function $V(\cdot)$ in (1.2), and (ii) an optimal stopping time $\tau^* \in \mathcal{S}$ that attains the supremum in (1.2), if such a stopping time exists.

By enlarging the probability space, if necessary, every expectation in (1.2) can be expressed as

$$\mathbb{E}_x [e^{-A_\tau} h(X_\tau) 1_{\{\tau < \infty\}}] = \mathbb{E}_x [1_{\{\zeta > \tau\}} h(X_\tau)] = \mathbb{E}_x [h(\widehat{X}_\tau)]$$

in terms of a unit-rate exponentially distributed random variable E independent of the diffusion X , the process

$$\widehat{X}_t \triangleq \begin{cases} X_t, & \text{if } A_t < E \\ \partial, & \text{if } A_t \geq E \end{cases} \quad \text{for every } t \geq 0$$

obtained by killing X at rate A_t , and killing time $\zeta \triangleq \inf\{t \geq 0; \widehat{X}_t = \partial\}$ for some fixed $\partial \notin \mathcal{I}$; see, for example, Borodin and Salminen [5, p. 28, II.23]. Then $\widehat{X}_t = X_t$ for every $t \in [0, \zeta)$, and \widehat{X} is sent at time ζ to a fixed ‘‘cemetery state’’ ∂ , where it stays forever. We extend every function $f(\cdot)$ defined on \mathcal{I} to $\mathcal{I}^\partial = \mathcal{I} \cup \{\partial\}$ by setting $f(\partial) = 0$.

A Borel measurable function $U : \mathcal{I} \mapsto [0, \infty)$ is called *excessive* for the process X killed at rate $A(\cdot)$ if $U(x) \geq \mathbb{E}_x [e^{-A_\tau} U(X_\tau) 1_{\{\tau < \infty\}}]$ for every $x \in \mathcal{I}$ and $\tau \in \mathcal{S}$. To solve (1.2), we first establish a concave characterization of excessive functions, and then use it to transform (1.2) to an equivalent undiscounted optimal stopping problem with a suitable reward function for a standard Brownian motion. Finally, this latter problem has an elegant solution described by Dynkin and Yushkevich [14, pp. 112-126].

At the time of writing this work, we were unaware of Dynkin’s [13, Volume II; pp. 146, 155; Theorems 15.10, 16.4] own concave characterizations of excessive functions of one-dimensional diffusions. Although statements of our characterization (Proposition 3.1) and Dynkin’s characterizations are similar, they are not the same. While one of Dynkin’s two characterizations does not give information at the boundaries unlike ours, the other differs from our characterization in the way the killing in the interior of the state space is handled. E. Dynkin allows it, and we do not. In Section 5, we

show that the killing inside the state space can be avoided by a suitable h -transform, and that our characterization takes care of this implicitly with more elementary arguments. In the same section, we argue that, by disallowing killing inside state space, our concave characterization of excessive functions leads to a more effective and natural solution of optimal stopping problems, which *is* the main concern of this work.

Our interest to the optimal stopping of one-dimensional diffusions killed at the rate of a continuous additive process was raised by Beibel and Lerche's work [1; 2], where they describe a novel solution method inspired by some treatment of generalized parking problems. Their method determines an optimal strategy by applying, at *every* point in the state-space of the underlying diffusion, five tests. Each test amounts to sliding a parameter back and forth until one of three conclusions is arrived: either an optimal strategy does not exist, or it exists and is either a one-sided or a two-sided stopping rule; in the latter case, another search is required for the critical threshold(s) of the stopping rule. The search for the correct value of the sliding parameter and critical thresholds (especially, of two-sided optimal stopping rules) can be quite demanding, and repeating those tasks at every point in the state-space limits the effectiveness of the method.

In contrast with Beibel and Lerche's method, our proposed method determines an optimal strategy everywhere at once by a simple inspection of a transformation of the terminal reward function and its smallest nonnegative concave majorant, both of which are straightforward to calculate. To help for a comparison of two methods, we applied in Section 4 our method to some of Beibel and Lerche's [2] examples. The examples suggest that by sliding the parameter in Beibel and Lerche's method, one is executing a local search, in our terms, for the value of the smallest nonnegative concave majorant of the transformed reward function. On the contrary, our proposed method constructs the value function globally, by utilizing concave characterization of value function, and avoids difficulties of local search. This is especially evident from the ease with which our method determines two-sided optimal stopping rule in Section 4.3.

We start with an overview of one-dimensional diffusions in Section 2. In Section 3, we show that value function of (1.2) and excessive functions in general are concave in some generalized sense, and describe the solution method for the optimal stopping problem. In Section 4, we exhibit the method on several examples. We examine the connection between our and Dynkin's characterization of excessive functions in Section 5.

2. ONE-DIMENSIONAL REGULAR DIFFUSION PROCESSES

We assume that X is a one-dimensional regular diffusion of the type (1.1) on an interval \mathcal{I} . If an endpoint is included in the state-space \mathcal{I} , we shall assume that it is either absorbing, or instantaneously reflecting. If it is not contained in \mathcal{I} , then we assume that it is natural (see, for example, Karlin and Taylor [27]). If the left-boundary point a is absorbing, then we set $\mathcal{B}_{\text{Abs}}^\ell \triangleq \{a\}$, and $\mathcal{B}_{\text{Abs}}^\ell \triangleq \emptyset$ otherwise. Similarly, $\mathcal{B}_{\text{Abs}}^r \triangleq \{b\}$ if the right-boundary point b is absorbing, and $\mathcal{B}_{\text{Abs}}^r \triangleq \emptyset$ otherwise.

We define $\tau_r \triangleq \inf\{t \geq 0 : X_t = r\}$ for every $r \in \mathcal{I}$ and denote the interior of the state space \mathcal{I} by $\text{int}(\mathcal{I})$. A one-dimensional diffusion X is *regular*, if for every $x \in \text{int}(\mathcal{I})$ and $y \in \mathcal{I}$, we have $\mathbb{P}_x(\tau_y < \infty) > 0$. Therefore, the interior of the state-space \mathcal{I} cannot be decomposed into smaller sets from which X could not exit.

Let $A(\cdot)$ be a continuous additive functional of X . Then it is also *strongly* additive; see Revuz and Yor [32, page 403, Remark 1]). Namely, if T is a stopping time of \mathbb{F} , then $A_{T+S} = A_T + A_S(\theta_T)$ \mathbb{P}_x -a.s., for every $x \in \mathcal{I}$ and positive random variable S . If T and S are two stopping times, then $T + S \circ \theta_T$ is also a stopping time. The strong additivity of $A(\cdot)$ implies

$$A_{T+S \circ \theta_T} = A_T + A_S \circ \theta_T \quad \mathbb{P}_x\text{-a.s. } \forall x \in \mathcal{I}. \quad (2.1)$$

It is important to note that $A_S \circ \theta_T$ is the mapping $\omega \mapsto A_{S(\theta_T(\omega))}(\theta_T(\omega))$, whereas $A_S(\theta_T)$ is the mapping $\omega \mapsto A_{S(\omega)}(\theta_T(\omega))$.

Let us also introduce the functions

$$\psi(x) \triangleq \begin{cases} \mathbb{E}_x[e^{-A_{\tau_c}} 1_{\{\tau_c < \infty\}}], & x \leq c \\ \frac{1}{\mathbb{E}_c[e^{-A_{\tau_x}} 1_{\{\tau_x < \infty\}}]}, & x > c \end{cases}, \quad \varphi(x) \triangleq \begin{cases} \frac{1}{\mathbb{E}_c[e^{-A_{\tau_x}} 1_{\{\tau_x < \infty\}}]}, & x \leq c \\ \mathbb{E}_x[e^{-A_{\tau_c}} 1_{\{\tau_c < \infty\}}]}, & x > c \end{cases}, \quad x \in \mathcal{I} \quad (2.2)$$

for some arbitrary but fixed $c \in \text{int}(\mathcal{I})$. If $x < y < z$ are in \mathcal{I} , then \mathbb{P}_x -a.s. $\tau_z = \tau_y + \tau_z \circ \theta_{\tau_y}$, and the strong Markov property of X and (2.1) imply $\mathbb{E}_x[e^{-A_{\tau_z}} 1_{\{\tau_z < \infty\}}] = \mathbb{E}_x[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}}] \cdot \mathbb{E}_y[e^{-A_{\tau_z}} 1_{\{\tau_z < \infty\}}]$, and $\mathbb{E}_z[e^{-A_{\tau_x}} 1_{\{\tau_x < \infty\}}] = \mathbb{E}_z[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}}] \cdot \mathbb{E}_y[e^{-A_{\tau_x}} 1_{\{\tau_x < \infty\}}]$. Therefore,

$$\mathbb{E}_x[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}}] = \begin{cases} \psi(x)/\psi(y), & x \leq y \\ \varphi(x)/\varphi(y), & x > y \end{cases}, \quad x, y \in \mathcal{I}, \quad (2.3)$$

which also reveals that, by changing the reference point $c \in \text{int}(\mathcal{I})$ in (2.2), we obtain a multiple of the same functions by positive constants.

Lemma 2.1. *The process $\{e^{-A_{t \wedge \tau_y}} \psi(X_{t \wedge \tau_y}), \mathcal{F}_{t \wedge \tau_y} : t \geq 0\}$ is a \mathbb{P}_x -martingale for every $x < y$ in \mathcal{I} . Similarly, $\{e^{-A_{t \wedge \tau_y}} \varphi(X_{t \wedge \tau_y}), \mathcal{F}_{t \wedge \tau_y} : t \geq 0\}$ is a \mathbb{P}_x -martingale for every $x \geq y$ in \mathcal{I} .*

Proof. Fix $x < y$ in \mathcal{I} . Since \mathbb{P}_x -a.s. $X_{t \wedge \tau_y} \leq y$, we have \mathbb{P}_x -a.s. $\psi(X_{t \wedge \tau_y}) = \psi(y) \mathbb{E}_{X_{t \wedge \tau_y}}[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}}]$ by (2.3). Using the strong Markov property of X and the strong additivity of $A(\cdot)$, we find $e^{-A_{t \wedge \tau_y}} \psi(X_{t \wedge \tau_y}) = \psi(y) \mathbb{E}_x[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}} | \mathcal{F}_{t \wedge \tau_y}]$ \mathbb{P}_x -a.s.; i.e., $\{e^{-A_{t \wedge \tau_y}} \psi(X_{t \wedge \tau_y}), \mathcal{F}_{t \wedge \tau_y} : t \geq 0\}$ is a \mathbb{P}_x -martingale. The proof of the second part is similar. \square

We shall denote the *scale function* of X by $S(\cdot)$. It is the unique, up to affine transformations, strictly increasing and continuous function on \mathcal{I} with the property

$$\mathbb{P}_x(\tau_y < \tau_z) = \frac{S(z) - S(x)}{S(z) - S(y)} \quad \text{and} \quad \mathbb{P}_x(\tau_y > \tau_z) = \frac{S(x) - S(y)}{S(z) - S(y)}, \quad x \in [y, z] \subseteq \mathcal{I}; \quad (2.4)$$

see, for example, Revuz and Yor [32], Karlin and Taylor [27]. If $F : \mathcal{I} \mapsto \mathbb{R}$ is any strictly increasing, then a function $U : \mathcal{I} \mapsto \mathbb{R}$ is called *F-concave* if

$$U(x) \geq U(y) \frac{F(z) - F(x)}{F(z) - F(y)} + U(z) \frac{F(x) - F(y)}{F(z) - F(y)}, \quad x \in [y, z] \subseteq \mathcal{I}.$$

For the properties of F -concave functions; Dynkin [13, Volume II, p. 231-240], Revuz and Yor [32, pp. 544-547]. The following corollary will be useful later in proving the smooth-fit principle; see Proposition 3.8.

Corollary 2.1. *The functions $\psi(\cdot)$ and $\varphi(\cdot)$ of (2.2) are S -convex on $\text{int}(\mathcal{I})$.*

Proof. Fix any $[y, z] \subseteq \text{int}(\mathcal{I})$. Lemma 2.1 implies $\psi(x) = \mathbb{E}_x[e^{-A_{\tau_y \wedge \tau_z}} \psi(X_{\tau_y \wedge \tau_z})] \leq \psi(y)\mathbb{P}_x(\tau_y < \tau_z) + \psi(z)\mathbb{P}_x(\tau_y > \tau_z)$ for every $x \in [y, z]$, and S -convexity of $\psi(\cdot)$ follows from (2.4). The proof is similar for $\varphi(\cdot)$. \square

Lemma 2.2. *The functions $\psi(\cdot)$ and $\varphi(\cdot)$ are continuous and strictly positive on $\text{int}(\mathcal{I})$. If $a \in \mathcal{I}$, then $\lim_{x \downarrow a} \psi(x) = \psi(a)$ and $\lim_{x \downarrow a} \varphi(x) = \varphi(a)$. If $b \in \mathcal{I}$, then $\lim_{x \uparrow b} \psi(x) = \psi(b)$ and $\lim_{x \uparrow b} \varphi(x) = \varphi(b)$. The function $\psi(\cdot)$ is nondecreasing, and $\varphi(\cdot)$ is nonincreasing on \mathcal{I} .*

The proof of the lemma is similar to the arguments of Itô and McKean [19, Section 4.6]. Let us now introduce the functions

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad \text{and} \quad G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -\frac{1}{F(x)}, \quad x \in \text{int}(\mathcal{I}). \quad (2.5)$$

Both $F(\cdot)$ and $G(\cdot)$ are nondecreasing and continuous on $\text{int}(\mathcal{I})$. If $a \in \mathcal{I}$, then we define $F(a) \triangleq \lim_{x \downarrow a} F(x)$ and $G(a) \triangleq \lim_{x \downarrow a} G(x)$. If $b \in \mathcal{I}$, then similarly $F(b) \triangleq \lim_{x \uparrow b} F(x)$ and $G(b) \triangleq \lim_{x \uparrow b} G(x)$. Next proposition shows that $F(\cdot)$ and $G(\cdot)$ are *strictly increasing*, either if X is transient (for example, if it has absorbing boundaries) or if $A(\cdot)$ does not vanish everywhere.

Proposition 2.1. *There are distinct states $x, y \in \text{int}(\mathcal{I})$ such that $F(x) = F(y)$ if and only if (i) X is recurrent, and (ii) $\mathbb{P}_z\{A_t = 0, t \geq 0\} = 1$ for every $z \in \mathcal{I}$.*

Proof. The sufficiency is clear. For the proof of the necessity, suppose $F(x) = F(y)$ for some $x < y$ in $\text{int}(\mathcal{I})$. By (2.3), $1 = F(x)/F(y) = \mathbb{E}_x[e^{-A_{\tau_y}} 1_{\{\tau_y < \infty\}}] \mathbb{E}_y[e^{-A_{\tau_x}} 1_{\{\tau_x < \infty\}}]$, and therefore, $\mathbb{P}_x\{\tau_y < \infty\} = \mathbb{P}_y\{\tau_x < \infty\} = \mathbb{P}_x\{A_{\tau_y} = 0\} = \mathbb{P}_y\{A_{\tau_x} = 0\} = 1$. Since X is regular, this implies $\mathbb{P}_u\{\tau_v < \infty\} = 1$ for every $u, v \in \mathcal{I}$ and proves (i). Part (ii) follows from the strong Markov property of the recurrent process X and strong additivity of A . \square

Lemma 2.3. *Suppose that $F(\cdot)$ is strictly increasing. For every $x \in [l, r] \subseteq \mathcal{I}$, we have*

$$\mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}] = \frac{\varphi(r)\psi(x) - \psi(r)\varphi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)}, \quad \mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}] = \frac{\psi(l)\varphi(x) - \varphi(l)\psi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)}.$$

Proof. By Lemma 2.1 and optional sampling theorem, the stopped processes $e^{-A_{t \wedge \tau_l \wedge \tau_r}} \psi(X_{t \wedge \tau_l \wedge \tau_r})$ and $e^{-A_{t \wedge \tau_l \wedge \tau_r}} \varphi(X_{t \wedge \tau_l \wedge \tau_r})$ are bounded continuous \mathbb{P}_x -martingales for every $x \in [l, r]$, and

$$\begin{aligned} \psi(x) &= \mathbb{E}_x[e^{-A_{\tau_l \wedge \tau_r}} \psi(X_{\tau_l \wedge \tau_r})] = \psi(l)\mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}] + \psi(r)\mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}], \\ \varphi(x) &= \mathbb{E}_x[e^{-A_{\tau_l \wedge \tau_r}} \varphi(X_{\tau_l \wedge \tau_r})] = \varphi(l)\mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}] + \varphi(r)\mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}]. \end{aligned}$$

Since $F(\cdot)$ is strictly increasing, $F(r) > F(l)$ and $\varphi(r)\psi(l) - \psi(r)\varphi(l) \neq 0$. Therefore, the equations can be solved for $\mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}]$ and $\mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}]$ uniquely as above. \square

Proposition 2.2. Let $U : \mathcal{I} \mapsto [0, \infty)$ be an $A(\cdot)$ -excessive function, and $Z_t \triangleq e^{-A_t}U(X_t)$, $t \geq 0$. Then $\{Z_t, \mathcal{F}_t\}_{t \geq 0}$ is a nonnegative \mathbb{P}_x -supermartingale, $x \in \mathcal{I}$ with the last element $Z_\infty \equiv 0$.

Proof. This follows from the definition of excessive functions, strong Markov property of X , and strong additivity of A ; see, e.g., Øksendal [29, Lemma 10.1.3(e)]. \square

3. OPTIMAL STOPPING PROBLEMS

Let X be a diffusion process described by (1.1), $A(\cdot)$ be a continuous additive functional of X , and $h : \mathcal{I} \mapsto \mathbb{R}$ be a locally bounded Borel function. The solution of (1.2) is trivially $V(x) = \sup_{y \in \mathcal{I}} h(y)$, $x \in \mathcal{I}$, if (i) X is recurrent, and (ii) $A(\cdot) \equiv 0$ almost surely. Therefore, we assume in the remainder that at least one of (i) and (ii) does not hold. Then $F(\cdot)$ and $G(\cdot)$ of (2.5) are *strictly* increasing by Proposition 2.1, and divisions by $F(r) - F(l)$ in (3.3) and (3.4) below always make sense.

In Proposition 3.1 below we give a new characterization of excessive functions. To motivate this key result, let $U : \mathcal{I} \mapsto [0, \infty)$ be an excessive function of the process X killed at the rate $A(\cdot)$; i.e.,

$$U(x) \geq \mathbb{E}_x[e^{-A_\tau}U(X_\tau)1_{\{\tau < \infty\}}], \quad \forall x \in \mathcal{I}, \forall \tau \in \mathcal{S}. \quad (3.1)$$

If $[l, r] \subseteq \mathcal{I}$ and $\tau \triangleq \tau_l \wedge \tau_r$, then $U(x) \geq U(l)\mathbb{E}_x[e^{-A_{\tau_l}}1_{\{\tau_l < \tau_r\}}] + U(r)\mathbb{E}_x[e^{-A_{\tau_r}}1_{\{\tau_l > \tau_r\}}]$, $x \in [l, r]$. Since $F(\cdot)$ is strictly increasing, Lemma 2.3 implies

$$U(x) \geq U(l) \frac{\varphi(r)\psi(x) - \psi(r)\varphi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)} + U(r) \frac{\psi(l)\varphi(x) - \varphi(l)\psi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)}, \quad x \in [l, r]. \quad (3.2)$$

By dividing both sides by $\varphi(x)$, respectively by $\psi(x)$, and rearranging terms afterwards gives

$$\frac{U(x)}{\varphi(x)} \geq \frac{U(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{U(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)}, \quad r \notin \mathcal{B}_{\text{Abs}}^r, \quad x \in [l, r], \quad (3.3)$$

respectively

$$\frac{U(x)}{\psi(x)} \geq \frac{U(l)}{\psi(l)} \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{U(r)}{\psi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}, \quad l \notin \mathcal{B}_{\text{Abs}}^\ell, \quad x \in [l, r]. \quad (3.4)$$

Hence $U(\cdot)/\varphi(\cdot)$ is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$, and $U(\cdot)/\psi(\cdot)$ is G -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$.

If the left-boundary a of the state-space \mathcal{I} is instantaneously reflecting (i.e., $a \in \mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$), then letting $\tau = \tau_r$ in (3.1) gives $U(x)/\psi(x) \geq U(r)/\psi(r)$, $x \in [a, r]$ because of (2.3). Hence, if a is instantaneously reflecting, then $U(\cdot)/\psi(\cdot)$ is *nonincreasing* on \mathcal{I} . Similarly, if the right-boundary b is instantaneously reflecting (i.e., $b \in \mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$), then $U(\cdot)/\varphi(\cdot)$ is *nondecreasing* on \mathcal{I} . According to Proposition 3.1, the concavity and monotonicity of $U(\cdot)/\varphi(\cdot)$ and $U(\cdot)/\psi(\cdot)$ are not only necessary but also sufficient for a nonnegative real-valued function $U(\cdot)$ to be excessive for the process X killed at rate $A(\cdot)$. We shall defer the proofs of the propositions to the end of the subsection.

Proposition 3.1. A function $U : \mathcal{I} \mapsto [0, \infty)$ is excessive for the one-dimensional diffusion X killed at rate $A(\cdot)$; i.e., (3.1) is satisfied, if and only if all of the followings are true:

- (i) $U(\cdot)/\varphi(\cdot)$ is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$,
- (ii) $U(\cdot)/\psi(\cdot)$ is G -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$,

- (iii) $U(\cdot)/\varphi(\cdot)$ is nondecreasing if b is instantaneously reflecting,
- (iv) $U(\cdot)/\psi(\cdot)$ is nonincreasing if a is instantaneously reflecting.

Remark 3.1. The conditions (i) and (ii) are essentially equivalent. The function $U(\cdot)/\varphi(\cdot)$ is F -concave on $\text{int}(\mathcal{I})$ if and only if $U(\cdot)/\psi(\cdot)$ is G -concave on $\text{int}(\mathcal{I})$. However, division by $\psi(a) = 0$ or $\varphi(b) = 0$ is not allowed if, respectively, the left-boundary a or the right-boundary b is absorbing; therefore, conditions (i) and (ii) complement each other when both boundaries are absorbing.

Remark 3.2. If $U(\cdot)/\varphi(\cdot)$ is F -concave, or $U(\cdot)/\psi(\cdot)$ is G -concave, then $D_F^-(U/\varphi)(\cdot)$ and $D_G^+(U/\psi)(\cdot)$ are nonincreasing. Therefore, functions $U(\cdot)/\varphi(\cdot)$ and $U(\cdot)/\psi(\cdot)$ are, respectively, nondecreasing and nonincreasing, if and only if, respectively,

$$D_F^-(U/\varphi)(b) \geq 0 \quad \text{and} \quad D_G^+(U/\psi)(a) \leq 0, \quad (3.5)$$

where the derivatives should be understood as their limits from left and from right, respectively, at b and a if $\varphi(b) = 0$ or $\psi(a) = 0$.

Remark 3.3. The functions $U(\cdot)/\varphi(\cdot)$ and $U(\cdot)/\psi(\cdot)$ are always nondecreasing and nonincreasing, respectively. Namely, the conditions (iii) and (iv), equivalently boundary conditions in (3.5), always hold, but they are implied by (i) and (ii), respectively, if the boundaries are natural or absorbing.

Suppose, for example, b is natural or absorbing, and let us show by using (i) that $U(\cdot)/\varphi(\cdot)$ is nondecreasing. In either case, we have $F(b-) = +\infty$. If $U(\cdot)/\varphi(\cdot)$ is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$, then $(U/\varphi) \circ F^{-1}(\cdot)$ is concave and has nonincreasing right-derivative on $(F(a+), +\infty)$. Because $(U/\varphi) \circ F^{-1}(\cdot)$ must stay nonnegative on the half-line $(F(a+), +\infty)$, its right-derivative must always be nonnegative. Therefore, $(U/\varphi) \circ F^{-1}(\cdot)$ is nondecreasing on $(F(a+), +\infty)$, and this implies that $U(\cdot)/\varphi(\cdot)$ is nondecreasing on \mathcal{I} .

In the argument above, we used, in addition to (i), two crucial information: $U(\cdot)$ is *nonnegative*, and that $F(b-)$ is *infinite*. The latter fails (i.e., $F(b)$ is finite), if b is a reflecting boundary; therefore, (iii) is no longer guaranteed by (i) and must be stated explicitly in that case. The relation between conditions (ii) and (iv) is similar.

Next proposition is the restatement of Beibel and Lerche's [2] Theorem 1 in terms of two new quantities; namely, ℓ_a and ℓ_b in (3.6). Later, we shall see that not only the finiteness of value function but also the existence of an optimal stopping time are determined completely by the values of ℓ_a and ℓ_b .

Proposition 3.2. The value function $V(\cdot)$ of (1.2) is either finite or infinite everywhere on $\mathcal{I} \setminus (\mathcal{B}_{\text{Abs}}^\ell \cup \mathcal{B}_{\text{Abs}}^r)$. Moreover, $V(\cdot) \equiv \infty$ on $\mathcal{I} \setminus (\mathcal{B}_{\text{Abs}}^\ell \cup \mathcal{B}_{\text{Abs}}^r)$ if and only if at least one of the limits

$$\ell_a \triangleq \overline{\lim}_{x \downarrow a} \frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad \ell_b \triangleq \overline{\lim}_{x \uparrow b} \frac{h^+(x)}{\psi(x)}, \quad (3.6)$$

is infinite ($h^+(\cdot) \triangleq \max\{0, h(\cdot)\}$).

Note that ℓ_a (respectively, ℓ_b) may become infinite if and only if a (respectively, b) is a natural boundary of the state space \mathcal{I} . The proof of the proposition is, therefore, similar to that of Proposition 5.2 in Dayanik and Karatzas [9]. According to Proposition 3.2, the optimal stopping problem in (1.2) has trivial solution, unless

$$\text{the quantities } \ell_a \text{ and } \ell_b \text{ are finite.} \quad (3.7)$$

Therefore, in the remainder we shall assume that (3.7) holds.

Proposition 3.3. The value function $V(\cdot)$ of (1.2) is the smallest excessive majorant of $h(\cdot)$ on \mathcal{I} . Equivalently, it is the smallest nonnegative majorant of $h(\cdot)$ on \mathcal{I} with the properties (i)-(iv) of Proposition 3.1.

Next proposition and two immediate corollaries are useful in calculating the value function $V(\cdot)$ in (1.2); see the examples in Section 4.

Proposition 3.4. Suppose that a is absorbing or natural, and that b is reflecting or natural. Let $W : F(\mathcal{I}) \mapsto \mathbb{R}$ be the smallest nonnegative concave (also, nondecreasing, if b is *reflecting*) majorant of the function $H(y) \triangleq (h/\varphi) \circ F^{-1}(y)$, $y \in F(\mathcal{I})$. Then $V(x) = \varphi(x)W(F(x))$ for every $x \in \mathcal{I}$.

Remark 3.4. If the roles of a and b are interchanged in Proposition 3.4, then we replace $\psi(\cdot)$, $\varphi(\cdot)$, $F(\cdot)$ and “nondecreasing” with $\varphi(\cdot)$, $\psi(\cdot)$, $G(\cdot)$ and “nonincreasing”, respectively.

Corollary 3.1. Let us define

$$\frac{\mathbf{h}}{\varphi}(x) \triangleq \sup_{z \leq x} \frac{h(z)}{\varphi(z)} \quad \text{and} \quad \frac{\mathbf{h}}{\psi}(x) \triangleq \sup_{z \geq x} \frac{h(z)}{\psi(z)}, \quad x \in \mathcal{I}.$$

Let $\mathbf{W}(\cdot)$ and $\widehat{\mathbf{W}}(\cdot)$ be the smallest nonnegative concave majorants of $\mathbf{H}(y) \triangleq (\frac{\mathbf{h}}{\varphi}) \circ F^{-1}(y)$, $y \in F(\mathcal{I})$ and $\widehat{\mathbf{H}}(y) \triangleq (\frac{\mathbf{h}}{\psi}) \circ G^{-1}(y)$, $y \in G(\mathcal{I})$, respectively. If b (resp., a) is reflecting, and a (resp., b) is absorbing or natural, then $V(x) = \varphi(x)\mathbf{W}(F(x))$ (resp., $V(x) = \psi(x)\widehat{\mathbf{W}}(G(x))$), $x \in \mathcal{I}$.

Corollary 3.2. Suppose both a and b are reflecting. Let $W : [F(a), F(b)] \mapsto \mathbb{R}$ be the smallest of the collection of all nonnegative, nondecreasing and concave majorants $\mathcal{W}(\cdot)$ of $H(\cdot) \triangleq (h/\varphi) \circ F^{-1}(\cdot)$, for which $\mathcal{W}(y)/y$ is nonincreasing. Then $V(x) = \varphi(x)W(F(x))$ for every $x \in [a, b]$.

Proof of the sufficiency in Proposition 3.1. Suppose $U(\cdot)$ is nonnegative, and (i)-(iv) hold. As a consequence of (i) and (ii), $U(\cdot)/\varphi(\cdot)$ and $U(\cdot)/\psi(\cdot)$ are lower semi-continuous on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$ and $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$, respectively. Since $\varphi(\cdot)$ and $\psi(\cdot)$ are continuous on \mathcal{I} , we conclude that $U(\cdot)$ is lower semi-continuous on \mathcal{I} . It is enough to prove that

$$U(x) \geq \mathbb{E}_x[e^{-At}U(X_t)], \quad \forall t \geq 0 \quad \text{and} \quad \forall x \in \mathcal{I}. \quad (3.8)$$

Since the strong Markov property of X , strong additivity of A , and (3.8) imply that $\{e^{-At}U(X_t), t \geq 0\}$ is a nonnegative \mathbb{F} -supermartingale, the function $U(\cdot)$ is excessive for X killed at rate $A(\cdot)$ by the optional sampling theorem for nonnegative supermartingales.

Here, we shall give the details for the proof of (3.8) when a is reflective, and b is natural. The proofs for the remaining cases are similar and can be found in Dayanik [8]. Let $[a, r_n]$, $n \geq 1$ be a sequence of subintervals, strictly increasing to $[a, b)$. Then

- E.1. the process $e^{-A_t \wedge \tau_{r_n}} \psi(X_{t \wedge \tau_{r_n}})$, $t \geq 0$ is a \mathbb{P}_x -martingale, $x \in [a, r_n]$, $n \geq 1$,
- E.2. the process $e^{-A_t \wedge \tau_a} \varphi(X_{t \wedge \tau_a})$, $t \geq 0$ is a \mathbb{P}_x -martingale, $x \in [a, b)$,
- E.3. the function $U(\cdot)/\varphi(\cdot)$ is F -concave on $[a, b)$,
- E.4. the function $U(\cdot)/\psi(\cdot)$ is nonincreasing on $[a, b)$,

thanks to Lemma 2.1, (i) and (iv). To prove (3.8) for $x = a$, note that E.4, E.1 and optional sampling imply $\mathbb{E}_a[e^{-A_t \wedge \tau_{r_n}} U(X_{t \wedge \tau_{r_n}})] \leq [U(a)/\psi(a)] \cdot \mathbb{E}_a[e^{-A_t \wedge \tau_{r_n}} \psi(X_{t \wedge \tau_{r_n}})] = U(a)$, $t \geq 0$. Using Fatou's Lemma and lower semi-continuity of $U(\cdot)$, we obtain (3.8). Suppose now $x \in (a, b)$. By E.3, there exists an affine transformation $L(\cdot) \triangleq c_1 F(\cdot) + c_2$ of $F(\cdot)$ such that $L(\cdot) \geq U(\cdot)/\varphi(\cdot)$ on $[a, b)$ and $L(x) = U(x)/\varphi(x)$. Using $L(\cdot)$, E.1 and E.2, we get $U(x) \geq \mathbb{E}_x[e^{-A_t \wedge \tau_a \wedge \tau_{r_n}} U(X_{t \wedge \tau_a \wedge \tau_{r_n}})]$, $t \geq 1$ for every large $n \geq 1$. Fatou's Lemma and the lower semi-continuity of $U(\cdot)$ imply $U(x) \geq \mathbb{E}_x[e^{-A_t \wedge \tau_a} U(X_{t \wedge \tau_a})] = \mathbb{E}_x[e^{-A_t} U(X_t) 1_{\{t \leq \tau_a\}}] + \mathbb{E}_x[e^{-A_{\tau_a}} U(X_{\tau_a}) 1_{\{t > \tau_a\}}] \equiv I + II$ for every $t \geq 0$. By E.1, $\{e^{-A_t} \psi(X_t); t \geq 0\}$ is a nonnegative continuous supermartingale. Optional sampling and E.4 imply that $II = [U(a)/\psi(a)] \cdot \mathbb{E}_x[e^{-A_{\tau_a}} \psi(X_{\tau_a}) 1_{\{t > \tau_a\}}] \geq [U(a)/\psi(a)] \cdot \mathbb{E}_x[e^{-A_t} \psi(X_t) 1_{\{t > \tau_a\}}] \geq \mathbb{E}_x[e^{-A_t} U(X_t) 1_{\{t > \tau_a\}}]$, $t \geq 0$. Finally, $U(x) \geq I + II \geq \mathbb{E}_x[e^{-A_t} U(X_t)]$, $t \geq 0$. \square

Proof of Proposition 3.3. The expression (1.2) trivially implies that $V(\cdot)$ is a nonnegative majorant of $h(\cdot)$. To prove that it is also excessive for X killed at rate $A(\cdot)$, fix any compact $[l, r] \subseteq \mathcal{I}$; and denote by σ_l^ε and σ_r^ε the stopping times such that $\mathbb{E}_y[e^{-A_{\sigma_y^\varepsilon}} h(X_{\sigma_y^\varepsilon}) 1_{\{\sigma_y^\varepsilon < \infty\}}] > V(y) - \varepsilon$, $y \in \{l, r\}$, $\varepsilon > 0$; and introduce the stopping time

$$\tau^\varepsilon \triangleq \begin{cases} \tau_l + \sigma_l^\varepsilon \circ \theta_{\tau_l}, & \text{on } \{\tau_l < \tau_r\}, \\ \tau_r + \sigma_r^\varepsilon \circ \theta_{\tau_r}, & \text{on } \{\tau_l > \tau_r\}. \end{cases}$$

We have $V(x) \geq \mathbb{E}_x[e^{-A_{\tau^\varepsilon}} h(X_{\tau^\varepsilon}) 1_{\{\tau^\varepsilon < \infty\}}] \geq V(l) \mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}] + V(r) \mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}] - \varepsilon$ for every $x \in [l, r]$. By letting $\varepsilon \downarrow 0$, we obtain $V(x) \geq V(l) \mathbb{E}_x[e^{-A_{\tau_l}} 1_{\{\tau_l < \tau_r\}}] + V(r) \mathbb{E}_x[e^{-A_{\tau_r}} 1_{\{\tau_l > \tau_r\}}]$, $x \in [l, r]$, which leads to (i) and (ii) of Proposition 3.1 because of (2.4). Next, suppose a is reflecting, and define the stopping times

$$\rho^\varepsilon \triangleq \begin{cases} \tau_r + \sigma_r^\varepsilon \circ \theta_{\tau_r}, & \text{on } \{\tau_r < \infty\}, \\ \infty, & \text{on } \{\tau_r = \infty\}. \end{cases}$$

Similarly, $V(x) \geq \mathbb{E}_x[e^{-A_{\rho^\varepsilon}} h(X_{\rho^\varepsilon}) 1_{\{\rho^\varepsilon < \infty\}}] \geq V(r) \psi(x)/\psi(r) - \varepsilon$ for every $x \in [a, r]$. Letting $\varepsilon \downarrow 0$ proves (iv). The proof of (iii) is similar, and Proposition 3.1 implies that $V(\cdot)$ is excessive for X killed at rate $A(\cdot)$. Let $U : \mathcal{I} \mapsto [0, \infty)$ be another excessive majorant of $h(\cdot)$. Then $U(x) \geq \mathbb{E}_x[e^{-A_\tau} U(X_\tau) 1_{\{\tau < \infty\}}] \geq \mathbb{E}_x[e^{-A_\tau} h(X_\tau) 1_{\{\tau < \infty\}}]$, $x \in \mathcal{I}$ for every $\tau \in \mathcal{S}$. By taking supremum over $\tau \in \mathcal{S}$, we obtain $U(x) \geq V(x)$, $x \in \mathcal{I}$. \square

Proof of Proposition 3.4. Define $\tilde{V}(x) \triangleq \varphi(x)W(F(x))$, $x \in \mathcal{I}$. Then $\tilde{V}(\cdot)$ is a nonnegative majorant of $h(\cdot)$ such that $\tilde{V}(\cdot)/\varphi(\cdot)$ is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r \equiv \mathcal{I}$. The latter also implies that

$\tilde{V}(\cdot)/\psi(\cdot)$ is G -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$. Moreover, $\tilde{V}(\cdot)/\varphi(\cdot)$ is nondecreasing if b is reflecting. Therefore, $\tilde{V}(\cdot) \geq V(\cdot)$ by [Proposition 3.3](#). Next, let $\tilde{W}(y) \triangleq (V/\varphi) \circ F^{-1}(y)$, $y \in F(\mathcal{I})$. Then $\tilde{W}(\cdot)$ is a nonnegative concave majorant of $H(\cdot)$ on $F(\mathcal{I})$, which is also nondecreasing if b is reflecting. Therefore, $\tilde{W}(\cdot) \geq W(\cdot)$ on $F(\mathcal{I})$, and $\tilde{V}(x) = \varphi(x)W(F(x)) \leq \varphi(x)\tilde{W}(F(x)) = V(x)$, $x \in \mathcal{I}$. Hence $\tilde{V}(\cdot) = V(\cdot)$. \square

3.1. The properties of the value function $V(\cdot)$. We shall need next three propositions to study the existence of optimal stopping rules in [Section 3.2](#). They are also of interest on their own. [Propositions 3.5](#) and [3.6](#) show that the value function $V(\cdot)$ inherits some of its important properties from the reward function $h(\cdot)$. [Proposition 3.7](#) gives further some geometric insight. Note that excessive functions of a linear diffusion are in general discontinuous at absorbing boundaries.

Proposition 3.5. If a (resp., b) is natural, then $\lim_{x \downarrow a} V(x)/\varphi(x) = \ell_a$ (resp., $\lim_{x \uparrow b} V(x)/\psi(x) = \ell_b$), where ℓ_a and ℓ_b are the quantities defined by [\(3.6\)](#).

Proof. It is similar to the proof of [Proposition 5.4](#) in [Dayanik and Karatzas \[9\]](#). \square

Proposition 3.6. The value function $V(\cdot)$ is continuous on $\mathcal{I} \setminus (\mathcal{B}_{\text{Abs}}^\ell \cup \mathcal{B}_{\text{Abs}}^r)$, and $V(a) \leq \underline{\lim}_{x \downarrow a} V(x)$ if a is absorbing, and $V(b) \leq \underline{\lim}_{x \uparrow b} V(x)$, if b is absorbing. If $h(\cdot)$ is continuous on \mathcal{I} , then $V(\cdot)$ is also continuous on \mathcal{I} .

Proof. Since $V(\cdot)/\varphi(\cdot)$ ($V(\cdot)/\psi(\cdot)$, respectively) is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$ (G -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^\ell$, respectively), and $\varphi(\cdot)$ and $\psi(\cdot)$ are continuous on \mathcal{I} , $V(\cdot)$ is continuous on $\text{int}(\mathcal{I})$, and $V(a) \leq \underline{\lim}_{x \downarrow a} V(x)$ if $a \in \mathcal{I}$, and $V(b) \leq \underline{\lim}_{x \uparrow b} V(x)$ if $b \in \mathcal{I}$. If $a \in \mathcal{I}$ is reflecting, then $V(a)/\psi(a) \geq V(x)/\psi(x)$ for every $x \geq a$, and $V(a) \geq \overline{\lim}_{x \downarrow a} V(x)$. If $b \in \mathcal{I}$ is reflecting, then $V(b)/\varphi(b) \geq V(x)/\varphi(x)$ for every $x \leq b$, and $V(b) \geq \overline{\lim}_{x \uparrow b} V(x)$. Hence, $V(\cdot)$ is continuous on $\mathcal{I} \setminus (\mathcal{B}_{\text{Abs}}^\ell \cup \mathcal{B}_{\text{Abs}}^r)$. If $h(\cdot)$ is continuous on \mathcal{I} , and the left-boundary $a \in \mathcal{I}$ is absorbing, one can show $V(a) \geq \overline{\lim}_{x \downarrow a} V(x)$ as in [Dynkin and Yushkevich \[14, pages 112-119\]](#). \square

In the remainder of this section, we shall assume that the reward function $h(\cdot)$ is continuous on \mathcal{I} . Therefore, the value function $V(\cdot)$ will be continuous on \mathcal{I} by [Proposition 3.6](#). Define

$$\mathbf{\Gamma} \triangleq \{x \in \mathcal{I} : V(x) = h(x)\} \quad \text{and} \quad \mathbf{C} \triangleq \mathcal{I} \setminus \mathbf{\Gamma} = \{x \in \mathcal{I} : V(x) > h(x)\}.$$

Since both $h(\cdot)$ and $V(\cdot)$ are continuous, \mathbf{C} is a countable union of open subintervals of \mathcal{I} .

Proposition 3.7. Suppose $(l, r) \subseteq \mathbf{C}$ for some $l, r \in \mathcal{I}$. Then

$$\frac{V(x)}{\varphi(x)} = \frac{V(l)}{\varphi(l)} \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{V(r)}{\varphi(r)} \frac{F(x) - F(l)}{F(r) - F(l)}, \quad x \in [l, r], r \notin \mathcal{B}_{\text{Abs}}^r, \quad (3.9)$$

$$\frac{V(x)}{\psi(x)} = \frac{V(l)}{\psi(l)} \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{V(r)}{\psi(r)} \frac{G(x) - G(l)}{G(r) - G(l)}, \quad x \in [l, r], l \notin \mathcal{B}_{\text{Abs}}^\ell. \quad (3.10)$$

Moreover, $V(x) = \mathbb{E}_x[e^{-A\tau_l \wedge \tau_r} V(X_{\tau_l \wedge \tau_r})]$ for every $x \in [l, r]$.

Proof. Let $\tau \triangleq \inf\{t \geq 0 : X_t \notin (l, r)\}$ be the exit time of X from the interval (l, r) . Note that $\tau = \tau_l \wedge \tau_r$ if $X_0 \in (l, r)$, and $\tau = 0$ if $X_0 \in \mathcal{I} \setminus (l, r)$. Define $L : \mathcal{I} \mapsto \mathbb{R}$ as

$$L(x) \triangleq \mathbb{E}_x[e^{-A\tau}V(X_\tau)] \\ = \begin{cases} V(l)\frac{\varphi(r)\psi(x) - \psi(r)\varphi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)} + V(r)\frac{\psi(l)\varphi(x) - \varphi(l)\psi(x)}{\varphi(r)\psi(l) - \psi(r)\varphi(l)}, & \text{if } x \in (l, r) \\ V(x), & \text{if } x \notin (l, r) \end{cases},$$

by using [Proposition 2.3](#). Because $V(\cdot)$ is excessive for X killed at rate $A(\cdot)$, we have $V(\cdot) \geq L(\cdot)$ on \mathcal{I} . On the other hand, $L(\cdot)$ is also excessive for X killed at rate $A(\cdot)$, and the reverse inequality follows, once we prove that $L(\cdot)$ majorizes $h(\cdot)$ on \mathcal{I} . We already have $L(x) = V(x) \geq h(x)$, $x \in \mathcal{I} \setminus (l, r)$. Assume contrarily that $\delta \triangleq \max_{x \in [l, r]} [h(x) - L(x)]/\varphi(x) > 0$. Then $L(x) + \delta\varphi(x)$ is an excessive *majorant* of $h(\cdot)$ on \mathcal{I} , and $L(x) + \delta\varphi(x) \geq V(x) \geq h(x)$, $x \in \mathcal{I}$. Since $L(\cdot)$ is continuous, and $h(y) - L(y) \leq 0$ for $y \in \{l, r\}$, δ is attained at some $x^* \in (l, r)$. Therefore, $h(x^*) = L(x^*) + \delta\varphi(x^*) \geq V(x^*) \geq h(x^*)$, i.e., $h(x^*) = V(x^*)$. Hence, $x^* \in \mathbf{\Gamma} \cap (l, r)$; but this contradicts with $(l, r) \subseteq \mathbf{C}$. Finally, the righthand sides of [\(3.9\)](#) and [\(3.10\)](#) are the same as $L(x)/\varphi(x)$ and $L(x)/\psi(x)$, respectively, for $x \in [l, r]$. \square

Proposition 3.8 (The smooth-fit principle). Let $x \in \text{int}(\mathcal{I})$. Suppose that the left- and right-derivatives $D_F^\pm h(x)$ of $h(\cdot)$ with respect to $F(\cdot)$ at x exist, and $h(x) = V(x)$. Then

$$D_F^-(h/\varphi)(x) \geq D_F^-(V/\varphi)(x) \geq D_F^+(V/\varphi)(x) \geq D_F^+(h/\varphi)(x). \quad (3.11)$$

If $D_F h(x)$ and $D_F \varphi(x)$ also exist, then $D_F h(x) = D_F V(x)$.

Proof. By the definition of $F(\cdot)$ and [Corollary 2.1](#), $D_F^\pm \varphi(\cdot)$ exist everywhere on $\text{int}(\mathcal{I})$. Therefore, $D_F^\pm(h/\varphi)(x)$ exist. For every $a < l < x < r < b$, we have

$$\frac{(h/\varphi)(x) - (h/\varphi)(l)}{F(x) - F(l)} \geq \frac{(V/\varphi)(x) - (V/\varphi)(l)}{F(x) - F(l)} \geq \frac{(V/\varphi)(r) - (V/\varphi)(x)}{F(r) - F(x)} \geq \frac{(h/\varphi)(r) - (h/\varphi)(x)}{F(r) - F(x)},$$

since $V(\cdot)$ majorizes $h(\cdot)$ on \mathcal{I} , and $V(\cdot)/\varphi(\cdot)$ is F -concave on $\mathcal{I} \setminus \mathcal{B}_{\text{Abs}}^r$. As l and r tend to x , [\(3.11\)](#) follows. If $D_F(h/\varphi)(x)$ also exists, then $D_F(V/\varphi)(x)$ exists and is equal to $D_F(h/\varphi)(x)$. \square

3.2. Existence of an optimal stopping time. Let us define the stopping time

$$\bar{\tau} \triangleq \inf\{t \geq 0 : X_t \in \mathbf{\Gamma}\}, \quad \text{and} \quad U(x) \triangleq \mathbb{E}_x[e^{-A\bar{\tau}}h(X_{\bar{\tau}})1_{\{\bar{\tau} < \infty\}}], \quad x \in \mathcal{I}. \quad (3.12)$$

If there is an optimal stopping time $\tau^* \in \mathcal{S}$, then it can be shown that $\bar{\tau}$ is also optimal; see, e.g., El Karoui [\[15\]](#), Karatzas and Shreve [\[25, Appendix D\]](#). Therefore, we shall investigate here the necessary and sufficient conditions for the optimality of $\bar{\tau}$. Since $V(\cdot) \geq U(\cdot)$, the stopping time $\bar{\tau}$ is optimal if and only if $U(x) \geq V(x)$ for every $x \in \mathcal{I}$.

Proposition 3.9. The function $U(\cdot)$ of [\(3.12\)](#) is continuous and excessive for X killed at rate $A(\cdot)$.

Proof. Observe that $U(x) = \mathbb{E}_x[e^{-A\bar{\tau}}V(X_{\bar{\tau}})1_{\{\bar{\tau} < \infty\}}]$, $x \in \mathcal{I}$. Since the value function $V(\cdot)$ is excessive for X killed at rate $A(\cdot)$, the same for $U(\cdot)$ follows from the strong additivity of $A(\cdot)$, strong Markov property of X , and the optional sampling theorem for nonnegative supermartingales. The proof of the continuity of $U(\cdot)$ on $\mathcal{I} \setminus (\mathcal{B}_{\text{Abs}}^\ell \cup \mathcal{B}_{\text{Abs}}^r)$ is the same as that of $V(\cdot)$; see [Proposition 3.6](#). If a is absorbing, then $U(a) = V(a)$. Since $V(\cdot)$ is excessive for X killed at rate $A(\cdot)$ and continuous, we have $U(a) = V(a) = \lim_{x \downarrow a} V(x) \geq \overline{\lim}_{x \downarrow a} \mathbb{E}_x[e^{-A\bar{\tau}}V(X_{\bar{\tau}})1_{\{\bar{\tau} < \infty\}}] = \overline{\lim}_{x \downarrow a} U(x)$. Finally, excessivity of $U(\cdot)$ implies $\underline{\lim}_{x \downarrow a} U(x) \geq U(a)$. The proof of the continuity of $U(\cdot)$ at an absorbing right-boundary point b is similar. \square

Propositions [3.9](#) and [3.3](#) imply that $U(\cdot) \geq V(\cdot)$ if and only if $U(\cdot)$ majorizes the reward function $h(\cdot)$ on \mathcal{I} . By [Proposition 3.10](#), the latter is always true, if either (i) \mathcal{I} is closed and bounded, or (ii) for every natural boundary of X , the corresponding limit ℓ_a or ℓ_b of [\(3.6\)](#) is zero.

Proposition 3.10. Suppose that the quantity ℓ_a (resp., ℓ_b) of [\(3.6\)](#) is zero, if the left-boundary point a (resp., the right-boundary point b) is natural. Then the stopping time $\bar{\tau}$ of [\(3.12\)](#) is optimal.

What happens if ℓ_a or ℓ_b is nonzero? Suppose that the left-boundary point a is natural and $\ell_a > 0$. Suppose also that $(a, r) \subseteq \mathbf{C}$ for some $r \in \text{int}(\mathcal{I})$. Then $\mathbb{P}_x(\bar{\tau} > \tau_r) = 1$, and $U(x) = \mathbb{E}_x[e^{-A\bar{\tau}}V(X_{\bar{\tau}})1_{\{\bar{\tau} < \infty\}}] \leq \mathbb{E}_x[e^{-A\bar{\tau}}V(X_{\bar{\tau}})1_{\{\tau_r < \bar{\tau}\}}] \leq \mathbb{E}_x[e^{-A\tau_r}V(X_{\tau_r})1_{\{\tau_r < \bar{\tau}\}}] \leq V(r)\psi(x)/\psi(r)$, $x \in (a, r)$. Thus, $\overline{\lim}_{x \downarrow a} U(x)/\varphi(x) \leq \lim_{x \downarrow a} [V(r)/\psi(r)] \cdot [\psi(x)/\varphi(x)] = 0$ since a is natural. Thus, $\lim_{x \downarrow a} U(x)/\varphi(x) = 0 < \ell_a = \lim_{x \downarrow a} V(x)/\varphi(x)$ by [Proposition 3.5](#), and $U(x) \neq V(x)$ for some $x \in \mathcal{I}$. Therefore, $\bar{\tau}$ cannot be an optimal stopping time. Similarly, if the right-boundary point b is natural with $\ell_b > 0$, and $(l, b) \subseteq \mathbf{C}$ for some $l \in \text{int}(\mathcal{I})$, then $\bar{\tau}$ cannot be an optimal stopping time. Our next result, however, shows that these are the only cases without an optimal stopping time.

Proposition 3.11. Suppose that at least one of the boundary points is natural with nonzero limit ℓ_a or ℓ_b . The stopping time $\bar{\tau}$ of [\(3.12\)](#) is optimal, if and only if

- (i) $\{r \in \mathcal{I} : (a, r) \subseteq \mathbf{C}\} = \emptyset$ if the left-boundary point a is natural with $\ell_a > 0$, and
- (ii) $\{l \in \mathcal{I} : (l, b) \subseteq \mathbf{C}\} = \emptyset$ if the right-boundary point b is natural with $\ell_b > 0$.

Proof of Proposition 3.10. It is enough to prove $U(\cdot) \geq h(\cdot)$. Assume on the contrary that

$$\delta \triangleq \sup_{x \in \mathcal{I}} \frac{h(x) - U(x)}{\psi(x) + \varphi(x)} > 0. \quad (3.13)$$

We claim that δ is attained at some $x^* \in \mathcal{I}$. This is clear from the continuity of $h(\cdot)$, $U(\cdot)$, $\psi(\cdot)$ and $\varphi(\cdot)$ on \mathcal{I} , if both boundary points are contained in \mathcal{I} . If a is natural, then $\overline{\lim}_{x \downarrow a} [h(x) - U(x)]/[\psi(x) + \varphi(x)] \leq \overline{\lim}_{x \downarrow a} h^+(x)/\varphi(x) = \ell_a = 0$. If b is natural, then $\overline{\lim}_{x \uparrow b} [h(x) - U(x)]/[\psi(x) + \varphi(x)] \leq \overline{\lim}_{x \uparrow b} h^+(x)/\psi(x) = \ell_b = 0$. Therefore, the function $(h(x) - U(x))/(\psi(x) + \varphi(x))$ vanishes, or becomes negative near natural boundaries, and we can still find some $x^* \in \mathcal{I}$ where δ is attained.

Define $\overline{U}(x) \triangleq U(x) + \delta[\psi(x) + \varphi(x)]$, $x \in \mathcal{I}$. According to [Proposition 3.1](#), $\psi(\cdot)$ and $\varphi(\cdot)$ are excessive for X killed at rate $A(\cdot)$. Therefore, $\overline{U}(\cdot)$ is an excessive majorant of $h(\cdot)$ on \mathcal{I} . By [Proposition 3.3](#), $\overline{U}(\cdot) \geq V(\cdot)$. In particular, $h(x^*) \leq V(x^*) \leq \overline{U}(x^*) = U(x^*) + \delta[\psi(x^*) + \varphi(x^*)] =$

$h(x^*)$. This implies $x^* \in \mathbf{\Gamma}$, and $U(x^*) = h(x^*)$; equivalently $\delta = 0$, which contradicts with (3.13). \square

Proof of the sufficiency in Proposition 3.11. Note that the same proof of Proposition 3.10 will work if we can show that δ of (3.13) would be attained in \mathcal{I} , should it be strictly positive. By means of (i) and (ii), we shall prove that $U(\cdot) \geq h(\cdot)$ in some neighborhood of each natural boundary with positive limit, ℓ_a or ℓ_b . Thus, $\{x \in \mathcal{I} : h(x) - U(x) > 0\}$ should still be contained in a closed and bounded subinterval of \mathcal{I} , and δ should be attained in \mathcal{I} , if it were positive. Suppose that the left-boundary point a is natural with $\ell_a > 0$, and there is no $r \in \mathcal{I}$ such that $(a, r) \subseteq \mathbf{C}$. Therefore, there exists a sequence $(a_n)_{n \geq 1} \subseteq \mathbf{\Gamma}$, strictly decreasing to a . For every $x < a_1$ in \mathcal{I} , there are two cases. Case I: $x \in \mathbf{\Gamma}$. Then $U(x) = V(x) \geq h(x)$. Case II: $x \in \mathbf{C}$. Because \mathbf{C} is open and $a_n < x < a_1$ for some $n \geq 1$, x is contained in a subinterval $(\alpha, \beta) \subseteq \mathbf{C}$ for some $\alpha, \beta \in \mathbf{\Gamma}$. Then Proposition 3.7 implies that $U(x) = \mathbb{E}_x[e^{-A_{\tau_\alpha \wedge \tau_\beta}} V(X_{\tau_\alpha \wedge \tau_\beta})] = V(x) \geq h(x)$. Hence, for some $a_1 \in \mathcal{I}$, $U(x) = V(x) \geq h(x)$ for every $x < a_1$ if a is natural with $\ell_a > 0$. Similarly, (ii) guarantees that, for some $b_1 \in \mathcal{I}$, $U(x) = V(x) \geq h(x)$ for every $x > b_1$ if b is natural with $\ell_b > 0$. \square

4. EXAMPLES

Examples 4.1, 4.2, and 4.4 are from Beibel and Lerche [2]. Example 4.3 is a modification of Example 4.2 and turns out to have a nontrivial two-sided optimal stopping region.

4.1. Standard Brownian motion: State-dependent discounting. Let X be a standard Brownian motion. For some constants $r > 0$ and $\alpha > 0$, consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x \left[\exp \left\{ -r \int_0^\tau X_t^2 dt \right\} (X_\tau^+)^{\alpha} 1_{\{\tau < \infty\}} \right], \quad x \in \mathbb{R}, \quad (4.1)$$

with the continuous additive functional $A(t) \triangleq r \int_0^t X_s^2 ds$, $t \geq 0$ and reward function $h(x) \triangleq (x^+)^{\alpha}$, $x \in \mathbb{R}$. First, we shall determine the functions $\psi(\cdot)$ and $\varphi(\cdot)$ of (2.2). Beibel and Lerche [2] notice that the nonnegative function

$$\phi(x) \triangleq e^{-x^2/4} \frac{2^{5/4}}{\Gamma(1/2)} \int_0^\infty e^{xt-t^2/2} \frac{1}{\sqrt{t}} dt, \quad x \in \mathbb{R}$$

satisfies $\phi''(x) = \frac{1}{4}x^2\phi(x)$;

$$\phi(0) = 1, \quad \lim_{x \rightarrow -\infty} \phi(x) = 0, \quad \lim_{x \rightarrow \infty} \phi(x) \frac{\sqrt{x}}{e^{x^2/4}} = K \quad (4.2)$$

for some constant $K > 0$, and the function $u(x) \triangleq \phi((8r)^{1/4}x)$, $x \in \mathbb{R}$, solves $\frac{1}{2}u''(x) = rx^2u(x)$. Since $\phi(\cdot)$ is also bounded on $(-\infty, y]$ for every $y \in \mathbb{R}$, the process $\{e^{-A_t}u(X_t); t \geq 0\}$ is a positive local martingale; by optional sampling, $\mathbb{E}_0[e^{-A_{\tau_y}}] = 1/u(y)$ and $\mathbb{E}_x[e^{-A_{\tau_0}}] = u(x)$, $x \leq 0 \leq y$. Therefore, $\psi(x) = u(x)$, $x \in \mathbb{R}$ if $c = 0$ in (2.2).

On the other hand, the positive function $v(x) \triangleq u(-x)$, $x \in \mathbb{R}$ satisfies $\frac{1}{2}v''(x) = rx^2v(x)$, $x \in \mathbb{R}$. Since $\lim_{x \rightarrow \infty} v(x) = \lim_{x \rightarrow -\infty} u(x) = 0$ by (4.2), it is bounded on every $[y, \infty)$, $y \in \mathbb{R}$; therefore, $\{e^{-A_t}v(X_t); t \geq 0\}$ is a positive local martingale. By optional sampling, we have $\mathbb{E}_0[e^{-A_{\tau_x}}] =$

$1/u(x)$ and $\mathbb{E}_y[e^{-A\tau_0}] = u(y)$, $x \leq 0 \leq y$. Therefore, $\varphi(x) = v(x) = \phi(-(8r)^{1/4}x)$, $x \in \mathbb{R}$ if $c = 0$ in (2.2).

The boundaries of the state-space $\mathcal{I} = (-\infty, \infty)$ are natural. By (4.2), we have

$$\ell_{-\infty} \triangleq \overline{\lim}_{x \rightarrow -\infty} \frac{h^+(x)}{\varphi(x)} = 0, \quad \ell_{\infty} \triangleq \overline{\lim}_{x \rightarrow \infty} \frac{h^+(x)}{\psi(x)} = K \cdot \lim_{x \rightarrow \infty} x^\alpha \frac{\sqrt{(8r)^{1/4}x}}{e^{(8r)^{1/2}x^2/4}} = 0.$$

By Propositions 3.2 and 3.10, the value function $V(\cdot)$ of (4.1) is finite, and the stopping time $\bar{\tau}$

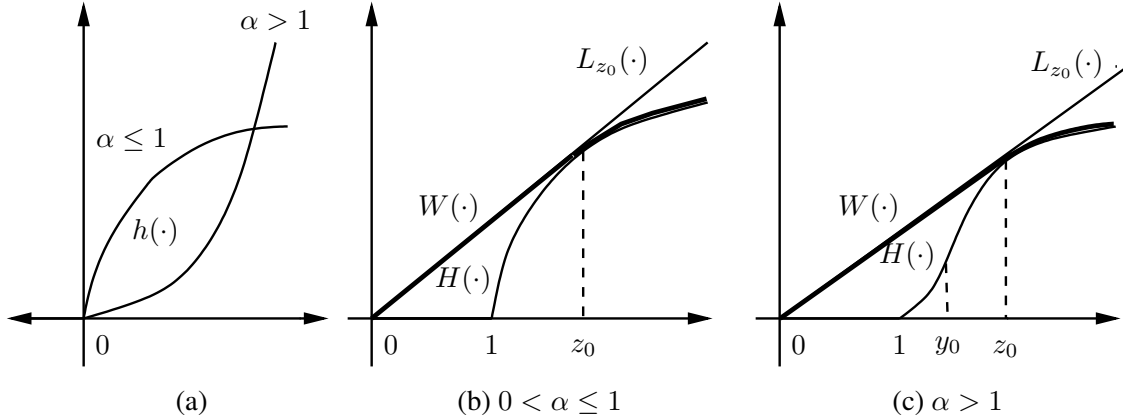


FIGURE 1. (**Brownian Motion: State-dependent discounting**) The sketch of (a) the reward function $h(\cdot)$, the function $H(\cdot)$, $W(\cdot)$ if (b) $0 < \alpha \leq 1$, and (c) $\alpha > 1$.

of (3.12) is optimal. By Proposition 3.4, we have $V(x) = \varphi(x)W(F(x))$, $x \in \mathbb{R}$, where $F(\cdot) = \psi(\cdot)/\varphi(\cdot)$, and $W(\cdot)$ is the smallest nonnegative concave majorant of $H(y) \triangleq (h/\varphi) \circ F^{-1}(y)$, $y \in [0, \infty)$. The function $H(\cdot)$ vanishes on $[0, 1]$ and is twice-differentiable everywhere on $(0, \infty)$ except $y = 1$. If $0 < \alpha \leq 1$, then it is strictly concave on $[1, \infty)$. If $\alpha > 1$, then it is convex on $[1, y_0]$ and concave on $[y_0, \infty)$ with $y_0 \triangleq F([\alpha(\alpha - 1)/2r]^{1/4}) > 1$ (cf. Figure 1(b,c)). In both cases, there is unique $z_0 > 1$ such that the line $L_{z_0}(\cdot)$, tangent to $H(\cdot)$ at z_0 , passes through the origin. This point is the unique solution $z_0 > 1$ of $H(z_0)/z_0 = H'(z_0)$. Therefore,

$$W(y) = \begin{cases} L_{z_0}(y), & 0 \leq y \leq z_0 \\ H(y), & y \geq z_0 \end{cases} \quad \text{and} \quad V(x) = \begin{cases} h(x_0) \frac{F(x)}{F(x_0)}, & x \leq x_0 \\ h(x), & x > x_0 \end{cases},$$

where $x_0 \triangleq F^{-1}(z_0) > 0$ is unique solution of $x_0\psi'(x_0) = \alpha\psi(x_0)$. The optimal stopping region is $\Gamma \triangleq \{x \in \mathbb{R} : V(x) = h(x)\} = [x_0, \infty)$, and the stopping time $\bar{\tau} = \inf\{t \geq 0 : X_t \geq x_0\}$ is optimal.

4.2. Brownian motion: Discounting with respect to local time. Let X be a standard Brownian motion, and L be its local time at zero, which is a continuous additive functional of X . Consider the optimal stopping problem

$$V(x) = \sup_{\tau \geq 0} \mathbb{E}_x [e^{-rL_\tau} (X_\tau^+)^{\alpha} \mathbf{1}_{\{\tau < \infty\}}], \quad x \in \mathbb{R}$$

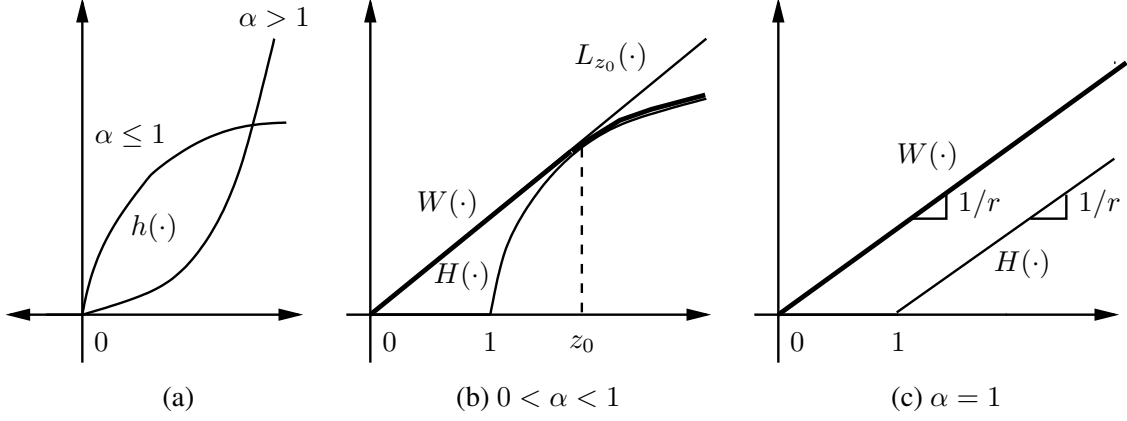


FIGURE 2. (**Brownian motion: Discounting with respect to the local time at zero**) The sketch of (a) the reward function $h(\cdot)$, and the functions $H(\cdot)$ and $W(\cdot)$ when (b) $0 < \alpha < 1$, and (c) $\alpha = 1$.

with the reward function $h(x) = (x^+)^{\alpha}$, $x \in \mathbb{R}$, where $r > 0$ and $\alpha > 0$ are constant. Because L increases only when X hits 0, we have $\mathbb{E}_x[e^{-rL\tau_0}] = 1$, and we know from Borodin and Salminen [5, p. 199] that $E_0[e^{-rL\tau_y}] = 1/(1 + r|y|)$ for every $y \in \mathbb{R}$. Therefore,

$$\psi(x) = \begin{cases} 1, & x \leq 0 \\ 1 + rx, & x > 0 \end{cases} \quad \text{and} \quad \varphi(x) = \begin{cases} 1 - rx, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

if we take $c = 0$ in (2.2). Both boundaries of the state-space $\mathcal{I} = (-\infty, \infty)$ are natural. Since

$$\ell_{-\infty} \triangleq \overline{\lim}_{x \rightarrow -\infty} \frac{h^+(x)}{\varphi(x)} = 0, \quad \text{and} \quad \ell_{\infty} \triangleq \overline{\lim}_{x \rightarrow \infty} \frac{h^+(x)}{\psi(x)} = \begin{cases} \infty, & \alpha > 1 \\ 1/r, & \alpha = 1 \\ 0, & \alpha < 1 \end{cases},$$

Propositions 3.2 and 3.10 imply that

$$\left. \begin{cases} V \equiv \infty, & \text{if } \alpha > 1 \\ V < \infty, \text{ and an optimal stopping time may not exist,} & \text{if } \alpha = 1 \\ V < \infty, \text{ and } \bar{\tau} \text{ of (3.12) is optimal,} & \text{if } 0 < \alpha < 1 \end{cases} \right\}.$$

(By using Proposition 3.10, we shall show below that no optimal stopping rule exists when $\alpha = 1$). In the remainder of this subsection, we shall assume $0 < \alpha \leq 1$. Then the value function is $V(x) = \psi(x)W(F(x))$, where $F(x) = \psi(x)/\varphi(x) = (1 - rx)^{-1}1_{(-\infty, 0]}(x) + (1 + rx)1_{(0, \infty)}(x)$, and $W(\cdot)$ is the smallest nonnegative concave majorant of $H(y) \triangleq (h/\varphi) \circ F^{-1}(y)$, $y \in (0, \infty)$. The function $H(\cdot)$ vanishes on $(-\infty, 1]$ and is twice-differentiable everywhere on $(1, \infty)$, strictly increasing and concave on $[1, \infty)$ (cf. Figure 2).

Suppose $0 < \alpha < 1$. Then $H(\cdot)$ is strictly concave on $[1, \infty)$, and there exists a unique $z_0 > 1$ such that the line $L_{z_0}(\cdot)$, tangent to $H(\cdot)$ at z_0 , passes through the origin. The point z_0 is the unique solution of the equation $H(z_0)/z_0 = H'(z_0)$. The smallest nonnegative concave function

$W(\cdot)$ of $H(\cdot)$ and the value function become

$$W(y) = \begin{cases} \frac{H(z_0)}{z_0} y, & 0 \leq y \leq z_0 \\ H(y), & y > z_0 \end{cases}, \quad \text{and} \quad V(x) = \begin{cases} \frac{x_0^\alpha}{1 + rx_0}, & x \leq 0 \\ x_0^\alpha \frac{1 + rx}{1 + rx_0}, & 0 < x \leq x_0 \\ x^\alpha, & x > x_0 \end{cases},$$

thanks to [Proposition 3.4](#), where $x_0 \triangleq F^{-1}(z_0) = \alpha/[r(1 - \alpha)] > 0$. The optimal stopping region is $\Gamma = [x_0, \infty)$, and $\bar{\tau} = \inf\{t \geq 0 : X_t \geq x_0\}$ is an optimal stopping time.

If $\alpha = 1$, then $H(y) = (y - 1)^+/r$, $y \geq 0$ (cf. [Figure 2\(c\)](#)). Therefore, the smallest nonnegative concave majorant of $H(\cdot)$ on $[0, \infty)$ becomes $W(y) = y/r$, $y \geq 0$, and [Proposition 3.4](#) implies $V(x) = \varphi(x)W(F(x)) = (1/r) + x^+$. Since $\mathbf{C} \triangleq \{x \in \mathbb{R} : V(x) > h(x)\} = \mathbb{R}$ and $\ell_\infty > 0$, there is no optimal stopping time by [Proposition 3.11](#).

Observe that $V(x) = V(0)$, $x \leq 0$ in all cases. This is intuitively clear since the discounting does not start before the process reaches the origin, which happens with probability one.

4.3. Continuation. Let us replace the reward function in [Example 4.2](#) with $h(x) \triangleq |x|^\beta 1_{(-\infty, 0)}(x) + x^\alpha 1_{[0, \infty)}(x)$ for some constants $0 < \beta \leq \alpha < \infty$, and consider the optimal stopping problem $V(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}_x [e^{-rL_\tau} h(X_\tau) 1_{\{\tau < \infty\}}]$, $x \in \mathbb{R}$. The functions $\psi(\cdot)$, $\varphi(\cdot)$ and $F(\cdot)$ do not change; we have

$$\ell_{-\infty} = \begin{cases} 0, & 0 < \beta < 1 \\ 1/r, & \beta = 1 \\ \infty, & \beta > 1 \end{cases} \quad \text{and} \quad \ell_\infty = \begin{cases} 0, & 0 < \alpha < 1 \\ 1/r, & \alpha = 1 \\ \infty, & \alpha > 1 \end{cases}.$$

The value function $V(\cdot)$ is infinite everywhere by [Proposition 3.2](#) when $\alpha > 1$ and/or $\beta > 1$. Otherwise, it is finite, and $V(x) = \varphi(x)W(F(x))$ by [Proposition 3.4](#), where $W(\cdot)$ is the smallest nonnegative concave majorant of $H(y) \triangleq (h/\varphi) \circ F^{-1}(y)$, $y \in [0, \infty)$.

If $0 < \beta \leq \alpha < 1$, then $H(\cdot)$ is strictly concave on $[0, 1]$ and $[1, \infty)$, strictly increasing on $[1, \infty)$, and $H(0+) = H(1) = 0$ ([Figure 3\(a\)](#)). One can therefore find two unique numbers $0 < z_1 < 1 < z_2 < \infty$ such that $H'(z_1) = [H(z_2) - H(z_1)]/[z_2 - z_1] = H'(z_2)$. If $L(\cdot)$ is the line tangent to $H(\cdot)$ at z_1 and z_2 , then $W(\cdot)$ coincides with $L(\cdot)$ on (z_1, z_2) and with $H(\cdot)$ everywhere else. If $x_i \triangleq F^{-1}(z_i)$, $i = 1, 2$, then the value function $V(\cdot)$ is given by

$$V(x) = \begin{cases} \varphi(x) \left[\frac{h(x_1)}{\varphi(x_1)} \cdot \frac{F(x_2) - F(x)}{F(x_2) - F(x_1)} + \frac{h(x_2)}{\varphi(x_2)} \cdot \frac{F(x) - F(x_1)}{F(x_2) - F(x_1)} \right], & x \in (x_1, x_2) \\ h(x), & x \notin (x_1, x_2) \end{cases}, \quad (4.3)$$

and $x_1 < 0 < x_2$ solve

$$-\frac{\beta}{r}(-x_1)^{\beta-1} + (1 - \beta)(-x_1)^\beta = \frac{(1 - rx_1)x_2^\alpha - (-x_1)^\beta}{(1 + rx_2)(1 - rx_1) - 1} = \alpha x_2^{\alpha-1}.$$

The stopping time $\tau \triangleq \inf\{t \geq 0 : X_t \notin (x_1, x_2)\}$ is optimal by [Proposition 3.10](#).

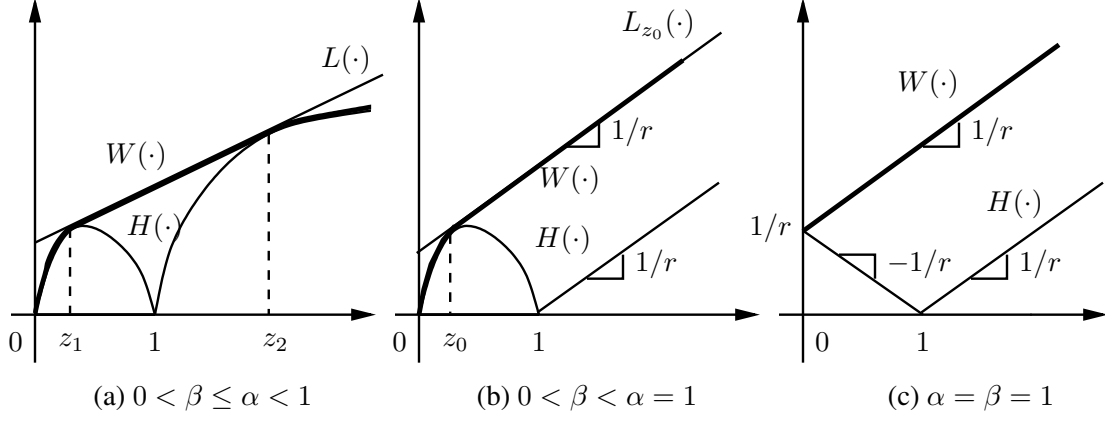


FIGURE 3. (**Example 4.2 continued**) The sketch of functions $H(\cdot)$ and $W(\cdot)$ when (a) $0 < \beta \leq \alpha < 1$, (b) $0 < \beta < \alpha = 1$, and (c) $\alpha = \beta = 1$.

If $0 < \beta < \alpha = 1$, then $H(\cdot)$ is strictly concave on $[0, 1]$, and it is a straight line with slope $1/r$ on $[1, \infty)$. Let z_0 be the unique solution of $H'(z) = 1/r$, $0 < z \leq 1$, and $L_{z_0}(\cdot)$ be the tangent line of $H(\cdot)$ at z_0 (Figure 3(b)). The function $W(\cdot)$ is equal to $H(\cdot)$ on $[0, z_0]$ and to $L_{z_0}(\cdot)$ on $[z_0, \infty)$. Moreover, $x_0 \triangleq F^{-1}(z_0) < 0$ solves $1 = |x|^{\beta-1}(\beta + r(1 - \beta)|x|)$, and we have $V(x) = |x \wedge x_0|^\beta + (x - x_0)^+/r$, $x \in \mathbb{R}$. Since $\Gamma \triangleq \{x \in \mathbb{R} : V(x) = h(x)\} = (-\infty, x_0]$ and $\ell_\infty > 0$, an optimal stopping time does not exist according to Proposition 3.11.

Finally, if $\alpha = \beta = 1$, then $H(y) = |y - 1|/r$, and $W(y) = (1/r)(1 + y)$, $y \geq 0$ (Figure 3(c)). Therefore $V(x) = \varphi(x)W(F(x)) = (2/r) + |x|$, $x \geq 0$. Since $\Gamma \triangleq \{x \in \mathbb{R} : V(x) = h(x)\} = \emptyset$ and $\ell_\mp > 0$, no stopping time is optimal according to Proposition 3.11.

4.4. Geometric Brownian motion with reflecting boundary. Let X be a geometric Brownian motion in $\mathcal{I} = [1, \infty)$ with dynamics $dX_t = X_t(\mu dt + \sigma dB_t)$, $X_0 \geq 1$ for constant $\mu, \sigma \in \mathbb{R}$. Assume that the left-boundary 1 of the state-space \mathcal{I} is instantaneously reflecting. Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x \left[e^{-\beta\tau} X_\tau^\alpha 1_{\{\tau < \infty\}} \right], \quad x \geq 1$$

with the reward function $h(x) = x^\alpha$, $x \geq 1$, where $\alpha \geq 0$ and $\beta \geq 0$ are constant. Note that $A_t = \beta t$, $t \geq 0$ in (1.2).

Let $u : \mathcal{I} \rightarrow \mathbb{R}$ (resp., $v : \mathcal{I} \rightarrow \mathbb{R}$) be a nondecreasing (resp., nonincreasing) solution of $\mathcal{A}u = \beta u$, $u'(1) = 0$ (resp., $\mathcal{A}v = \beta v$, $v(\infty) = 0$), where $\mathcal{A}u(x) \triangleq (1/2)\sigma^2 x^2 u''(x) + \mu x u'(x)$ is the infinitesimal generator of X . Then $\psi(\cdot)$ and $\varphi(\cdot)$ of (2.2) are positive multiples of $u(\cdot)$ and $v(\cdot)$, respectively; see also Borodin and Salminen [5, II.1.7-10].

The functions $w_1(x) = x^{\eta_1}$ and $w_2(x) = x^{\eta_2}$ are decreasing and increasing solutions of $\mathcal{A}w = \beta w$, respectively, where $\eta_1 < 0 < \eta_2$ are the roots of $\sigma^2 m^2 - (\sigma^2 - 2\mu)m - 2\beta = 0$. Then the functions $u(x) \triangleq w_2(x) - [w_2'(1)/w_1'(1)]w_1(x) = x^{\eta_2} - (\eta_2/\eta_1)x^{\eta_1}$ and $v(x) \triangleq w_1(x)$, $x \geq 1$ have the desired boundary behaviours: $u'(0) = v(\infty) = 0$. We shall set

$$\psi(x) \triangleq -\eta_1 u(x) = \eta_2 x^{\eta_1} - \eta_1 x^{\eta_2}, \quad \text{and} \quad \varphi(x) \triangleq v(x) = x^{\eta_1}, \quad x \geq 1$$

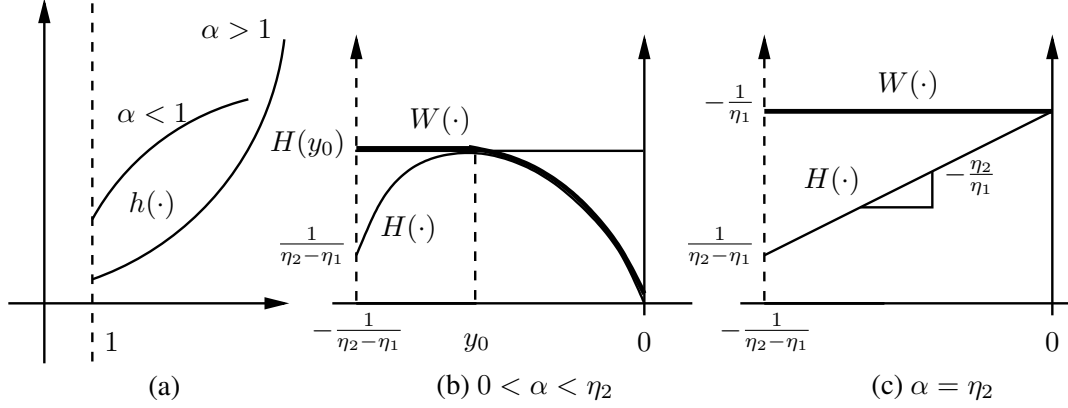


FIGURE 4. **(Geometric Brownian motion with reflecting boundary)** The sketch of (a) the reward function $h(\cdot)$, and the functions $H(\cdot)$ and $W(\cdot)$ when (b) $0 < \alpha < \eta_2$, and (c) $\alpha = \eta_2$. In (b), $H(\cdot)$ is strictly concave on $[-(\eta_2 - \eta_1)^{-1}, 0]$ with unique maximum at $-(\eta_2 - \eta_1)^{-1} < y_0 < G(+\infty)$. Therefore, $W(\cdot)$ remains constant at level $H(y_0)$ on $[-(\eta_2 - \eta_1)^{-1}, y_0]$, and coincides with decreasing and concave $H(\cdot)$ on $[y_0, 0]$. In (c), $H(\cdot)$ is an increasing straight line. Since $W(\cdot)$ has to be nonincreasing, it is the flat line touching $H(\cdot)$ at $y = 0$.

for convenience. The right-boundary ∞ of the state-space \mathcal{I} is natural, and

$$\ell_\infty \triangleq \overline{\lim}_{x \rightarrow \infty} \frac{h^+(x)}{\psi(x)} = \lim_{x \rightarrow \infty} \frac{x^\alpha}{\eta_2 x^{\eta_1} - \eta_1 x^{\eta_2}} = \begin{cases} \infty, & \alpha > \eta_2 \\ -\eta_1, & \alpha = \eta_2 \\ 0, & \alpha < \eta_2 \end{cases}.$$

Therefore, Propositions 3.2 and 3.10 imply that

$$\begin{cases} V \equiv \infty, & \alpha > \eta_2 \\ V < \infty, \text{ and an optimal stopping time may not exist,} & \alpha = \eta_2 \\ V < \infty, \text{ and } \bar{\tau} \text{ of (3.12) is an optimal stopping time,} & \alpha < \eta_2 \end{cases}.$$

(Using Proposition 3.11, we prove below that there is no optimal stopping time when $\alpha = \eta_2$.) Suppose that $0 < \alpha \leq \eta_2$. If we define $G(x) \triangleq -\varphi(x)/\psi(x)$, $x \geq 1$ and $H(y) \triangleq (h/\psi) \circ G^{-1}(y)$, $y \in [G(1), G(\infty)]$ with $G(1) = -1/(\eta_2 - \eta_1)$, $G(\infty) = 0$, and $W(\cdot)$ is the smallest nonnegative, nonincreasing, concave majorant of $H(\cdot)$ on $[G(1), G(\infty)]$, then $V(x) = \psi(x)W(G(x))$, $x \geq 1$ by Remark 3.4.

If $0 < \alpha < \eta_2$, then it is increasing on $[G(1), y_0]$, decreasing on $[y_0, G(\infty)]$ and concave on $[G(1), G(\infty)]$, where $y_0 \triangleq G(x_0)$, and $x_0 \triangleq \{\eta_2(\alpha - \eta_1)/[\eta_1(\alpha - \eta_2)]\}^{1/(\eta_2 - \eta_1)}$; see Figure 4(b). By Remark 3.4 and Proposition 3.10, we have

$$W(y) = \begin{cases} H(y_0), & G(1) \leq y < y_0 \\ H(y), & y_0 \leq y \leq G(\infty) \end{cases}, \quad V(x) = \begin{cases} x_0^\alpha \frac{\eta_2 x^{\eta_1} - \eta_1 x^{\eta_2}}{\eta_2 x_0^{\eta_1} - \eta_1 x_0^{\eta_2}}, & 1 \leq x < x_0 \\ x^\alpha, & x \geq x_0 \end{cases};$$

the optimal stopping region is $\Gamma \triangleq \{x \geq 1 : V(x) = h(x)\} = [x_0, \infty)$, and the stopping time $\bar{\tau} = \inf\{t \geq 0 : X_t \geq x_0\}$ is optimal. If $\alpha = \eta_2$, then $H(\cdot)$ is a straight line with positive slope $-\eta_2/\eta_1$, see Figure 4(c); therefore, $W \equiv -1/\eta_1$ and $V(x) = \psi(x)W(G(x)) = x^{\eta_2} - (\eta_2/\eta_1)x^{\eta_1}$, $x \geq 1$. Since $\ell_\infty > 0$ and $\mathbf{C} \triangleq \{x \geq 1 : V(x) > h(x)\} = [1, \infty)$, there is no optimal stopping rule in this case by [Proposition 3.11](#).

Beibel and Lerche [2] discuss a special case, where $\mu < 0$, $\beta \triangleq r + \mu$ and $\alpha \triangleq 1$, in order to reproduce the price of the Russian option and the optimal exercise policy.

5. THE CONNECTION WITH DYNKIN'S CONCAVE CHARACTERIZATION OF EXCESSIVE FUNCTIONS

In this last section, we would like to recall Dynkin's concave characterizations of excessive functions and show their difference from ours. E. Dynkin gives two characterizations. In the first one, a one-dimensional diffusion X killed at some time ζ is regular on a closed state space $\mathcal{I} = [a, b]$; in our setting, this means that both a and b can only be instantaneously reflecting; see [Proposition 5.1](#). In the second characterization, the end-points a and b are excluded from the state space; namely, they are both natural in our setting; see [Proposition 5.2](#). E. Dynkin did not study the cases where the behavior of the process on the left and right boundaries of the state space are different.

Proposition 5.1 (Dynkin [13, Volume II, p. 146, Theorem 15.10]). *Suppose that the process X killed at rate $A(\cdot)$ is regular on the state space $\mathcal{I} = [a, b]$. Then a function $U : [a, b] \mapsto [0, \infty)$ is excessive for this process; namely, (3.1) is satisfied, if and only if*

- (i) *the mapping $x \mapsto U(x)/\pi(x)$ is $p(\cdot)$ -concave on $x \in [a, b]$, and*
- (ii) *the boundary conditions*

$$D_p^-(U/\pi)(b) \geq -\frac{1-\alpha_2}{\alpha_2}(U/\pi)(b) \quad \text{and} \quad D_p^+(U/\pi)(a) \leq \frac{1-\alpha_1}{\alpha_1}(U/\pi)(a) \quad (5.1)$$

are satisfied; here, $p(x) \triangleq \mathbb{P}_x\{\tau_a > \tau_b \mid \tau_a \wedge \tau_b < \zeta\}$ is the intrinsic scale function, and

$$\pi(x) \triangleq \mathbb{P}_x\{\tau_a \wedge \tau_b < \zeta\}, \quad \alpha_1 \triangleq \mathbb{P}_a\{\tau_b < \zeta\} > 0, \quad \alpha_2 \triangleq \mathbb{P}_b\{\tau_a < \zeta\} > 0.$$

If Dynkin's first characterization is compared with [Proposition 3.1](#), one notices that the roles of the functions (ψ, F) and (φ, G) are now being played by (π, p) defined above. To calculate $\pi(\cdot)$ and $p(\cdot)$, one still needs the harmonic functions $\psi(\cdot)$ and $\varphi(\cdot)$, and [Lemma 2.3](#) gives

$$\begin{aligned} \pi(x) &= \mathbb{E}_x \left[\exp\{-A_{\tau_a \wedge \tau_b}\} \mathbf{1}_{\{\tau_a \wedge \tau_b < \infty\}} \right] = \frac{\psi(x)\varphi(a) - \psi(a)\varphi(x) + \psi(b)\varphi(x) - \psi(x)\varphi(b)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)} \\ p(x) &= \mathbb{E}_x \left[\exp\{-A_{\tau_b}\} \mathbf{1}_{\{\tau_a > \tau_b\}} \right] / \pi(x) = \frac{\psi(x)\varphi(a) - \psi(a)\varphi(x)}{\psi(x)\varphi(a) - \psi(a)\varphi(x) + \psi(b)\varphi(x) - \psi(x)\varphi(b)}. \end{aligned} \quad (5.2)$$

It is not obvious if simple algebra is enough to show that p -concavity of $U(\cdot)/\pi(\cdot)$ is equivalent to F -concavity of $U(\cdot)/\varphi(\cdot)$, which must be the case since both propositions characterize the same objects. Moreover, unlike the boundary conditions in (3.5), which are equivalent by [Remark 3.4](#) to

(iii) and (iv) of Proposition 3.1, boundary conditions in (5.1) are not simple, since by (2.3)

$$\alpha_1 = \mathbb{E}_a[\exp\{-A_{\tau_b}\}1_{\{\tau_b < \infty\}}] = \frac{\psi(a)}{\psi(b)} \quad \text{and} \quad \alpha_2 = \mathbb{E}_b[\exp\{-A_{\tau_a}\}1_{\{\tau_a < \infty\}}] = \frac{\varphi(b)}{\varphi(a)}$$

are strictly between 0 and 1, and depend both end-points, a and b , of the state space.

However, a direct connection can be established between our and Dynkin's characterization if one applies first a suitable h -transformation. Let us define a new probability measure by

$$\mathbb{P}_x^\varphi(A) = \frac{1}{\varphi(x)} \mathbb{E}_x [1_{\{\zeta > t\}} \varphi(X_t) 1_A], \quad A \in \mathcal{F}_t \cap \{\zeta > t\}. \quad (5.3)$$

Under \mathbb{P}_x^φ , the process X is still a diffusion killed at time ζ , and (5.3) also holds when t is replaced with an \mathbb{F} -stopping time T ; see, e.g., Chung and Walsh [7, Chapter 11], Borodin and Salminen [5, pp. 33-34]; therefore, new characteristics of the process become

$$\begin{aligned} \pi^\varphi(x) &\triangleq \mathbb{P}_x^\varphi\{\zeta > \tau_a \wedge \tau_b\} = \frac{1}{\varphi(x)} \mathbb{E}_x [1_{\{\zeta > \tau_a \wedge \tau_b\}} \varphi(X_{\tau_a \wedge \tau_b})] \\ &= \frac{1}{\varphi(x)} \mathbb{E}_x [\exp\{-A_{\tau_a \wedge \tau_b}\} \varphi(X_{\tau_a \wedge \tau_b}) 1_{\{\tau_a \wedge \tau_b < \infty\}}] = 1 \quad \text{by Lemma 2.1,} \\ p^\varphi(x) &\triangleq \mathbb{P}_x^\varphi\{\tau_a > \tau_b \mid \zeta > \tau_a \wedge \tau_b\} = \mathbb{P}_x^\varphi\{\tau_a > \tau_b, \zeta > \tau_a \wedge \tau_b\} \\ &= \frac{1}{\varphi(x)} \mathbb{E}_x [1_{\{\zeta > \tau_a \wedge \tau_b\}} \varphi(X_{\tau_a \wedge \tau_b}) 1_{\{\tau_a > \tau_b\}}] = \frac{\varphi(b)}{\varphi(x)} \mathbb{E}_x [e^{-A_{\tau_b}} 1_{\{\tau_a > \tau_b\}}] \\ &= \frac{\varphi(b)}{\varphi(x)} \cdot \frac{\psi(x)\varphi(a) - \psi(a)\varphi(x)}{\psi(b)\varphi(a) - \psi(a)\varphi(b)} = \frac{F(x) - F(a)}{F(b) - F(a)} \quad \text{by Lemma 2.3,} \\ \alpha_1^\varphi &\triangleq \mathbb{P}_a^\varphi\{\tau_b < \zeta\} = \frac{1}{\varphi(a)} \mathbb{E}_a [1_{\{\zeta > \tau_b\}} \varphi(X_{\tau_b})] = \frac{1}{\varphi(a)} \mathbb{E}_a [e^{-A_{\tau_b}} \varphi(X_{\tau_b}) 1_{\{\tau_b < \infty\}}] \\ &= \frac{\varphi(b)}{\varphi(a)} \mathbb{E}_a [e^{-A_{\tau_b}} 1_{\{\tau_b < \infty\}}] = \frac{\varphi(b)}{\varphi(a)} \cdot \frac{\psi(a)}{\psi(b)} = \frac{F(a)}{F(b)} \quad \text{by (2.3),} \\ \alpha_2^\varphi &\triangleq \mathbb{P}_b^\varphi\{\tau_a < \zeta\} = \frac{1}{\varphi(b)} \mathbb{E}_b [1_{\{\zeta > \tau_a\}} \varphi(X_{\tau_a})] = \frac{1}{\varphi(b)} \mathbb{E}_b [e^{-A_{\tau_a}} \varphi(X_{\tau_a}) 1_{\{\tau_a < \infty\}}] \\ &= \frac{\varphi(a)}{\varphi(b)} \mathbb{E}_b [e^{-A_{\tau_a}} 1_{\{\tau_a < \infty\}}] = \frac{\varphi(a)}{\varphi(b)} \cdot \frac{\varphi(b)}{\varphi(a)} = 1 \quad \text{by (2.3) again.} \end{aligned}$$

Namely, under probability measure \mathbb{P}^φ the process X is almost surely never killed on $(a, b]$; the right boundary b is reflecting, and the *intrinsic* scale function is an affine transformation of $F(\cdot) = \psi(\cdot)/\varphi(\cdot)$.

It is easy to check that a function $U(\cdot)$ is \mathbb{P} -excessive for the killed process X if and only if $U^\varphi(\cdot) \triangleq U(\cdot)/\varphi(\cdot)$ is \mathbb{P}^φ -excessive for the same process. Therefore, Proposition 5.1 implies that a function $U(\cdot)$ is \mathbb{P} -excessive for the killed process X if and only if (i) the function $U^\varphi(\cdot)/\pi^\varphi(\cdot) \equiv U(\cdot)/\varphi(\cdot)$ is concave with respect to $F(\cdot)$, and (ii) the boundary conditions

$$D_F^-(U/\varphi)(b) \geq 0 \quad \text{and} \quad D_F^+(U/\varphi)(a) \leq (U/\psi)(a) \quad (5.4)$$

are satisfied, which are obtained from (5.1) by substituting for $U(\cdot)$ the function $U^\varphi(\cdot) \equiv U(\cdot)/\varphi(\cdot)$ and for $\pi(\cdot), p(\cdot), \alpha_1, \alpha_2$ the \mathbb{P}^φ -characteristics $\pi^\varphi(\cdot) \equiv 1, p^\varphi(\cdot), \alpha_1^\varphi, \alpha_2^\varphi = 1$, calculated above explicitly. Note that the first inequalities of (3.5) and (5.4) are the same, and if we show that

the second inequalities are also identical, then this will complete by Remark 3.2 the proof of equivalence between Proposition 3.1 and Proposition 5.1 in this special case. However, since $G(\cdot) = -\varphi(\cdot)/\psi(\cdot) = -1/F(\cdot) \leq 0$, we have $D_F^+G = D_F^+(-1/F) = 1/F^2 = G^2$, $(D_F^+G)(D_G^+F) = 1$,

$$D_F^+\left(\frac{U}{\varphi}\right) = (D_F^+G)D_G^+\left(\frac{U}{\psi}F\right) = (D_F^+G)\left[D_G^+\left(\frac{U}{\psi}\right)F + \frac{U}{\psi}D_G^+F\right] = (-G)D_G^+\left(\frac{U}{\psi}\right) + \frac{U}{\psi};$$

therefore, the second inequality in (5.4) becomes $-G(a)D_G^+(U/\psi)(a) + (U/\psi)(a) \leq (U/\psi)(a)$, equivalently $D_G^+(U/\psi)(a) \leq 0$, which is the same as the second inequality in (3.5).

The concave characterization of excessive functions in terms of $\pi(\cdot)$ and $p(\cdot)$ is very illuminating, but computing these functions by using (5.2) makes this characterization less useful than that is given by Proposition 3.1.

Dynkin's second concave characterization is for one-dimensional killed diffusions X that are regular in an open interval $\mathcal{I} = (a, b)$ and is quite close to ours in Proposition 3.1. Note that Dynkin's result below does not provide information at accessible boundary points (absorbing or reflecting) and when the diffusion's behavior at left and right boundaries are different (natural, absorbing, reflecting).

Proposition 5.2 (Dynkin [13, Volume II, pp. 149, 155, Theorems 16.1, 16.4]). *Let X be a one-dimensional continuous strong Markov process regular in (a, b) , and $q_1(\cdot) > 0$ and $q_2(\cdot)$ be any two linearly independent harmonic functions. Then $q_2(\cdot)/q_1(\cdot)$ is strictly increasing.*

A nonnegative function $U(\cdot)$ is excessive for X on (a, b) ; i.e., (3.1) holds with $\tau = \inf\{t \geq 0; X_t \notin (l, r)\}$ for every $[l, r] \subset (a, b)$, if and only if $U(\cdot)/q_1(\cdot)$ is concave with respect to $q_2(\cdot)/q_1(\cdot)$.

Proposition 5.2 suggests that any pair of linearly independent positive harmonic functions $q_1(\cdot)$ and $q_2(\cdot)$ may play the roles of $\varphi(\cdot)$ and $\psi(\cdot)$ in Propositions 3.2, 3.4, 3.5, 3.8, 3.10, and 3.11. Indeed, for every such pair the function, $q_1(\cdot)/q_2(\cdot)$ is always strictly monotone, and therefore, it is also invertible. In this generality, if the boundaries can be handled in a similar fashion that we did with $\varphi(\cdot)$ and $\psi(\cdot)$, Dynkin's interesting observation can be very useful in applications to optimal stopping problems, especially when certain harmonic functions are easier to calculate than others, or the transformations $(h/\varphi) \circ F^{-1}$ and $(h/\psi) \circ G^{-1}$ are easier to calculate, plot, or analyze for their smallest nonnegative concave majorants if $\varphi(\cdot), \psi(\cdot)$ are replaced with $q_1(\cdot), q_2(\cdot)$. E. Dynkin did not investigate the important implications of excessive functions' concave characterizations for the solution of optimal stopping problems (except the beautiful treatment of standard Brownian motion case in [14]), and in this paper we tried to fill this gap.

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