AN ADAPTIVE BAYESIAN REPLACEMENT POLICY WITH MINIMAL REPAIR

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Abstract. In this study, an adaptive Bayesian decision model is developed to determine the optimal replacement age for the systems maintained according to a general age replacement policy. It is assumed that when a failure occurs, it is either *critical* with probability p or *non-critical* with probability $1-p$, independently. A maintenance policy is considered where the non-critical failures are corrected with minimal repair and the system is replaced either at the first critical failure or at age τ whichever occurs first. The aim is to find the optimal value of τ which minimizes the expected cost per unit time. Two adaptive Bayesian procedures which utilize different levels of information are proposed for sequentially updating the optimal replacement times. Posterior density/mass functions of the related variables are derived when the time to failure for the system can be expressed as a Weibull random variable. Some simulation results are also presented for illustration purposes.

1. Introduction and Preliminaries

For systems which are subject to random failures, effective maintenance policies are needed to avoid high system costs and/or low reliability. Age replacement and block replacement are two main policies employed for the maintenance of non-repairable systems and their properties are well studied. For repairable systems, several repair actions have been discussed in the literature, among which *minimal* and *imperfect* repair have received the most attention. In this paper we consider a system which can be minimally repaired. The concept of minimal repair was first introduced in the celebrated paper of Barlow and Hunter [1] and was followed by many others including Park [10], Cleroux, Dubuc and Tilquin [5], Nakagawa and Kowada [9] and Block, Borges and Savits [3]. A recent review of several replacement policies with minimal repair can be found in Beichelt [2]. Under minimal repair, it is assumed that the repair action returns the system to an operational state but the system characteristics are the same as they were just before the failure. Minimal repair is an appropriate model for complex systems such as computers, airplanes and large motors, where system failures may occur due to component failures and the system can be made operational by replacing the failed component with a new one. Most of the existing studies regarding minimal repair employ

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a classical approach which assumes that the parameters of the failure time distribution are known in advance and the aim is to find the optimal values of the decision variables. The standard approach is to minimize the long run average cost function obtained by the renewal reward theorem. Availability of precise data about the failure structure of the system which allows reliable prediction of the failure parameters is therefore a crucial issue for the classical approach. However, if the system under consideration is relatively new so that sufficient information has not accumulated yet to estimate the system parameters with a high confidence, it is more appropriate to consider a policy that adapts itself to the observed data in the course of maintenance actions.

In this study, we propose an *adaptive Bayesian* approach which incorporates the information provided by the observed performance of the system into the decision process for the future maintenance activities. The parameters of the system failure time distribution are assumed random and in the course of the system operation, the observed data is used for updating the posterior distribution of these parameters. The system considered is subject to random failures which are classified as *critical* (Type 2) or *non-critical* (Type 1). A failure can be critical with probability $0 \leq p \leq 1$, independent of the other failures. A Bayesian analysis for a system which is a special case ($p = 0$) of the one studied in this paper can be found in Mazzuchi and Soyer [8], [7]. This type of classification for the failures may be appropriate if for instance it is based on the estimated repair cost. In a more general setting, the cost of minimal repair may also depend on the age at failure, in which case the probability of a critical failure is described by $p(t)$. Although, for certain $p(t)$ functions, the results of the present paper can be extended with minor modifications, the analysis with an arbitrary function becomes intractable. An extension when p is random is discussed in Section 4. The following control policy is considered:

Control Policy: A critical failure is corrected by a replacement, whereas a non-critical failure is corrected by minimal repair. In addition, the system is replaced at age τ.

A replacement brings the system to a good as new state and a minimal repair brings to a good as old state. The cost for a minimal repair is c_m, for a planned replacement at τ is c_p and for corrective replacement at critical failures is c_r. It is assumed that $c_r > c_p > c_m$. According to the control policy, each replacement starts a renewal epoch, and hence a *cycle* is defined as the time between two consecutive system replacements. The adaptive Bayes policy proposed in this paper considers the problem in a finite horizon and the exact cost per unit time within a cycle is minimized with respect to the replacement age τ. Let Y be the time until a critical failure occurs. Then the cycle length L can be written as $L = \min(Y, \tau)$. Let N_t be the number of non-critical failures in the interval $(0, Y \wedge t]$, $t > 0$, where $a \wedge b = \min(a, b)$. Then, N_τ corresponds to the number of non-critical failures in a replacement cycle. Suppose f, F and λ denote the density, distribution and the hazard rate function of the system lifetime. If the system is observed over $(0, x_0]$ and all the failures in the system are repaired minimally, then the joint density of the times of the first n failures is given as (see Beichelt[2])

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} \lambda(x_1)\lambda(x_2)\cdots\lambda(x_{n-1}) f(x_n) & \text{if} \quad x_1 < x_2 < \ldots < x_n < x_0 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

Under the control policy given above, the distribution function G of Y is given as $G(t) = 1 - \left(\overline{F}(t)\right)^p$ and for $k = 0, 1, 2, \ldots,$ the conditional distribution of N_t given Y is

$$P\{N_t = k \mid Y = t\} = P\{N_t = k \mid Y \geq t\} = e^{-\xi(t)} \frac{(\xi(t))^k}{k!} \tag{1.2}$$

where $\xi(t) = q\Lambda(t) = q\int^t \lambda(u)du$. Given $Y = t$, N_t is a non-homogeneous Poisson Process (NHPP) with cumulative intensity $\xi(t)$. These results are utilized to derive the expected cost function and the posterior densities.

The paper is organized as follows: In Section 2, the adaptive Bayesian approach is introduced and a one-step Bayesian analysis is discussed. In Section 3, the adaptive method which uses the number of non-critical failures is introduced. In Section 4, the use of failure times and the cycle lengths for updating purposes is discussed. Numerical results and comparison of the two methods are also included in this section. Concluding remarks and future extensions are stated in Section 5.

2. Bayesian Approach

Consider the system introduced in the previous section. In the Bayesian analysis presented below, Y is assumed to have a Weibull distribution with scale parameter α and shape parameter $\beta > 1$ which indicates an increasing failure rate function (IFR). For $t, \alpha, \beta > 0$, the density and the hazard rate functions of Y are given as

$$f(t \mid \alpha, \beta) = \alpha\beta t^{\beta-1} e^{-\alpha t^\beta}; \quad \lambda(t \mid \alpha, \beta) = \alpha\beta t^{\beta-1} \tag{2.3}$$

According to the control policy, the maintenance cost per unit time, $C(\tau)$, in a cycle is

$$C(\tau) = \frac{c_m N_Y + c_r}{Y} I(Y < \tau) + \frac{c_m N_\tau + c_p}{\tau} I(Y \geq \tau) \tag{2.4}$$

where $I(\cdot)$ is the indicator function. The conditional expectation of $C(\tau)$ for given α and β is found as:

Proposition 1: The expected maintenance cost per unit time in a cycle is given as

$$\begin{aligned} E\left(C\left(\tau\right) \mid \alpha,\beta\right) &= c_m q p \alpha^2 \beta \int_0^\tau t^{2\beta-2} e^{-p\alpha t^\beta} dt + c_r p \alpha \beta \int_0^\tau t^{\beta-2} e^{-p\alpha t^\beta} dt \\ &+ \frac{c_m q \alpha \tau^\beta + c_p}{\tau} e^{-p\alpha \tau^\beta} \end{aligned} \tag{2.5}$$

In many practical situations, partial information about the main characteristics of the failure process, α and β, may be available from the past data or the experience. We assume that according to such information, α can be characterized as a continuous random variable with a gamma distribution with parameters $a > 0$ and $b > 0$, and β is a discrete random variable which takes n different values, $\beta_j > 1, j = 1, 2, \ldots, n$, with probabilities P_j. Furthermore, it is assumed that α and β are independent random variables. The unconditional expectation of the cycle cost per unit time is given below which follows from (2.5).

Proposition 2: The expected total maintenance per unit time in a cycle is given as

$$E_{\alpha,\beta}\left[C\left(\tau\right)\right] = \sum_{l=1}^{n} P_l \cdot C_l \tag{2.6}$$

where

$$\begin{aligned} C_l &= (a+1)\, a b^a c_m q p \beta_l \int_0^\tau \frac{t^{2\beta_l-2}}{\left(b + p t^{\beta_l}\right)^{a+2}} dt + a b^a c_r p \beta_l \int_0^\tau \frac{t^{\beta_l-2}}{\left(b + p t^{\beta_l}\right)^{a+1}} dt \\ &+ a b^a c_m q \frac{\tau^{\beta_l-1}}{\left(b + p\tau^{\beta_l}\right)^{a+1}} + \frac{c_p}{\tau}\left(\frac{b}{b + p\tau^{\beta_l}}\right)^a \end{aligned}$$

2.1. **One-step Bayesian Analysis:** The optimal replacement age τ^*, for the first replacement cycle can be found by minimizing (2.6) with respect to τ, which does not yield a closed form solution and requires numerical methods, for which the first order condition can easily be found. If the scale parameter β is either known or can be estimated precisely, the cost function is simplified significantly. For $\beta = \beta_o$ fixed, (2.6) is minimized at

$$\tau^* = \left[\frac{b c_p}{a\left[q\left(\beta_o - 1\right) c_m + p\beta_o\left(c_r - c_p\right)\right] - p c_p}\right]^{1/\beta_o}, \tag{2.7}$$

which can be used as a simple one-step procedure if a good estimate β_o of β is available. However, since τ^* is based on the prior information about α and β, changes in the perception of these quantities in the course of maintenance actions will not be utilized in the decision process. It is therefore desirable to modify the model parameters by using the accumulated information. We propose an adaptive Bayesian decision model which incorporates such data in the next section.

2.2. **Adaptive Bayesian Decision Model.** Consider a system which is subject to failures and which is maintained according to the control policy discussed above. Let $\mathbf{D}$ denote the information obtained during a replacement cycle. In our study $\mathbf{D}$ will refer to the number of non-critical failures or to the system failure and replacement times. For cycle i, $i = 1, 2, ...$, let τ_i be the time of the ith preventive replacement, and $P^{(i)}$ be the i'th posterior marginal probability mass function (p.m.f.), where $0i =$ correspond to the prior distributions. Also denote by $f^{(i)}(\alpha, \beta)$ and $f^{(i)}(\alpha \mid \beta)$ the i'th posterior joint density of α, β and posterior conditional density of α given β respectively. The i'th posterior density or the mass function is computed from the $(i-1)$'st posterior density or mass function by the Bayes rule after data, $\mathbf{D}^{(i)}$, has been collected during the i'th replacement cycle. More explicitly, for $i = 1, 2, 3, \ldots$; and, $j = 1, 2, \ldots, n$ we have

$$f^{(i)}(\alpha, \beta_j) \equiv f\left(\alpha, \beta_j \mid \mathbf{D}^{(i)}\right) \tag{2.8}$$

$$P_j^{(i)} \equiv P\left(\beta = \beta_j \mid \mathbf{D}^{(i)}\right) \tag{2.9}$$

$$f^{(i)}(\alpha \mid \beta_j) \equiv f\left(\alpha \mid \beta = \beta_j, \mathbf{D}^{(i)}\right) \tag{2.10}$$

The *Maintenance Cost Per Unit Time* in the s'th replacement cycle is defined as

$$\Phi\left(\tau \mid f^{(s-1)}\right) = \Phi^{(s)}(\tau) = E_{\alpha,\beta}\left[C(\tau) \mid f^{(s-1)}\right] \tag{2.11}$$

where the expectation is taken with respect to $(s-1)$st posterior joint density function of α and β. The adaptive Bayesian decision model computes the optimal replacement age, τ_1^* by minimizing (2.6) and the first system replacement takes place either at the time of the first critical failure or at time τ_1^*, whichever occurs first. During the first cycle, the data $\mathbf{D}^{(\mathbf{1})}$ is observed and the first posterior joint density function, $f^{(1)}$, of α and β is computed, from which the optimal replacement age, τ_2^* is found by minimizing $\Phi^{(2)}$ for the second replacement cycle and the process continues the same way. In the following sections two data types will be considered for the implementation of the proposed adaptive procedure.

3. Count Data on Number of Minimal Repairs

In this section, $\mathbf{D}$ corresponds to the number of minimal repairs/non-critical failures observed in a cycle and $\mathbf{D}^{(i)}$ is equivalent to N_{τ_i} for the i'th cycle. For $j = 1, 2, \ldots, n$, let k_i denote the number of non-critical failures observed in i'th cycle, set $\kappa_0 = k_0 \equiv 0$, $b_j^{(0)} \equiv b$ and define $\kappa_i = \sum_{j=1}^{i} k_j$ and $b_j^{(i)} = b_j^{(i-1)} + \tau_i^{\beta_j}$. Further notation and definitions introduced below will be needed in the sequel.

Definition: For $j = 1, 2, \ldots, n$; $i = \kappa_s, \kappa_s + 1, \ldots,$ $s = 1, 2, 3, \ldots$, and $l = k_1, k_1 + 1, k_1 + 2, \ldots$, let $R_{0j}^{(0)} = 1$, $R_{lj}^{(0)} = 0$ and define

$$r^{(s)}(a,i,j) = \frac{\Gamma(a+i)}{\Gamma(a)\, i!} \left(\frac{b_j^{(s-1)}}{b_j^{(s)}} \right)^a \left(1 - \frac{b_j^{(s-1)}}{b_j^{(s)}} \right)^i [1 + (p-1) I(i \neq k_s)] \tag{3.12}$$

$$I^{(s)}(i,j) = \sum\nolimits_{m=k_s}^{\infty} r^{(s)}(a+i, m, j) \tag{3.13}$$

$$R_{lj}^{(1)} = \frac{r^{(1)}(a,l,j)}{\sum_{i=k_1}^{\infty} r^{(1)}(a,i,j)} \tag{3.14}$$

$$R_{lj}^{(s)} = \frac{\sum_{i=\kappa_{s-1}}^{l-k_s} R_{ij}^{(s-1)} r^{(s)}(a+i, l-i, j)}{\sum_{u=\kappa_s}^{\infty} \sum_{i=\kappa_{s-1}}^{u-k_s} R_{ij}^{(s-1)} r^{(s)}(a+i, u-i, j)} \tag{3.15}$$

For $\tau > 0$ and $k = 0, 1, 2, \ldots$, the probability mass function of the number of non-critical failures in a cycle is given by:

$$P\{N_\tau = k \mid \alpha, \beta\} = q^k \frac{\left(\alpha \tau^\beta\right)^k}{k!} e^{-\alpha \tau^\beta} + q^k p \sum\nolimits_{i=k+1}^{\infty} \frac{\left(\alpha \tau^\beta\right)^i}{i!} e^{-\alpha \tau^\beta}. \tag{3.16}$$

The proofs of the following results on the posterior probability density/mass functions are done by induction and can be found in Dayanik and Gürler [6]. For $s = 1, 2, \ldots$, the unconditional probability mass function of N_{τ_s} is given as

$$P_{\alpha,\beta}\{N_{\tau_s} = k_s\} = q^{k_s} \sum\nolimits_{j=1}^{n} P_j^{(s-1)} \sum\nolimits_{l=\kappa_{s-1}}^{\infty} R_{lj}^{(s-1)} I^{(s)}(l,j)$$

and for and $j = 1, 2 \ldots, n$ the s'th posterior marginal probability mass function of β is

$$P_j^{(s)} = \frac{P_j^{(s-1)} \sum_{l=\kappa_{s-1}}^{\infty} R_{lj}^{(s-1)} I^{(s)}(l,j)}{\sum_{i=1}^{n} P_i^{(s-1)} \sum_{l=\kappa_{s-1}}^{\infty} R_{li}^{(s-1)} I^{(s)}(l,i)}. \tag{3.17}$$

Let $\gamma(\alpha \mid a, b)$ correspond to the gamma density function with shape parameter a and scale parameter b. Then, for $s = 1, 2, 3, \ldots$, the s'th posterior conditional probability density function of α given β is

$$f^{(s)}(\alpha \mid \beta_j) = \sum\nolimits_{l=\kappa_s}^{\infty} R_{lj}^{(s)} \gamma\left(\alpha \mid a+l, b_j^{(s)}\right) \tag{3.18}$$

Note that since $R_{lj}^{(s)} > 0$ and $\sum_{l=\kappa_s}^{\infty} R_{lj}^{(s)} = 1$, $f^{(s)}(\alpha \mid \beta_j)$ is in the form of a mixture of gamma densities, where the mixing weights $R_{lj}^{(s)}$'s are updated at each cycle in accordance with the observed data.

Objective function: For $l = \kappa_s, \kappa_s + 1, \ldots,$ $s = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$, define

$$f_l^{(s)}(\alpha, \beta_j) = P_j^{(s)} R_{lj}^{(s)} \gamma\left(\alpha \mid a+l, b_j^{(s)}\right)$$

Recall that $\Phi^{(s)}(\tau)$ is the maintenance cost per unit time in the s'th cycle and let $\Phi_l^{(s)}(\tau) = \Phi\left(\tau \mid f_l^{(s)}\right)$. Then the maintenance cost per unit time in the s-th replacement cycle is given by

$$\Phi^{(s)}(\tau) = \sum\nolimits_{l=\kappa_{s-1}}^{\infty} \Phi_l^{(s-1)}(\tau) \tag{3.19}$$

Special Cases of p: The special cases $p = 0, 1$ are interesting since they correspond to the *age replacement with minimal repair*, and the classical *age replacement* policies respectively. For $p = 0$, we have

Proposition 3: For $j = 1, 2, \ldots, n$ and $s = 1, 2, 3, \ldots$,

$$P_j^{(s)} = \frac{P_j^{(s-1)} r^{(s)}\left(a + \kappa_{s-1}, k_s, j\right)}{\sum_{l=1}^{n} P_l^{(s-1)} r^{(s)}\left(a + \kappa_{s-1}, k_s, l\right)} \tag{3.20}$$

and

$$f^{(s)}\left(\alpha \mid \beta_j\right) = \gamma\left(\alpha \mid a + \kappa_s, b_j^{(s)}\right) \tag{3.21}$$

$$P_{\alpha,\beta}\left\{N_{\tau_s} = k_s\right\} = \sum\nolimits_{j=1}^{n} P_j^{(s-1)} r^{(s)}\left(a + \kappa_{s-1}, k_s, j\right) \tag{3.22}$$

For the case $p = 1$, the number of minimal repairs is always zero and the adaptive policy should be modified to describe **D** differently. In this case the procedure can be based on the times of system replacements in the previous cycles as discussed in Section 4.

3.1. **Experimental Results.** In this section, simulation results are presented for the proposed model, where the simulation of replacement cycles is based on the sample paths of a NHPP and a Bernoulli Process (see e.g. Çınlar[4]). A Weibull distribution with $\alpha = 3$ and $\beta = 2.6$ is used for the failure time, and a gamma distribution with $a = 1$ and $b = 0.25$ is used for the prior density of α. The prior p.m.f of β is obtained by discretizing the Beta density function with support on $(2, 3)$, and parameters $c = d = 1$ at $n = 50$ equally spaced points on the interval $(2, 3)$ (see Dayanik and Gürler [6] for details). The cost parameters are taken as $c_m = 5$, $c_p = 50$, and $c_r = 100$.

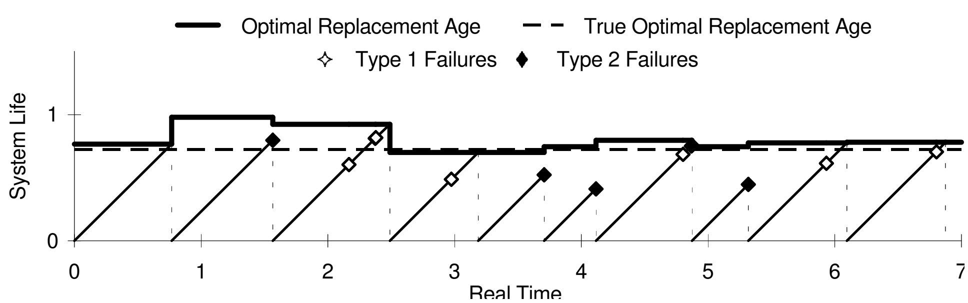

FIGURE 3.1. A sample path for count data

Figure 3.1 displays a sample path of system failures under the proposed policy with $p = 0.25$ for the first ten replacement cycles. The 1st, 3rd, 4th, 9th and 10th cycles are terminated with a preventive replacement, and the rest with critical failures. The dark path in Figure 3.1 shows how the optimal replacement age evolves with respect to the number of Type 1 failures. Generally speaking, long replacement ages induce a critical failure and a cost $c_r > c_p$ is incurred, whereas shorter replacement ages result in more often than necessary preventive replacements with a cost of c_p. It is seen from the generated example that the optimal replacement age resolves this trade-off by increasing τ^* slightly when no non-critical failures occur before the system is replaced upon a critical failure. This is the case for cycles 5, 6 and 8. When the system is replaced upon a critical failure, and has already been repaired minimally several times before the critical failure, τ^* for the next cycle decreases slightly which happens in Cycle 7. Finally, if the system is kept in operation with several minimal repairs until it is preventively replaced, the replacement age for the next cycle does not change much. Cycles 4, 9 and 10 are the examples. Also, observe that the optimal replacement times become stable and close to 0.7247 which is the optimal replacement time if the failure time has a Weibull distribution with parameters $\alpha = 3$ and $\beta = 2.6$.

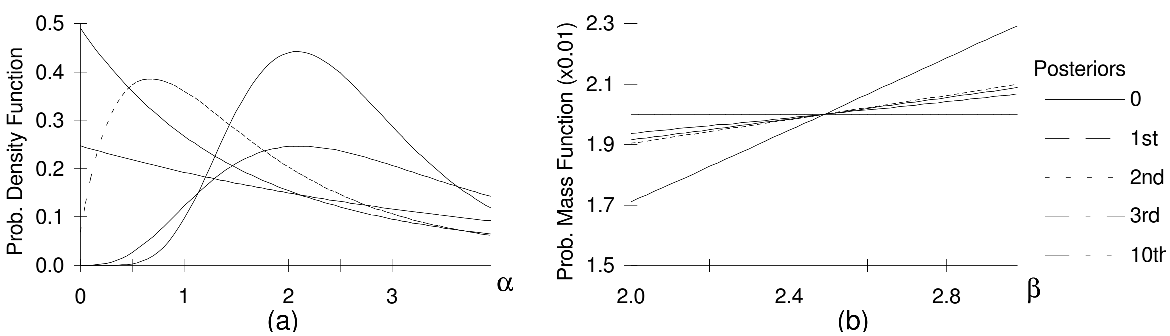

FIGURE 3.2. $p = 0.25$, (a) Marginal posterior density of α, (b) Marginal posterior probability mass function of β. For clarity of exposition the mass functions of β at 50 points are displayed as connected lines.

In Figure 3.2 marginal posterior density/mass functions of α and β are displayed. A faster stabilization is observed with the β p.m.f. and the density of α gets more concentrated about the true value 3 as the process continues. The impact of p is investigated by simulated samples with $p = 0.25, 0.50, 0.75$. We observed that as p increases, the convergence becomes slower. This can be explained by the fact that if very few non-critical failures occur in a cycle, it takes more time to learn about the characteristics of the failure process.

4. Failure Time Data

In this section an adaptive approach is introduced which utilizes the failure times and the length of the replacement cycles as well as the number of minimal repairs. More precisely, for the s'th cycle we have the following data

$$\mathbf{D^{(s)}} \equiv \left\{N_{\tau_s}, X_1^{(s)}, X_2^{(s)}, \ldots, X_{N_{\tau_s}}^{(s)}, Y^{(s)}\right\} \equiv \left(N_{\tau_s}, \widetilde{X}^{(s)}, Y^{(s)}\right)$$

where $X_i^{(s)}$ denote the time of the i'the failure in the s'th cycle. A replacement takes place either after a critical failure or at age τ. In the first case the system is replaced upon a critical failure before the system age reaches τ and $N_{\tau_s} = k_s \geq 0$ non-critical failures occur before a critical one. System fails at $0 < x_1^{(s)} < x_2^{(s)} < \ldots < x_{k_s+1}^{(s)} < \tau_s$ and $Y^{(s)} = x_{k_s+1}^{(s)}$. In the second case, the system is replaced at age τ before a critical failure occurs and $N_{\tau_s} = k_s \geq 0$. The system fails at $0 < x_1^{(s)} < x_2^{(s)} < \ldots < x_{k_s}^{(s)} < \tau_s$ and $Y^{(s)} = \tau_s$. In order to write the overall likelihood function, let us define Σ_s as the total number of system failures (both Type-1 and Type-2) in the s'th replacement cycle and

$$\pi_j^{(s)} = \begin{cases} \left[\prod_{i=1}^{\Sigma_s} x_i^{(s)}\right]^{\beta_j - 1} & \text{if} \quad \Sigma_s > 0 \\ 1, & \text{if} \quad \Sigma_s = 0 \end{cases} \tag{4.23}$$

Also let $b_j^{(s)} = b_j^{(s-1)} + \left(Y^{(s)}\right)^{\beta_j}$. Then the joint density function of $\left(N_{\tau_s}, \widetilde{X}^{(s)}, Y^{(s)}\right)$ is given as

$$h^{(s)}\left(\tau_s, \widetilde{x}^{(s)}, y^{(s)} \mid \alpha, \beta_j\right) = q^{k_s} p^{\Sigma_s - k_s} \left(\alpha\beta_j\right)^{\Sigma_s} \pi_j^{(s)} e^{-\alpha\left(b_j^{(s)} - b_j^{(s-1)}\right)} \tag{4.24}$$

Writing $a^0 = a,\ a^{(s)} = a^{(s-1)} + \Sigma_s$, we have by the Bayes theorem

$$f^{(s)}\left(\alpha, \beta_j\right) = \frac{h^{(s)}\left(\tau_s, \widetilde{x}^{(s)}, y^{(s)} \mid \alpha, \beta_j\right) f^{(s-1)}\left(\alpha, \beta_j\right)}{h_{\alpha,\beta}^{(s)}\left(\tau_s, \widetilde{x}^{(s)}, y^{(s)}\right)} \tag{4.25}$$

with

$$f^{(0)}\left(\alpha, \beta_j\right) = P_j^{(0)} \gamma\left(\alpha \mid a, b\right) \equiv P_j^{(0)} \gamma\left(\alpha \mid a^{(0)}, b_j^{(0)}\right) \tag{4.26}$$

Also, for $j = 1, 2, \ldots, n$, and $s = 1, 2, 3, \ldots$ it holds that

$$P_j^{(s)} = \frac{\pi_j^{(s)} \beta_j^{\Sigma_s} \dfrac{\left(b_j^{(s-1)}\right)^{a^{(s-1)}}}{\left(b_j^{(s)}\right)^{a^{(s)}}}}{\sum_{l=1}^{n} \pi_l^{(s)} \beta_l^{\Sigma_s} \dfrac{\left(b_l^{(s-1)}\right)^{a^{(s-1)}}}{\left(b_l^{(s)}\right)^{a^{(s)}}} P_l^{(s-1)}} P_j^{(s-1)} \tag{4.27}$$

and

$$f^{(s)}\left(\alpha \mid \beta_j\right) = \gamma\left(\alpha \mid a^{(s)}, b_j^{(s)}\right), \quad \alpha > 0 \tag{4.28}$$

The adaptive procedure of Section 2.2 can now be implemented by using these new expressions for the posterior distributions.

4.1. **Extension to Random p.** In the foregoing discussions, it is assumed that the probability of a non-critical failure p is known. We illustrate below that this assumption can be relaxed somewhat. Suppose p is a beta random variable independent of α and β, with parameters $u > 0$ and $v > 0$. Also, let $u^{(0)} = u$, $v^{(0)} = v$, $u^{(s)} = u^{(s-1)} + \Sigma_s - k_s$ and $v^{(s)} = v^{(s-1)} + k_s$ refer to the updated parameters. Then the expressions derived in the previous sections remain valid provided that they are interpreted as conditional probabilities or expectations given p. For $j = 1, \ldots, n$ and $s = 1, 2, \ldots$, the s'th posterior joint probability density function of α, β and p is

$$f^{(s)}\left(\alpha, \beta_j, p\right) = P_j^{(s)} \gamma\left(\alpha \mid a^{(s)}, b_j^{(s)}\right) \zeta\left(p \mid u^{(s)}, v^{(s)}\right), \quad \alpha > 0, \quad 0 < p < 1 \tag{4.29}$$

where

$$P_j^{(s)} = \frac{\beta_j^{\Sigma_s} \pi_j^{(s)} \dfrac{\left(b_j^{(s-1)}\right)^{a^{(s-1)}}}{\left(b_j^{(s)}\right)^{a^{(s)}}}}{\sum_{j=1}^{n} \beta_j^{\Sigma_s} \pi_j^{(s)} \dfrac{\left(b_j^{(s-1)}\right)^{a^{(s-1)}}}{\left(b_j^{(s-1)}\right)^{a^{(s-1)}}} P_j^{(s-1)}} P_j^{(s-1)} \tag{4.30}$$

The expected cycle cost function is given below, the evaluation of which requires numerical integration methods.

$$\begin{aligned} E_{\alpha,\beta,p}\left[C(\tau)\right] = \int_0^1 \zeta\left(p \mid u^{(s)}, v^{(s)}\right) \sum_{l=1}^{n} P_l \cdot \Bigg[& (a+1)\, a b^a c_m q p \beta_l \int_0^\tau \frac{t^{2\beta_l - 2}}{\left(b + p t^{\beta_l}\right)^{a+2}} dt \\ & + a b^a c_r p \beta_l \int_0^\tau \frac{t^{\beta_l - 2}}{\left(b + p t^{\beta_l}\right)^{a+1}} dt + a b^a c_m q \frac{\tau^{\beta_l - 1}}{\left(b + p \tau^{\beta_l}\right)^{a+1}} + \frac{c_p}{\tau} \left(\frac{b}{b + p\tau^{\beta_l}}\right)^a \Bigg] dp \end{aligned}$$

4.2. **Experimental Results.** Figure 4.1 illustrates a sample path of system failures when the failure time data is used to update the system replacement age with the numerical set-up of previous section.

In general a system performance similar to the count data case is observed in the first ten cycles. However, the availability of failure time data led to a faster convergence to the true replacement age as expected. Sensitivity to p is investigated for $p = 0.25, 0.50, 0.75$. It is observed that in comparison to the count data case, the replacement ages are generally closer to the true one and the replacement policy is less sensitive to p values. This agrees with intuition since the number of non-critical failures is the essential information for the

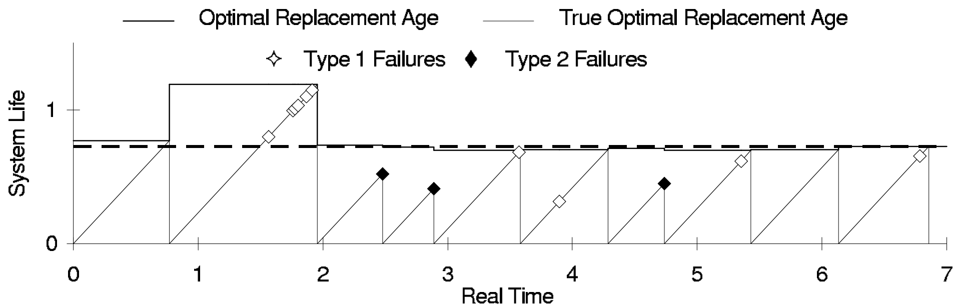

FIGURE 4.1. A sample path for failure time data.

count data and it directly depends on p, whereas the availability of the failure time data reduces the relative significance of p.

Impact of the Data Types: The two different data types discussed so far have obvious advantages and disadvantages in terms of the cost of data collection and processing, which we do not further discuss here. However their impact on the performance of the policy is of interest and to investigate this, the convergence rates of the optimal replacement age and the optimal maintenance cost to their true values are compared in the first ten cycles by a small simulation study. The distributions and parameters described in Section 3.1 are used with three values of minimal repair cost, set as $c_m = 5, 20, 40$. The percentage deviations of the optimal replacement age and the optimal maintenance cost from their true values are considered as performance measures and their average over 1000 simulation runs are used for comparisons. Figure 4.2 displays the case $c_m = 5, p = 0.25$, where relatively large

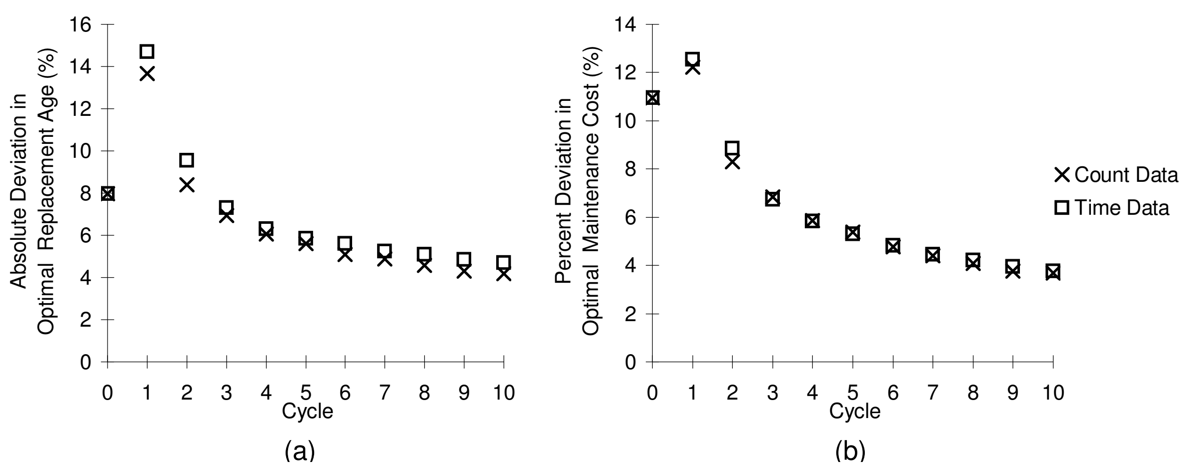

FIGURE 4.2. $c_m = 5$, (a) Deviation of replacement age from the true one, (b) Deviation of the optimal cost from the true one.

number of minimal repairs are observed and the count data seems to perform slightly better. The difference in terms of the cost function seems quite insensitive to the difference in the replacement ages. Note also that the convergence of both the optimal replacement age and the optimal maintenance costs get slower as c_m increases for both data types. The average number of observed non-critical failures per cycle were 6.55, 1.66, and 0.82, for $c_m = 5, 20, 40$ respectively. It is also observed that the count data yields a better performance as c_m gets smaller relative to c_p, and the opposite is observed as c_m gets closer to c_p.

5. Conclusion

In this paper a generalized age replacement policy for repairable systems is studied from a Bayesian perspective. The independent system failures are classified as critical and non-critical with a certain fixed probability. The system is replaced at a critical failure or at time τ, whichever occurs first and the non-critical failures are minimally repaired. An adaptive Bayesian approach is introduced which adjusts the optimal replacement time τ based on the accumulated data. Two data types, the number of non-critical failures and the failure times together with lengths of the replacement cycles are used for updating purposes. The Weibull distribution is assumed for the system lifetime. Although the choice of parameters for this distribution provides a flexible family, it would be of interest to see the impact of other distributions.

The Bayesian framework presented in this study can in principal be applied to other maintenance settings. In particular, it can be considered to include a generalized block replacement policy (see Policy 8 of Beichelt), which is not studied here since block replacement policies are more suitable for multicomponent systems.

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