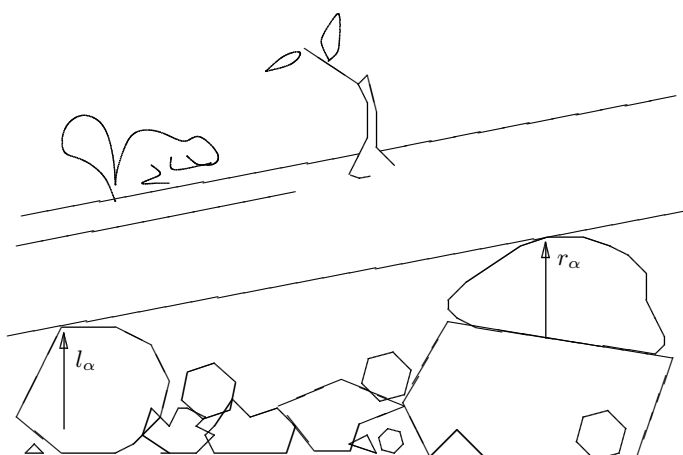# Contributions to the Theory of Optimal Stopping for One–Dimensional Diffusions

Savas Dayanik

Submitted in partial fulfillment of the

requirements for the degree

of Doctor of Philosophy

in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2002

2002

ABSTRACT

Contributions to the Theory of Optimal Stopping
for One–Dimensional Diffusions

Savas Dayanik

Advisor: Ioannis Karatzas

We give a new characterization of excessive functions with respect to arbitrary one–dimensional regular diffusion processes, using the notion of concavity. We show that excessive functions are essentially concave functions, in some generalized sense, and vice–versa.

This, in turn, allows us to characterize the value function of the optimal stopping problem, for the same class of processes, as "the smallest nonnegative concave majorant of the reward function". In this sense, we generalize results of Dynkin–Yushkevich for the standard Brownian motion. Moreover, we show that there is essentially one class of optimal stopping problems, namely, the class of *undiscounted* optimal stopping problems for the standard Brownian motion. Hence, optimal stopping problems for arbitrary diffusion processes are not inherently more difficult than those for Brownian motion.

The concavity of the value functions also allows us to draw sharper conclusions about their smoothness, thanks to the nice properties of concave functions. We can therefore offer a new perspective and new facts about the smooth–fit principle and the method of variational inequalities in the context of optimal stopping.

The results are illustrated in detail on a number of non–trivial, concrete optimal stopping problems, both old and new.

Contents

List of Figures

Acknowledgement

First and foremost, I would like to thank Ioannis Karatzas, my advisor. I am indebted for his patience with my endless stream of questions, when I was a student in his courses. It was those lengthy hours of discussions, after classes, that he offered with the same enthusiasm every single week, which nurtured my joy in studying probability.

His guidance, support and patience continued, after I started my research under his direction. This thesis would not be possible, unless he constantly showed his confidence in me, especially during my tough times at Columbia.

Not only his outstanding expertise has fed my curiosity for the unknown, but also he has become my role model with his dedication to teaching and research. Although I regret not being able to benefit the most out of his *Thursday Lunch Group*, I will always remember what a wonderful idea it was, in order to have young researchers got connected to each other, and to keep their enthusiasm at the top.

I am grateful to Donald Goldfarb, the chairman of the IEOR Department, for his understanding, encouragement and support throughout my Ph.D. study. Without his approval and his confidence in me, I would not be able to make a fast start with my research in my early years, and to gain an invaluable teaching experience as an instructor in my last year at Columbia.

My thanks also go to Mark Broadie, Guillermo Gallego, Ioannis Karatzas, Steven Kou and Daniel Ocone, the members of my dissertation committee, for their time to read this thesis, and for their insightful comments on the results and their possible

extensions.

I would like to thank Jing-Sheng (Jeannette) Song, Susan H. Xu and Steven Kou for extending opportunities to work together on independent research projects, which were sources of invaluable experience for me.

I would like to express my gratitude to Gennady Samorodnitsky and James Renegar for their support during my brief presence in the School of Operations Research and Industrial Engineering at Cornell University, before coming to Columbia.

My deepest thanks and gratitude go to my wife Aynur, and my parents who have constantly supported me in all of my endeavors.

Last, but not least, my special gratitude goes to Lisa Fleischer, Michael D. Grigoriadis, Ülkü Gürler, Serkan Hoşten, Garud Iyengar, Daniel Ocone, Victor de la Peña, Duong H. Phong, the members of the Thursday Lunch group, especially Olympia Hadjiliadis, Christian Pasarica, Hui Wang, Mona Zamifirescu, Gordan Žitković for invaluable discussions on various occasions.

To

my wife *Aynur*,

and

my parents *Naciye* and *Ali Dayanık*

Chapter 1

Introduction

The focus of this study is the optimal stopping of one–dimensional diffusion processes. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a standard one–dimensional Brownian motion $B = \{B_t; t \geq 0\}$. Suppose that we have a one–dimensional diffusion process X with state space $\mathcal{I} \subseteq \mathbb{R}$, on the same probability space, and with dynamics

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \tag{1.1}$$

for some Borel functions $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to (0, \infty)$ defined on $\mathcal{I}$. We assume that $\mathcal{I}$ is an interval with endpoints $-\infty \leq a < b \leq +\infty$. Throughout this study, we shall also assume that X is regular in (a, b), i.e. X reaches y with positive probability starting at x, for every x and y in (a, b). We shall denote by $\mathbb{F} = \{\mathcal{F}_t\}$ the natural filtration of X.

Let $\beta \geq 0$ be a constant, and $h(\cdot)$ be a Borel function such that $\mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big]$ is well–defined for every $\mathbb{F}$–stopping time τ and $x \in \mathcal{I}$. By convention, we assume

$$f(X_\tau(\omega)) = 0 \ \text{ on } \ \{\tau = +\infty\}, \quad \text{for every Borel function } f(\cdot).$$

Finally, we denote by

$$V(x) \triangleq \sup_{\tau \geq 0} \ \mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big], \qquad x \in \mathcal{I}, \tag{1.2}$$

the *value function* of the optimal stopping problem with *reward function* $h(\cdot)$ and *discount rate* β, where the supremum is taken over all $\mathbb{F}$–stopping times. The optimal stopping problem is to find the value function, as well as an optimal stopping time τ^* for which the supremum is attained, if such a stopping time exists.

One of the best known characterizations of the value function $V(\cdot)$ is given in terms of β–excessive functions (with respect to the process X), namely, the nonnegative functions $f(\cdot)$ that satisfy

$$f(x) \geq \mathbb{E}_x\big[e^{-\beta\tau}f(X_\tau)\big], \quad \text{for every } \mathbb{F}\text{–stopping time } \tau \text{ and } x \in \mathcal{I}. \qquad (1.3)$$

For every β–excessive function $f(\cdot)$ *majorizing* $h(\cdot)$, (1.3) implies that $f(x) \geq V(x)$, $x \in \mathcal{I}$. On the other hand, thanks to the strong Markov property of diffusion processes, it is not hard to show that $V(\cdot)$ is itself a β–excessive function. Namely,

Theorem 1.1 (Dynkin [2]). *The value function $V(\cdot)$ of (1.2) is the smallest β–excessive (with respect to X) majorant of $h(\cdot)$ on $\mathcal{I}$, if $h(\cdot)$ is lower semi–continuous.*

This characterization of the value function often serves as a verification tool. It does not however describe how to calculate the value function explicitly for a general diffusion process. The common practice in the literature is therefore to guess the value function, and then to put it to the test using Theorem 1.1.

One special optimal stopping problem, whose solution for arbitrary reward functions is perfectly known, was studied by Dynkin and Yushkevich [3]. These authors study the optimal stopping problem of (1.2) under the following assumptions:

$$\left\{ \begin{array}{l} X \text{ is a standard Brownian motion starting in a closed} \\ \text{bounded interval } [a, b], \text{ and is absorbed at the boundaries} \\ \text{(i.e. } \mu(\cdot) \equiv 0 \text{ on } [a, b], \sigma(\cdot) \equiv 1 \text{ on } (a, b), \text{ and } \sigma(a) = \sigma(b) = \\ 0, \text{ and } \mathcal{I} \equiv [a, b] \text{ for some } -\infty < a < b < \infty). \text{ Moreover,} \\ \beta = 0, \text{ and } h(\cdot) \text{ is a bounded Borel function on } [a, b]. \end{array} \right. \qquad \text{(DY)}$$

Their solution relies on the following key theorem, which characterizes the excessive functions with respect to one–dimensional Brownian motion.

Theorem 1.2 (Dynkin and Yushkevich [3])**.** *Every 0–excessive (or simply, excessive) function for one–dimensional Brownian motion X, i.e., every nonnegative function with the property (1.3), is concave and vice–versa.*

In conjunction with Theorem 1.1, this result implies the following

Corollary 1.1. *The value function $V(\cdot)$ of (1.2) is the smallest nonnegative concave majorant of $h(\cdot)$ under the assumptions (DY).*

In this study, we generalize the findings of Dynkin and Yushkevich for the standard Brownian motion to arbitrary one–dimensional, regular diffusion processes.

We shall show that the collection of excessive functions for a diffusion process X coincides with the collection of concave functions in some suitably generalized sense (cf. Proposition 3.1). A similar concavity result will also be established for the β–excessive functions (cf. Proposition 4.1 and Proposition 5.1).

Those explicit characterizations of excessive functions allow us to describe the value function $V(\cdot)$ of (1.2) in terms of generalized concave functions, in a manner similar to Theorem 1.2 (cf. Proposition 3.2 and Proposition 4.2). The new characterization of the value function, in turn, has important consequences.

The straight–forward connection between generalized concave functions and ordinary concave functions, will reduce the optimal stopping problem for arbitrary diffusion processes to those for the standard Brownian motion (cf. Proposition 3.3). Therefore, the "special" solution of Dynkin and Yushkevich, in fact, becomes a fundamental technique for solving the optimal stopping problems for any general one–dimensional regular diffusion process.

The nice properties of concave functions, summarized in Appendix A, will help us state and prove necessary and sufficient conditions about the finiteness of value functions, and the existence and the characterization of optimal stopping times, when the diffusion process is not contained in a compact interval, or when the boundaries

are not absorbing (cf. Proposition 5.2 and Proposition 5.8)

We shall also show that concavity and minimality properties of the value function determine its smoothness. This will let us understand the major components of the method of Variational Inequalities, which we briefly review in Subsection 7.2.1. We offer, for example, a new exposition and, we believe, a better understanding, of the smooth–fit principle, which is crucial to this method. It is again the concavity of the value function that helps to unify many of the existing results in the literature about the smoothness of $V(\cdot)$ and the smooth–fit principle.

Preview: We overview the basic facts about one–dimensional diffusion processes in Chapter 2. Basic facts about concave functions and their generalizations are collected in Appendix A. It is shown there that "generalized" concave functions possess all the well–known properties of "ordinary" concave functions, under suitable conditions.

We solve undiscounted and discounted stopping problems for a regular diffusion process, stopped at the first exit from a given closed and bounded interval, in Chapter 3 and Chapter 4, respectively. In Chapter 5, we study the same problem when the state–space of the diffusion process is an unbounded interval, or its boundaries are not absorbing.

The results are used in Chapter 6 to treat optimal stopping problems recently published in the literature. Chapter 7 is where we discuss further consequences of the new characterization for the value functions. We especially address the smoothness of the value function, and take a new look at the associated variational inequalities.

Chapter 2

One–Dimensional Regular Diffusion Processes

We assume that X is a one-dimensional regular diffusion of the type (1.1), on an interval $\mathcal{I}$. We shall assume that (1.1) has a (weak) solution, which is unique in the sense of the probability law. This is, for example, guaranteed, if $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the condition

$$\int_{(x-\varepsilon,x+\varepsilon)} \frac{1+|\mu(y)|}{\sigma^2(y)}\, dy < \infty, \quad \text{for some } \varepsilon > 0, \tag{2.1}$$

at every $x \in \text{int}(\mathcal{I})$ (Karatzas and Shreve [7, 329–393]), together with precise description of the behavior of the process at the boundaries of the state–space $\mathcal{I}$. If killing is allowed at time ζ, then the dynamics in (1.1) are valid for $0 \le t < \zeta$. We shall however assume that X can only be killed at the endpoints of $\mathcal{I}$ which do not belong to $\mathcal{I}$.

Define $\tau_r \triangleq \inf\{t \ge 0 : X_t = r\}$ for every $r \in \mathcal{I}$. A one–dimensional diffusion process X is called *regular*, if for any $x \in \text{int}(\mathcal{I})$ and $y \in \mathcal{I}$, we have $\mathbb{P}_x(\tau_y < +\infty) > 0$. Hence, the state–space $\mathcal{I}$ cannot be decomposed into smaller sets from which X could not exit (the sufficient condition (2.1) for the existence of the unique weak solution of (1.1) also guarantees that X is a regular process). The major consequences of this

assumption on X are listed below: their proofs can be found in Revuz and Yor [11, pages 300–312].

Let $J \triangleq (l, r)$ be a subinterval of $\mathcal{I}$ such that $[l, r] \subseteq \mathcal{I}$, and σ_J is the exit time of X from J. If $x \in J$, then $\sigma_J = \tau_l \wedge \tau_r$, $\mathbb{P}_x$–a.s. For $x \notin J$, then $\sigma_J = 0$, $\mathbb{P}_x$–a.s.

Proposition 2.1. *If J is bounded, then the function $m_J(x) \triangleq \mathbb{E}_x[\sigma_J]$, $x \in I$ is bounded on J. In particular, σ_J is a.s. finite, and $\mathbb{P}_x(\tau_l < \tau_r) + \mathbb{P}_x(\tau_l > \tau_r) = 1$, $x \in J$.*

Proposition 2.2. *There exists a continuous, strictly increasing function $S(\cdot)$ on $\mathcal{I}$ such that for any l, r, x in $\mathcal{I}$, with $a \leq l < x < r \leq b$, we have*

$$\mathbb{P}_x(\tau_r < \tau_l) = \frac{S(x) - S(l)}{S(r) - S(l)}. \tag{2.2}$$

Any other function $\widetilde{S}$ with the same properties is an affine transformation of S, i.e., $\widetilde{S} = \alpha S + \beta$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$. The function S is unique in this sense, and is called the "scale function" of X.

If the killing time ζ is finite with positive probability, and $\lim_{t \uparrow \zeta} X_t = a$ (say), then $\lim_{x \to a} S(x)$ is finite. We shall define $S(a) \triangleq \lim_{x \to a} S(x)$, and we will say $S(X_\zeta) = S(l)$. With this in mind, we have

Proposition 2.3. *A locally bounded Borel function f is a scale function, if and only if the process $Y_t^f \triangleq f(X_{t \wedge \zeta \wedge \tau_a \wedge \tau_b})$, $t \geq 0$, is a local martingale. If X can be further represented by the stochastic differential equation (1.1), then*

$$S(x) = \int_c^x \exp\left\{ -\int_c^y \frac{2\mu(z)}{\sigma^2(z)} dz \right\} dy, \qquad x \in \mathcal{I}, \tag{2.3}$$

for any arbitrary but fixed $c \in \mathcal{I}$.

The scale function $S(\cdot)$ satisfies

$$S'(x) = \exp\left\{ -2\int_c^x \frac{\mu(u)}{\sigma^2(u)} du \right\}, \quad \text{on int}(\mathcal{I}),$$

and we shall define

$$S''(x) \triangleq -\frac{2\mu(x)}{\sigma^2(x)} S'(x), \quad x \in \text{int}(\mathcal{I}).$$

Let us now introduce the second–order differential operator

$$\mathcal{A}u(\cdot) \triangleq \frac{1}{2}\sigma^2(\cdot)\frac{d^2u}{dx^2}(\cdot) + \mu(\cdot)\frac{du}{dx}(\cdot), \quad \text{on } \mathcal{I}, \tag{2.4}$$

associated with the infinitesimal generator of X. As an ordinary differential equation,

$$\mathcal{A}u = \beta u \tag{2.5}$$

always has two linearly independent, positive solutions. These are uniquely deter-
mined up to multiplication, if we require one of them to be strictly increasing and
the other to be strictly decreasing. We shall denote the *increasing* solution by $\psi(\cdot)$
and the *decreasing* solution by $\varphi(\cdot)$. In fact, we have

$$\psi(x) = \begin{cases} \mathbb{E}_x[e^{-\beta\tau_c}], & \text{if } x \le c \\ \frac{1}{\mathbb{E}_c[e^{-\beta\tau_x}]}, & \text{if } x > c \end{cases}, \text{ and } \varphi(x) = \begin{cases} \frac{1}{\mathbb{E}_c[e^{-\beta\tau_x}]}, & \text{if } x \le c \\ \mathbb{E}_x[e^{-\beta\tau_c}], & \text{if } x > c \end{cases}, \quad \forall x \in \mathcal{I},$$

for any arbitrary but fixed $c \in \mathcal{I}$ (cf. Ito and McKean [5, pages 128–129]). Solutions of
(2.5) in the domain of infinitesimal operator $\mathcal{A}$ are obtained as linear combinations of
$\psi(\cdot)$ and $\varphi(\cdot)$, subject to appropriate boundary conditions imposed on the process X.
If an endpoint is contained in the state–space $\mathcal{I}$, we shall assume that it is absorbing;
and if it is not contained in $\mathcal{I}$, we shall assume that X is killed if it can reach the
boundary with positive probability. In either case, the boundary conditions on $\psi(\cdot)$
and $\varphi(\cdot)$ are $\psi(a) = \varphi(b) = 0$. For the complete characterization of $\psi(\cdot)$ and $\varphi(\cdot)$
corresponding to other types of boundary behavior, refer to Ito and McKean [5, pages
128–135]. Note that the *Wronskian determinant*

$$W(\psi, \varphi) \triangleq \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x) \tag{2.6}$$

of $\psi(\cdot)$ and $\varphi(\cdot)$ is a positive constant. One last useful expression is

$$\mathbb{E}_x\left[e^{-\beta\tau_y}\right] = \left\{ \begin{array}{ll} \dfrac{\psi(x)}{\psi(y)}, & x \le y \\[2mm] \dfrac{\varphi(x)}{\varphi(y)}, & x > y \end{array} \right\}. \tag{2.7}$$

Chapter 3

Undiscounted Optimal Stopping

Suppose we start the diffusion process X of (1.1) in the closed and bounded interval $[c, d]$ contained in the interior of the state–space $\mathcal{I}$, and stop it as soon as it reaches one of the boundaries c or d. Let the function $h : [c, d] \to \mathbb{R}$ be bounded, and set

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[h(X_\tau)], \quad x \in [c, d]. \tag{3.1}$$

We want to characterize V and find an optimal stopping time τ^* such that $V(x) = \mathbb{E}_x[h(X_{\tau^*})]$, $x \in [c, d]$, if such τ^* exists. If $h \leq 0$, then trivially $V \equiv 0$, and $\tau \equiv \infty$ is an optimal stopping time. Therefore, we shall assume $\sup_{x \in [c,d]} h(x) > 0$.

Following the spirit of Dynkin and Yushkevich [3, pages 112–126], we shall first characterize the class of excessive functions. As shown in Theorem 1.1, these play a fundamental role in optimal stopping problems.

To motivate what follows, let $U : [c, d] \to \mathbb{R}$ be an excessive function of X. Then, for any stopping time τ of X, and $x \in [c, d]$, we have $U(x) \geq \mathbb{E}_x[U(X_\tau)]$. In particular, if $x \in [l, r]$, some subinterval of $[c, d]$, then we may take $\tau = \tau_l \wedge \tau_r$, where $\tau_r \triangleq \inf\{t \geq 0 : X_t = r\}$. Thanks to the regularity of X, we have

$$U(x) \geq \mathbb{E}_x[U(X_{\tau_l \wedge \tau_r})] = U(l) \cdot \mathbb{P}_x(\tau_l < \tau_r) + U(r) \cdot \mathbb{P}_x(\tau_l > \tau_r), \qquad x \in [c, d].$$

One can argue that $u_1(x) \triangleq \mathbb{P}_x(\tau_l < \tau_r)$ and $u_2(x) \triangleq \mathbb{P}_x(\tau_l > \tau_r)$ are the unique

solutions of $\mathcal{A}u = 0$ in (l, r), with boundary conditions $u_1(l) = 1 - u_1(r) = 1$ and $1 - u_2(l) = u_2(r) = 1$, respectively. The scale function $S(\cdot)$ of X and the constant function 1 are two familiar independent solutions of the second order ODE, $\mathcal{A}u = 0$. Using the boundary conditions, one can then show, in terms of $S(\cdot)$ and 1, that

$$\mathbb{P}_x(\tau_l < \tau_r) = \frac{S(r) - S(x)}{S(r) - S(l)}, \quad \text{and} \quad \mathbb{P}_x(\tau_l > \tau_r) = \frac{S(x) - S(l)}{S(r) - S(l)}, \qquad x \in [l, r].$$

After plugging these identities into the above inequality, we finally witness that

$$U(x) \geq U(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + U(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}, \qquad x \in [l, r].$$

This tells us that every excessive function of X is S–concave on $[c, d]$ (see Appendix A for a detailed discussion). In the case that X is a standard Brownian motion, Dynkin and Yushkevich [3] showed that the reverse is also true. We shall next show that the reverse is true for an *arbitrary* diffusion process X.

Let $S(\cdot)$ be the scale function of X as above. Remember that $S(\cdot)$ is strictly increasing and continuous on $\mathcal{I}$. Since, by our choice of c and d, the process reaches c and d in finite time with positive probability, we have $-\infty < S(c) < S(d) < \infty$.

Proposition 3.1 (Characterization of Excessive Functions). *A function* $U : [c, d] \to \mathbb{R}$ *is nonnegative and S–concave on $[c, d]$, if and only if*

$$U(x) \geq \mathbb{E}_x[U(X_\tau)] \tag{3.2}$$

holds for every $x \in [c, d]$ and every stopping time τ of X.

This, in turn, allows us to conclude the main result of this chapter, namely

Proposition 3.2 (Characterization of the Value Function). *The value function $V(\cdot)$ of (3.1) is the smallest nonnegative, S–concave majorant of $h(\cdot)$ on $[c, d]$, in the sense that, for any other nonnegative S–concave majorant $U(\cdot)$ of $h(\cdot)$ on $[c, d]$, we have $U(\cdot) \geq V(\cdot)$.*

The S–concavity of the value function $V(\cdot)$ of (3.1) was already noticed by Karatzas and Sudderth [8]. We defer the proofs of Proposition 3.1 and Proposition 3.2 to the end of the chapter, and discuss their implications first.

It is usually a simple matter to find the smallest nonnegative concave majorant of a bounded function on some closed bounded interval. Geometrically, it coincides with the rope stretched from above the graph of function, with both ends pulled to the ground. On the contrary, it is hard to visualize the nonnegative S–concave majorant of a function. The following Proposition has therefore some importance, when we need to calculate $V(\cdot)$ explicitly.

Proposition 3.3. *Let* $W : [S(c), S(d)] \to \mathbb{R}$ *be the smallest nonnegative concave majorant of* $H : [S(c), S(d)] \to \mathbb{R}$, *given by* $H(y) \triangleq h\big(S^{-1}(y)\big)$. *Then we have* $V(x) = W\big(S(x)\big)$, *for every* $x \in [c, d]$.

Remark 3.1. Since the standard Brownian motion B is on natural scale, i.e. $S(x) = x$ up to some affine transformation, $W(\cdot)$ of Proposition 3.3 is itself the value function of some optimal stopping problem of the standard Brownian motion, namely

$$W(y) = \sup_{\tau \geq 0} \mathbb{E}_y[H(B_\tau)] = \sup_{\tau \geq 0} \mathbb{E}_y\Big[h\Big(S^{-1}\big(B_\tau\big)\Big)\Big], \quad y \in [S(c), S(d)]. \tag{3.3}$$

where the supremum is taken over all stopping times of B. Therefore, solving the original optimal stopping problem is the same as solving another, with a different reward function, for a standard Brownian motion. This supports our claim in the introduction, that there is essentially one class of optimal stopping problems for diffusion processes, namely those for the one–dimensional Brownian motion (see Remark 4.1 also).

The concave characterization of Proposition 3.2 for the value function $V(\cdot)$, together with the well–known properties of concave functions, allows us to arrive at important conclusions about the smoothness of $V(\cdot)$ and the existence of an optimal stopping time.

Proposition 3.4. *If h is continuous on $[c, d]$, then V is continuous on $[c, d]$.*

We shall characterize now the optimal stopping time, when it exists. Consider *the optimal stopping region*

$$\Gamma \triangleq \{x \in [c, d] : V(x) = h(x)\}, \text{ and define } \tau^* \triangleq \inf\{t \geq 0 : X_t \in \Gamma\}, \qquad (3.4)$$

the time of first–entry into this region. For the proof of the next Proposition, we need the following

Lemma 3.1. *Suppose $c \leq l < x < r \leq d$. Then*

$$\mathbb{E}_x[h(X_{\tau_l \wedge \tau_r})] = h(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + h(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}$$

$$\mathbb{E}_x[h(X_{\tau_r})] = h(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)} \quad and \quad \mathbb{E}_x[h(X_{\tau_l})] = h(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)}.$$

Proof. By definition (see Revuz and Yor [11, pages 300–310]), S is the unique (up to affine transformations) strictly increasing, continuous function defined in $\mathcal{I}$ such that, for every $x \in [l, r] \subset \text{int}(\mathcal{I})$, we have

$$\mathbb{P}_x(\tau_l < \tau_r) = \frac{S(r) - S(x)}{S(r) - S(l)}.$$

Using this and the convention $h(X_\tau) \equiv 0$ on $\{\tau = \infty\}$, it is easy to obtain expressions above. $\qquad\qquad\square$

Proposition 3.5. *If h is continuous on $[c, d]$, then τ^* of (3.4) is an optimal stopping time.*

Proof. Let $U(x) \triangleq \mathbb{E}_x[h(X_{\tau^*})]$, for every $x \in [c, d]$. Obviously $V \geq U$, so we need to prove that $V \leq U$. Since V is the smallest nonnegative S–concave majorant of h on $[c, d]$, it is enough to show that U is a nonnegative, S–concave majorant of h on $[c, d]$.

Since h is continuous, V is continuous by Proposition 3.4. Therefore Γ is a closed set relative to $[c, d]$, and $h(X_{\tau^*}) = V(X_{\tau^*})$ on $\{\tau^* < \infty\}$. Thus $U(x) = \mathbb{E}_x[h(X_{\tau^*})] = \mathbb{E}_x[V(X_{\tau^*})]$, for every $x \in [c, d]$. Because V is nonnegative, U is also nonnegative.

1°   We shall show that U is S–*concave* and *continuous* on $[c, d]$. First, observe that, for $x \in \mathbf{\Gamma}$, we have $\mathbb{P}_x(\tau^* = 0) = 1$, and

$$U(x) = V(x), \quad \forall\, x \in \mathbf{\Gamma}. \tag{3.5}$$

What happens on $\mathbf{C} \triangleq [c, d] \backslash \mathbf{\Gamma}$? Since $\mathbf{\Gamma}$ is closed relative to $[c, d]$, $\mathbf{C}$ is open relative to $[c, d]$. Therefore, $\mathbf{C}$ is the union of a countable family, $\{J_\alpha\}_{\alpha \in \Lambda}$, of disjoint open (relative to $[c, d]$) subintervals of $[c, d]$. There are two types of intervals:

Type 1 Interval: $J_\alpha = (l, r)$, $l < r$ for some $l, r \in \mathbf{\Gamma}$. Note that for every $x \in J_\alpha$, we have $\tau^* = \tau_l \wedge \tau_r$ if the process starts at $X_0 = x$. Therefore, Lemma 3.1 implies

$$U(x) = \mathbb{E}_x[V(X_{\tau_l \wedge \tau_r})] = V(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + V(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}, \quad \forall\, x \in J_\alpha = (l, r). \tag{3.6}$$

We shall define the S–concave function $L_\alpha : [c, d] \to \mathbb{R}$ by

$$L_\alpha(x) \triangleq V(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + V(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}, \quad \forall x \in [c, d].$$

(See Figure 3.1). Clearly $U = L_\alpha$ in J_α. Since $L_\alpha(y) = V(y)$, for $y \in \{l, r\}$, and V is S–concave, Proposition A.3 implies that

$$U = L_\alpha \leq V \text{ on } J_\alpha, \quad \text{and,} \quad L_\alpha \geq V \geq U \text{ outside } J_\alpha. \tag{3.7}$$

Introduce $U_\alpha : [c, d] \to \mathbb{R}$ defined by

$$U_\alpha(x) \triangleq L_\alpha(x) \wedge V(x), \quad \forall\, x \in [c, d]. \tag{3.8}$$

(See Figure 3.1). Since both L_α and V are S–concave, Proposition A.5 implies that U_α is also S–concave on $[c, d]$. Furthermore, (3.7) implies that

$$U_\alpha(x) = \begin{cases} U(x), & \text{if } x \in J_\alpha \\ V(x), & \text{if } x \notin J_\alpha \end{cases} \leq V(x), \quad \forall\, x \in [c, d]. \tag{3.9}$$

Type 2 Interval: $J_\alpha = [c, r)$ or $(l, d]$, for some $l, r \in \mathbf{\Gamma}$. Consider the case $J_\alpha = [c, r)$ for some $r \in \mathbf{\Gamma}$ (similar considerations apply to the case $J_\alpha = (l, d]$). For every
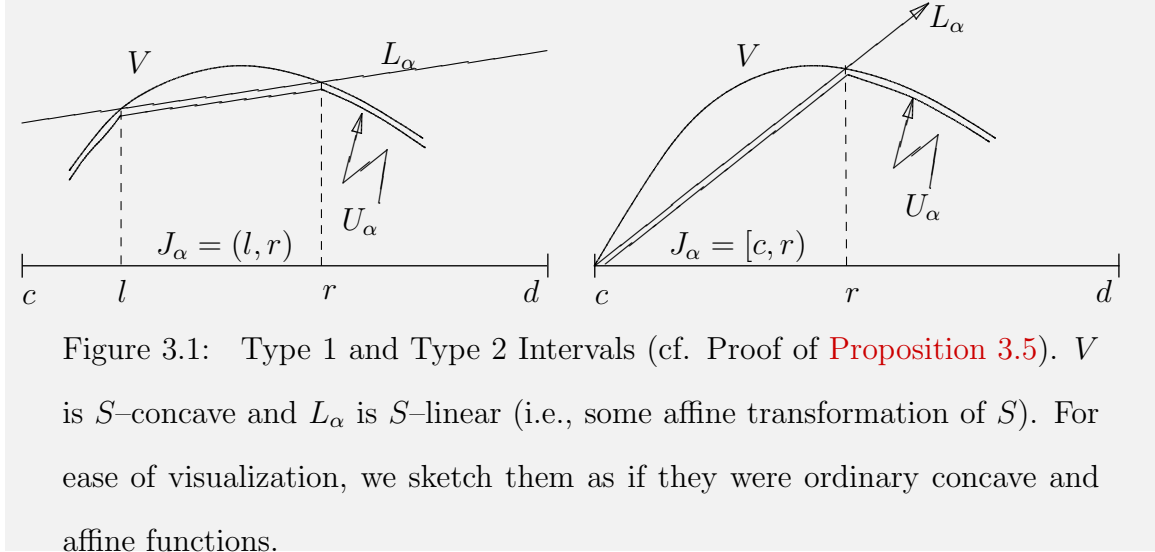
Figure 3.1: Type 1 and Type 2 Intervals (cf. Proof of Proposition 3.5). V is S–concave and L_α is S–linear (i.e., some affine transformation of S). For ease of visualization, we sketch them as if they were ordinary concave and affine functions.

$x \in J_\alpha = [c, r)$, $\tau^* = \tau_r$ on $\{X_0 = x\}$. Lemma 3.1 implies that

$$U(x) = \mathbb{E}_x[V(X_{\tau^*})] = \mathbb{E}_x[V(X_{\tau_r})] = V(r) \cdot \frac{S(x) - S(c)}{S(r) - S(c)}, \quad \forall\, x \in J_\alpha. \tag{3.10}$$

Since $c \in J_\alpha \subseteq \mathbf{C}$, we have $c \notin \mathbf{\Gamma}$, i.e. $V(c) > h(c)$. Because c is absorbing, we in fact have $V(c) = 0 \vee h(c)$. However, $0 \vee h(c) = V(c) > h(c)$ implies that $V(c) = 0 > h(c)$. Therefore, we can write

$$U(x) = V(c) \cdot \frac{S(r) - S(x)}{S(r) - S(c)} + V(r) \cdot \frac{S(x) - S(c)}{S(r) - S(c)}, \quad \forall\, x \in J_\alpha = [c, r). \tag{3.11}$$

We can define L_α and U_α as above, by replacing every occurrence of l with c. Since (3.11) is similar to (3.6), all results about L_α and U_α above can be extended to the second case. In particular, U_α is S–concave on $[c, d]$, and (3.7)–(3.9) remain in force.

Since $V \geq U$ everywhere, (3.5) and (3.9) imply that

$$U(x) = V(x) \wedge \left(\wedge_{\alpha \in \Lambda} U_\alpha(x) \right), \quad \forall\, x \in [c, d].$$

Because V and U_α, $\alpha \in \Lambda$, are S–concave on $[c, d]$, Proposition A.5 implies that U is also S–concave on $[c, d]$.

2° Because U is S–concave, and S is continuous on $[c, d]$, U is continuous in (c, d). To show continuity of U on $[c, d]$, it remains to prove that U is continuous at

the endpoints c and d. The expressions (3.5) and (3.11) imply that U coincides either with h or L_α, for some $\alpha \in \Lambda$, in some non-empty neighborhood of c. Since both h and L_α are continuous on $[c, d]$, U is continuous at c. Continuity of U at d can be argued similarly.

3°   To complete the proof of Proposition, it remains to show that U *majorizes* h *on* $[c, d]$. Assume on the contrary that

$$\theta \triangleq \max_{x \in [c,d]} h(x) - U(x) > 0.$$

Since h is bounded, θ is finite. $U + \theta$ is a nonnegative S–concave majorant of h on $[c, d]$. Therefore, Proposition 3.2 implies that $U + \theta \geq V$ on $[c, d]$. Because h and U are continuous, there exists some $x_0 \in [c, d]$ where θ is attained, i.e. $h(x_0) - U(x_0) = \theta$. Now observe that

$$V(x_0) \leq U(x_0) + \theta = U(x_0) + h(x_0) - U(x_0) = h(x_0) \leq V(x_0).$$

Hence $x_0 \in \Gamma$. However, (3.5) implies that $U(x_0) = V(x_0) = h(x_0)$, i.e. $\theta = h(x_0) - U(x_0) = 0$, contradicting the assumption $\theta > 0$. Therefore, $U \geq h$ on $[c, d]$. $\qquad\square$

Let W and H be as defined in Proposition 3.3. In Remark 3.1, we point out that W is the value function of Optimal Stopping problem (3.3) with reward function H. If h is continuous on $[c, d]$, then H is continuous on the closed bounded interval $[S(c), S(d)]$. Therefore, we can talk about the optimal stopping region $\widetilde{\Gamma} \triangleq \{y \in [S(c), S(d)] : W(y) = H(y)\}$ of (3.3).

Lemma 3.2. $\Gamma = S^{-1}\big(\widetilde{\Gamma}\big).$

Proof. By Proposition 3.3, we have

$$\Gamma = \{x \in [c, d] : V(x) = h(x)\} = \{x \in [c, d] : W\big(S(x)\big) = H\big(S(x)\big)\}$$
$$= \Big\{x \in [c, d] : S(x) \in \big\{y \in [S(c), S(d)] : W(y) = H(y)\big\}\Big\} = S^{-1}\big(\widetilde{\Gamma}\big). \quad \square$$

Let $\widetilde{\mathbf{C}} \triangleq [S(c), S(d)] \setminus \widetilde{\mathbf{\Gamma}}$. Since H and W are continuous, $\widetilde{\mathbf{C}}$ is open relative to $[S(c), S(d)]$. Therefore $\widetilde{\mathbf{C}}$ is the union of a countable family, $\{\widetilde{J}_\alpha\}_{\alpha \in \widetilde{\Lambda}}$, of disjoint open (relative to $[S(c), S(d)]$) subintervals of $[S(c), S(d)]$.

Lemma 3.2 implies that $\left\{ J_\alpha \triangleq S^{-1}(\widetilde{J}_\alpha) \right\}_{\alpha \in \widetilde{\Lambda}}$ is the countable family of disjoint open (relative to $[c, d]$) subintervals on $[c, d]$ whose union gives $\mathbf{C}$, as discussed in the proof of Proposition 3.5. In particular, $(l, r) \in \widetilde{\mathbf{C}}$ for some $l, r \in \widetilde{\mathbf{\Gamma}}$ if and only if $(S^{-1}(l), S^{-1}(r)) \subseteq \mathbf{C}$ and $S^{-1}(l) \in \mathbf{\Gamma}$, $S^{-1}(r) \in \mathbf{\Gamma}$. Similarly, $[S(c), r) \in \widetilde{\mathbf{C}}$ for some $r \in \widetilde{\mathbf{\Gamma}}$ if and only if $[c, S^{-1}(r)) \subseteq \mathbf{C}$ and $S^{-1}(r) \in \mathbf{\Gamma}$.

PROOF OF PROPOSITION 3.1. For the proof of sufficiency, if $x \in [l, r] \subseteq [c, d]$ and we set $\tau = \tau_l \wedge \tau_r$ in (3.2), Lemma 3.1 gives

$$U(x) \geq \mathbb{E}_x[U(X_{\tau_l \wedge \tau_r})] = U(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + U(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)}.$$

Thus, U is S–concave on $[c, d]$.

To prove the necessity, suppose $U : [c, d] \to [0, +\infty)$ is S–concave on $[c, d]$; then it is enough to show

$$U(x) \geq \mathbb{E}_x[U(X_t)], \quad \forall\, x \in [c, d],\ \forall\, t \geq 0. \tag{3.12}$$

Indeed, observe that for every $x \in [c, d]$ and $s, t \geq 0$, the inequality (3.12) and the Markov property of X imply

$$U(X_s) \geq \mathbb{E}_{X_s}[U(X_t)] = \mathbb{E}_x\big[U(X_{t+s}) \big| \sigma(X_u : 0 \leq u \leq s)\big], \qquad \text{a.s.}$$

Hence, $\{U(X_t)\}_{t \in [0, +\infty)}$ is a nonnegative supermartingale, and (3.2) follows then from Optional Sampling Theorem.

To prove (3.12), let us first show

$$U(x) \geq \mathbb{E}_x[U(X_{\rho \wedge t})], \quad \forall\, x \in [c, d],\ \forall\, t \geq 0. \tag{3.13}$$

where the stopping time $\rho \triangleq \tau_c \wedge \tau_d$ is the exit time of X from (c, d).

First note that, since $x = c$ and $x = d$ are absorbing, i.e. $\mathbb{P}_x\{\rho = 0\} = 1$, for $x \in \{c, d\}$, we have $E_x[U(X_{\rho \wedge t})] = U(x)$, for every $t \geq 0$ and $x \in \{c, d\}$. Hence (3.13) holds for $x = c$ and $x = d$.

Now fix any $x_0 \in (c, d)$. Since U is S–concave on $[c, d]$, Proposition A.6(ii) shows that there exists a function $L : [c, d] \to \mathbb{R}$ in the form of

$$L(x) = c_1 S(x) + c_2, \quad \text{for all } x \in [c, d],$$

where c_1 and c_2 are constants such that

$$L(x_0) = U(x_0), \quad \text{and,} \quad L(x) \geq U(x), \quad \forall \, x \in [c, d].$$

Thus, for any $t \geq 0$, we have

$$\mathbb{E}_{x_0}[U(X_{\rho \wedge t})] \leq \mathbb{E}_{x_0}[L(X_{\rho \wedge t})] = \mathbb{E}_{x_0}[c_1 S(X_{\rho \wedge t}) + c_2] = c_1 \mathbb{E}_{x_0}[S(X_{\rho \wedge t})] + c_2$$

Since $S(\cdot)$ is continuous on the closed and bounded interval $[c, d]$, and the process $S(X_t)$ is a continuous local martingale, the *stopped* process $\{S(X_{\rho \wedge t}), \, t \geq 0\}$ is a bounded martingale, and

$$\mathbb{E}_{x_0}[S(X_{\rho \wedge t})] = S(x_0), \quad \text{for every } t \geq 0.$$

Therefore,

$$\mathbb{E}_{x_0}[U(X_{\rho \wedge t})] \leq c_1 \mathbb{E}_{x_0}[S(X_{\rho \wedge t})] + c_2 = c_1 S(x_0) + c_2 = L(x_0) = U(x_0).$$

This proves (3.13). We can finally show (3.12). Since $X_t = X_\sigma$ on $\{t \geq \sigma\}$, (3.13) implies that, for every $x \in [c, d]$ and $t \geq 0$, we have $\mathbb{E}_x[U(X_t)] = \mathbb{E}_x[U(X_{\rho \wedge t})] \leq U(x)$. $\square$

PROOF OF PROPOSITION 3.2. Since $\tau \equiv \infty$ and $\tau \equiv 0$ are stopping times, we have $V \geq 0$ and $V \geq h$, respectively. Hence V is nonnegative and majorizes h. To show that V is S–concave, we shall fix some $x \in [l, r] \subseteq [c, d]$. Since h is bounded,

V is finite on $[c, d]$. Therefore, for any arbitrarily small $\varepsilon > 0$, we can find stopping times σ_l and σ_r such that

$$\mathbb{E}_y[h(X_{\sigma_y})] \geq V(y) - \varepsilon, \quad y = l, r.$$

Define a new stopping time

$$\tau \triangleq \begin{cases} \tau_l + \sigma_l \circ \theta_{\tau_l}, & \text{on } \{\tau_l < \tau_r\}, \\[2mm] \tau_r + \sigma_r \circ \theta_{\tau_r}, & \text{on } \{\tau_l > \tau_r\}, \end{cases}$$

where θ_t is the shift operator (see Ito and McKean [5], Karatzas and Shreve [7]). Using strong Markov property of X and Lemma 3.1, we obtain

$$V(x) \geq \mathbb{E}_x[h(X_\tau)] = \mathbb{E}_x\big[h(X_\tau)1_{\{\tau_l<\tau_r\}}\big] + \mathbb{E}_x\big[h(X_\tau)1_{\{\tau_l>\tau_r\}}\big]$$

$$= \mathbb{E}_x\big[h(X_{\tau_l+\sigma_l\circ\theta_{\tau_l}})1_{\{\tau_l<\tau_r\}}\big] + \mathbb{E}_x\big[h(X_{\tau_r+\sigma_r\circ\theta_{\tau_r}})1_{\{\tau_l>\tau_r\}}\big]$$

$$= \mathbb{E}_x\big[1_{\{\tau_l<\tau_r\}}\theta_{\tau_l}h(X_{\sigma_l})\big] + \mathbb{E}_x\big[1_{\{\tau_l>\tau_r\}}\theta_{\tau_r}h(X_{\sigma_r})\big]$$

$$(\text{Since } \tau_x + \sigma_x \circ \theta_{\tau_x} \geq \tau_x, \ X_{\tau_x+\sigma_x\circ\theta_{\tau_x}} = X_{\sigma_x} \circ \theta_{\tau_x}, \ x = l, r)$$

$$= \mathbb{E}_x\Big[1_{\{\tau_l<\tau_r\}}\mathbb{E}_{X_{\tau_l}}\big[h(X_{\sigma_l})\big]\Big] + \mathbb{E}_x\Big[1_{\{\tau_l>\tau_r\}}\mathbb{E}_{X_{\tau_r}}\big[h(X_{\sigma_r})\big]\Big]$$

$$(\text{From the strong Markov property of } X)$$

$$= \mathbb{E}_l\big[h(X_{\sigma_l})\big]\mathbb{P}_x\{\tau_l < \tau_r\} + \mathbb{E}_r\big[h(X_{\sigma_r})\big]\mathbb{P}_x\{\tau_l > \tau_r\}$$

$$= \mathbb{E}_l\big[h(X_{\sigma_l})\big]\frac{S(r) - S(x)}{S(r) - S(l)} + \mathbb{E}_r\big[h(X_{\sigma_r})\big]\frac{S(x) - S(l)}{S(r) - S(l)}$$

$$\geq \big[V(l) - \varepsilon\big]\frac{S(r) - S(x)}{S(r) - S(l)} + \big[V(r) - \varepsilon\big]\frac{S(x) - S(l)}{S(r) - S(l)}$$

$$= V(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + V(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that V is indeed a nonnegative S–concave majorant of h on $[c, d]$. To complete the proof of Proposition, we have to show that V is the smallest such function.

Let $U : [c, d] \to \mathbb{R}$ be any other nonnegative S–concave majorant of h on $[c, d]$. Then, Proposition 3.1 implies

$$U(x) \geq \mathbb{E}_x[U(X_\tau)] \geq \mathbb{E}_x[h(X_\tau)],$$

for every $x \in [c, d]$ and every stopping time τ. Therefore $U \geq V$ on $[c, d]$. This completes the proof. $\square$

PROOF OF PROPOSITION 3.3. Let $\widehat{V}(x) \triangleq W\big(S(x)\big)$, $x \in [c, d]$. Since W is nonnegative, and majorizes H on $[S(c), S(d)]$, $\widehat{V}$ is nonnegative, and

$$\widehat{V}(x) = W\big(S(x)\big) \geq H\big(S(x)\big) = h(x), \quad , \forall x \in [c, d],$$

i.e. $\widehat{V}$ majorizes h on $[c, d]$. Furthermore, if $x \in [l, r] \subseteq [c, d]$, $l < r$, then $S(l) < S(r)$, $S(x) \in [S(l), S(r)] \subseteq [S(c), S(d)]$, and

$$
\begin{aligned}
\widehat{V}(l) \cdot &\frac{S(r) - S(x)}{S(r) - S(l)} + \widehat{V}(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)} \\
&= W\big(S(l)\big) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + W\big(S(r)\big) \cdot \frac{S(x) - S(l)}{S(r) - S(l)} \leq W\big(S(x)\big) = \widehat{V}(x),
\end{aligned}
$$

since W is concave on $[S(c), S(d)]$. Hence $\widehat{V}$ is a nonnegative concave majorant of h on $[c, d]$. Therefore Proposition 3.2 implies $\widehat{V} \geq V$.

Next define $\widehat{W} : [S(c), S(d)] \to \mathbb{R}$ by $\widehat{W}(y) \triangleq V\big(S^{-1}(y)\big)$. Since V is nonnegative and majorizes h on $[c, d]$, $\widehat{W}$ is also nonnegative, and

$$\widehat{W}(y) = V\big(S^{-1}(y)\big) \geq h\big(S^{-1}(y)\big) = H(y), \; y \in [S(c), S(d)],$$

i.e. $\widehat{W}$ majorizes H on $[S(c), S(d)]$. Furthermore, if $y \in [L, R] \subseteq [S(c), S(d)]$ for some $L < R$, then $l \triangleq S^{-1}(L) < r \triangleq S^{-1}(R)$, $x \triangleq S^{-1}(y) \in [l, r] \subseteq [c, d]$, and

$$
\begin{aligned}
\widehat{W}(L) \cdot &\frac{R - y}{R - L} + \widehat{W}(R) \cdot \frac{y - L}{R - L} \\
&= V(l) \cdot \frac{S(r) - S(x)}{S(r) - S(l)} + V(r) \cdot \frac{S(x) - S(l)}{S(r) - S(l)} \leq V(x) = V\big(S(y)\big) = \widehat{W}(y),
\end{aligned}
$$

since, by Proposition 3.2, V is S–concave on $[c, d]$. Hence $\widehat{W}$ is a nonnegative concave majorant of H on $[S(c), S(d)]$. Since W is the smallest of such functions, we have $\widehat{W} \geq W$ on $[S(c), S(d)]$. Therefore, $V(x) = V\big(S^{-1}(y)\big) = \widehat{W}(y) \geq W(y) = W\big(S(x)\big) = \widehat{V}(x)$, for every $x \in [c, d]$. Together with the opposite inequality shown above, this proves $V(x) = \widehat{V}(x) = W\big(S(x)\big)$, for every $x \in [c, d]$. $\qquad\square$

PROOF OF PROPOSITION 3.4. Since S is continuous on $[c, d]$, and V is S–concave on $[c, d]$, Proposition A.1 implies that V is continuous in (c, d), and

$$V(c) \leq \liminf_{x \downarrow c} V(x), \text{ and } V(d) \leq \liminf_{x \uparrow d} V(x). \tag{3.14}$$

Since h is bounded on the closed bounded interval $[c, d]$, V is continuous in (c, d), and we already have (3.14) at the boundaries. To prove V is also continuous at c and d, it is enough to show

$$V(c) \geq \limsup_{x \downarrow c} V(x), \text{ and } V(d) \geq \limsup_{x \uparrow d} V(x).$$

Look at c first. Let $\widetilde{h}(y) \triangleq \max_{z \in [c, y]} h(y)$, for every $y \in [c, d]$. Since h is uniformly continuous on the closed bounded interval $[c, d]$, $\widetilde{h}$ is also continuous on $[c, d]$. Fix $c < y < d$. For any $x \in (c, y)$ and optimal stopping time τ, we have

$$\{X_\tau > y\} \subseteq \{\tau_y < \tau\} \subseteq \{\tau_y < \tau_c\}, \quad \text{on } \{X_0 = x\}.$$

Since $h(X_\tau) \equiv 0$ on $\{\tau = \infty\}$ by convention, we have

$$\mathbb{E}_x[h(X_\tau)] = \mathbb{E}_x\big[h(X_\tau)1_{\{X_\tau \leq y\}}\big] + \mathbb{E}_x\big[h(X_\tau)1_{\{X_\tau > y\}}\big] \leq \widetilde{h}(y) + \widetilde{h}(d)\mathbb{P}_x(\tau_y < \tau_c)$$

$$= \widetilde{h}(y) + \widetilde{h}(d) \cdot \frac{S(x) - S(c)}{S(y) - S(c)} \leq \big(0 \vee \widetilde{h}(y)\big) + \big(0 \vee \widetilde{h}(d)\big) \cdot \frac{S(x) - S(c)}{S(y) - S(c)}.$$

Since the right–hand side no longer depends on τ, we further get

$$V(x) \leq \big(0 \vee \widetilde{h}(y)\big) + \big(0 \vee \widetilde{h}(d)\big) \cdot \frac{S(x) - S(c)}{S(y) - S(c)}.$$

Since S is continuous at c, we obtain by taking limit supremum of both sides as x tends to c

$$\limsup_{x \downarrow c} V(x) \le 0 \vee \widetilde{h}(y), \quad \forall y \in (c, d).$$

Finally, by letting y tend to c, we get $\limsup_{x \downarrow c} V(x) \le 0 \vee \widetilde{h}(c) = 0 \vee h(c)$ since $\widetilde{h}$ is continuous on $[c, d]$. However, since c is absorbing, i.e. $\mathbb{P}_c(X_t = c, \forall t \ge 0) = 1$, we have $V(c) = 0 \vee h(c)$. This proves that V is continuous at c. Continuity of $V(\cdot)$ at d can be shown similarly. □

Chapter 4

Discounted Optimal Stopping

In this chapter, we shall study the *discounted* optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta\tau}h(X_\tau)], \qquad x \in [c,d], \tag{4.1}$$

with $\beta > 0$, where the diffusion process X and the reward function $h(\cdot)$ have the same properties as described in Chapter 3. Namely, X is started in a bounded closed interval $[c,d]$ contained in the interior of its state space $\mathcal{I}$, and is absorbed whenever it reaches c or d. Moreover, $h : [c,d] \to \mathbb{R}$ is a bounded Borel function such that $\sup_{x \in [c,d]} h(x) > 0$ (if $h \leq 0$ everywhere, then $\tau \equiv +\infty$ is trivially optimal, and $V \equiv 0$).

In order to motivate the key result of Proposition 4.1, let $U : [c,d] \to \mathbb{R}$ be a β–excessive function with respect to X. Namely, for every stopping time τ of X, and $x \in [c,d]$, we have $U(x) \geq \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)]$. We shall take any subinterval $[l,r]$ of $[c,d]$, and look closer to the same inequality for any $x \in [l,r]$ and for *the exit time* $\tau = \tau_l \wedge \tau_r$ of X from $[l,r]$. Since X is regular, we have

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\tau_l \wedge \tau_r)}U(\tau_l \wedge \tau_r)]$$

$$= U(l) \cdot \mathbb{E}_x[e^{-\beta\tau_l}1_{\{\tau_l < \tau_r\}}] + U(r) \cdot \mathbb{E}_x[e^{-\beta\tau_r}1_{\{\tau_l > \tau_r\}}], \quad x \in [l,r].$$

One can argue that $u_1(x) \triangleq \mathbb{E}_x[e^{-\beta\tau_l}1_{\{\tau_l < \tau_r\}}]$ and $u_2(x) \triangleq \mathbb{E}_x[e^{-\beta\tau_r}1_{\{\tau_l > \tau_r\}}]$ are the

unique solutions of $\mathcal{A}u = \beta u$ in (l, r), with the boundary conditions $u_1(l) = 1$, $u_1(r) = 0$ and $u_2(l) = 0$, $u_2(r) = 1$, respectively. From Chapter 2, let us recall that the second order differential equation $\mathcal{A}u = \beta u$ has two positive, linearly independent solutions $\psi(\cdot)$ and $\varphi(\cdot)$ on $\mathcal{I}$, which are strictly increasing and strictly decreasing, respectively. Using the boundary conditions, one calculates

$$u_1(x) = \frac{\psi(x)\varphi(r) - \psi(r)\varphi(x)}{\psi(l)\varphi(r) - \psi(r)\varphi(l)}, \quad u_2(x) = \frac{\psi(l)\varphi(x) - \psi(x)\varphi(l)}{\psi(l)\varphi(r) - \psi(r)\varphi(l)}, \quad x \in [l, r].$$

Substituting these into the inequality above, then dividing both sides of the inequality by $\varphi(x)$ (respectively, $\psi(x)$), we obtain

$$\frac{U(x)}{\varphi(x)} \geq \frac{U(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{U(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)} \qquad x \in [l, r], \qquad (4.2)$$

and

$$\frac{U(x)}{\psi(x)} \geq \frac{U(l)}{\varphi(l)} \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{U(r)}{\varphi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}, \qquad x \in [l, r], \qquad (4.3)$$

respectively, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad \text{and} \quad G(x) \triangleq -\frac{1}{F(x)} = -\frac{\varphi(x)}{\psi(x)}, \quad x \in [c, d]. \qquad (4.4)$$

The way we chose c and d guarantees that $\psi(\cdot)$ and $\varphi(\cdot)$ never vanish on $[c, d]$, so $F(\cdot)$ and $G(\cdot)$ are well-defined and strictly increasing. Therefore, we can talk about $F-$ and G–concave functions (cf. Appendix A) on $[c, d]$. Observe now that the inequalities (4.2) and (4.3) imply that $\frac{U(\cdot)}{\varphi(\cdot)}$ and $\frac{U(\cdot)}{\psi(\cdot)}$ are $F-$ and G–concave on $[c, d]$, respectively. In Proposition 4.1 below, we shall show that *the converse is also true*.

It is worth pointing out the correspondence between the roles of $S(\cdot)$ and 1 in the *undiscounted* optimal stopping, and the roles of $\psi(\cdot)$ and $\varphi(\cdot)$ in the *discounted* optimal stopping. The division between the pairs $(S(\cdot), 1)$ and $(\psi(\cdot), \varphi(\cdot))$ is, in fact, artificial: Both pairs consist of an increasing and a decreasing solution of the second order differential equation, $\mathcal{A}u = \beta u$ in $\mathcal{I}$, for the undiscounted (i.e. $\beta = 0$) and the

discounted (i.e. $\beta > 0$) versions of the same optimal stopping problems, respectively. Therefore, the results of Chapter 3 can be restated and proved with only minor (and obvious) changes.

The key result of the chapter is

Proposition 4.1 (The characterization of β–excessive functions). *A function $U :$ $[c, d] \to \mathbb{R}$ is nonnegative, and $\frac{U(\cdot)}{\varphi(\cdot)}$ is an F–concave function (equivalently, $\frac{U(\cdot)}{\psi(\cdot)}$ is a G–concave function), if and only if $U(\cdot)$ is β–excessive, i.e.,*

$$U(x) \geq \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)] \tag{4.5}$$

holds for every $x \in [c, d]$ and every stopping time τ of X.

We almost immediately conclude from this the following

Proposition 4.2 (The characterization of the value function). *The value function $V(\cdot)$ of (4.1) is the smallest nonnegative majorant of $h(\cdot)$ such that*

$$\frac{V(\cdot)}{\varphi(\cdot)} \text{ is } F\text{–concave } \left(equivalently, \frac{V(\cdot)}{\psi(\cdot)} \text{ is } G\text{–concave} \right), \quad on \ [c, d],$$

in the sense that, if $U(\cdot)$ is another function with the same properties, then $U \geq V$.

The equivalence of the characterizations, in Proposition 4.1 and Proposition 4.2 in terms of F and G, follows from

Lemma 4.1. *Let $U : [c, d] \to \mathbb{R}$ any function. $\frac{U}{\varphi}$ is F–concave on $[c, d]$ if and only if $\frac{U}{\psi}$ is G–concave on $[c, d]$.*

Proof. Follows from the definition of concave functions. □

Since is hard to visualize the nonnegative F– or G–concave majorant of a function geometrically, it will again be nice to describe them in terms of ordinary concave functions. We have the following result:

Proposition 4.3. *Let W be the smallest nonnegative concave majorant of*

$$H \triangleq \left(\frac{h}{\varphi}\right) \circ F^{-1} : [F(c), F(d)] \to \mathbb{R} \tag{4.6}$$

on $[F(c), F(d)]$, where $F^{-1}(\cdot)$ is the inverse of the strictly increasing function $F(\cdot)$.
Then

$$V(x) = \varphi(x) \, W\big(F(x)\big), \quad x \in [c, d]. \tag{4.7}$$

Similarly, let $\widetilde{W}$ be the smallest nonnegative concave majorant of

$$\widetilde{H} \triangleq \left(\frac{h}{\psi}\right) \circ G^{-1} : [G(c), G(d)] \to \mathbb{R} \tag{4.8}$$

on $[G(c), G(d)]$. Since $G(\cdot)$ is strictly increasing, $G^{-1}(\cdot)$ is also well-defined. Then

$$V(x) = \psi(x) \, \widetilde{W}\big(G(x)\big), \quad x \in [c, d]. \tag{4.9}$$

Proof. We sketch the proof of the first part only. Define $U : [c, d] \to \mathbb{R}$ as $U(x) \triangleq \varphi(x)W\big(F(x)\big)$. Since $W(\cdot)$ is a nonnegative, concave majorant of $H(\cdot)$, $U(\cdot)$ turns out to be a nonnegative majorant of $h(\cdot)$, such that $\frac{U(\cdot)}{\varphi(\cdot)}$ is F–concave on $[c, d]$. Proposition 4.2 therefore implies $U \geq V$ on $[c, d]$.

Next define $\widehat{W} : [F(c), F(d)] \to \mathbb{R}$ by

$$\widehat{W}(y) \triangleq \left(\frac{V}{\varphi}\right) \circ F^{-1}(y).$$

Using the properties of $V(\cdot)$ as stated in Proposition 4.2, it is straight–forward to show that $\widehat{W}(\cdot)$ is a nonnegative concave majorant of $H(\cdot)$ on $[F(c), F(d)]$. Therefore, we have $\widehat{W} \geq W$. This now implies the opposite inequality

$$U(x) = \varphi(x)W\big(F(x)\big) \leq \varphi(x)\widehat{W}\big(F(x)\big) = \varphi(x)\frac{V(x)}{\varphi(x)} = V(x).$$

Proof of the second part is similar. $\square$

Remark 4.1. Let B be a one–dimensional standard Brownian motion in $[F(c), F(d)]$ with absorbing boundaries. Let W and H be defined as in Proposition 4.3. By Proposition 3.2 of Chapter 3, we have

$$W(y) \equiv \sup_{\tau \geq 0} \mathbb{E}_y[H(B_\tau)], \quad y \in [F(c), F(d)]. \tag{4.10}$$

Similarly, if $\widetilde{B}$ is the one-dimensional standard Brownian Motion in $[G(c), G(d)]$ with absorbing boundaries, and $\widetilde{W}$ and $\widetilde{H}$ are as in Proposition 4.3, then we also have

$$\widetilde{W}(y) \equiv \sup_{\tau \geq 0} \mathbb{E}_y[\widetilde{H}(\widetilde{B}_\tau)], \quad y \in [G(c), G(d)]. \tag{4.11}$$

As we already pointed out in Remark 3.1, there is essentially only one class of optimal stopping problems, namely *the class of undiscounted optimal stopping problems for Brownian motion.*

Note that F is continuous on $[c, d]$. Since $\frac{V}{\varphi}$ is F–concave on $[c, d]$, Proposition A.1 implies that $\frac{V}{\varphi}$ is continuous in (c, d) and

$$\frac{V(c)}{\varphi(c)} \leq \liminf_{x \downarrow c} \frac{V(x)}{\varphi(x)}, \quad \text{and,} \quad \frac{V(d)}{\varphi(d)} \leq \liminf_{x \uparrow d} \frac{V(x)}{\varphi(x)}.$$

Because φ is itself continuous on $[c, d]$, we conclude that V is continuous in (c, d) and

$$V(c) \leq \liminf_{x \downarrow c} V(x), \quad \text{and,} \quad V(d) \leq \liminf_{x \uparrow d} V(x). \tag{4.12}$$

Lemma 4.2. *If h is continuous on $[c, d]$, then V is also continuous on $[c, d]$.*

Proof. Similar to that of Proposition 3.4. □

We shall next characterize the optimal stopping rule. Define

$$\boldsymbol{\Gamma} \triangleq \{x \in [c, d] : V(x) = h(x)\}, \quad \text{and,} \quad \tau^* \triangleq \inf\{t \geq 0 : X_t \in \boldsymbol{\Gamma}\}. \tag{4.13}$$

We shall need the following

Lemma 4.3. *Let* $\tau_r \triangleq \inf\{t \geq 0 : X_t = r\}$. *Then for every* $c \leq l < x < r \leq d$,

$$\mathbb{E}_x[e^{-\beta(\tau_l \wedge \tau_r)} h(X_{\tau_l \wedge \tau_r})] = \varphi(x)\left[\frac{h(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)}\right],$$

$$= \psi(x)\left[\frac{h(l)}{\psi(l)} \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{h(r)}{\psi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}\right].$$

Furthermore,

$$\mathbb{E}_x[e^{-\beta\tau_r} h(X_{\tau_r})] = \varphi(x)\frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(c)}{F(r) - F(c)} = \psi(x)\frac{h(r)}{\psi(r)} \cdot \frac{G(x) - G(c)}{G(r) - G(c)},$$

and

$$\mathbb{E}_x[e^{-\beta\tau_l} h(X_{\tau_l})] = \varphi(x)\frac{h(l)}{\varphi(l)} \cdot \frac{F(d) - F(x)}{F(d) - F(l)} = \psi(x)\frac{h(l)}{\psi(l)} \cdot \frac{G(d) - G(x)}{G(d) - G(l)}.$$

Proof. We have

$$\mathbb{E}_x[e^{-\beta(\tau_l \wedge \tau_r)} h(X_{\tau_l \wedge \tau_r})] = h(l)\mathbb{E}_x[e^{-\beta\tau_l} 1_{\{\tau_l < \tau_r\}}] + h(r)\mathbb{E}_x[e^{-\beta\tau_r} 1_{\{\tau_l > \tau_r\}}]$$

$$= h(l)\widetilde{\mathbb{E}}_x[e^{-\beta\widetilde{\tau}_l}] + h(r)\widetilde{\mathbb{E}}_x[e^{-\beta\widetilde{\tau}_r}],$$

where $\widetilde{\mathbb{E}}$ is the expected value under the probability measure induced by the finite-dimensional distribution of the stopped process $\widetilde{X}_t \triangleq X_{\tau_l \wedge \tau_r \wedge t}$ in $[l, r]$, and $\widetilde{\tau}_r$ are defined with respect to this stopped process. The last equality follows from the fact that both X and $\widetilde{X}$ are governed by the same dynamics in (l, r). By the same token, X and $\widetilde{X}$ have the same infinitesimal generator $\mathcal{A}$ in (l, r). Two processes are distinguished by the boundary conditions posed on the elements of the domain of generator for $\widetilde{X}$.

If we denote the increasing and decreasing solutions of $\mathcal{A}u = \beta u$, subject to the boundary conditions that uniquely determine $\widetilde{X}$, by $\widetilde{\psi}(\cdot, l)$ and $\widetilde{\varphi}(\cdot, r)$, then we must have $\widetilde{\psi}(l, l) = \widetilde{\varphi}(r, r) = 0$. Since ψ and φ spans all the solutions of $\mathcal{A}u = \beta u$, one can check that

$$\widetilde{\psi}(x, l) = \psi(x) - \varphi(x)\frac{\psi(l)}{\varphi(l)}, \quad \text{and,} \quad \widetilde{\varphi}(x, r) = \varphi(x) - \psi(x)\frac{\varphi(r)}{\psi(r)}, \quad \forall x \in [l, r].$$

According to Chapter 2, we have

$$\widetilde{\mathbb{E}}_x[e^{-\beta\widetilde{\tau}_l}] = \frac{\widetilde{\varphi}(x,r)}{\widetilde{\varphi}(l,r)}, \quad \text{and,} \quad \widetilde{\mathbb{E}}_x[e^{-\beta\widetilde{\tau}_r}] = \frac{\widetilde{\psi}(x,l)}{\widetilde{\psi}(r,l)}, \quad \forall x \in [l,r].$$

Therefore

$$\mathbb{E}_x[e^{-\beta(\tau_l\wedge\tau_r)}h(X_{\tau_l\wedge\tau_r})] = h(l)\frac{\varphi(x) - \psi(x)\frac{\varphi(r)}{\psi(r)}}{\varphi(l) - \psi(l)\frac{\varphi(r)}{\psi(r)}} + h(r)\frac{\psi(x) - \varphi(x)\frac{\psi(l)}{\varphi(l)}}{\psi(r) - \varphi(r)\frac{\psi(l)}{\varphi(l)}}.$$

Rearranging the terms will finally give the first two expressions of the Lemma. The remaining identities can be proved similarly. □

Proposition 4.4. *If h is continuous on $[c,d]$, then τ^* of (4.13) is an optimal stopping rule.*

Proof. Define $U(x) \triangleq \mathbb{E}_x[e^{-\beta\tau^*}h(X_{\tau^*})]$, for every $x \in [c,d]$. We obviously have $V \geq U$. To show the reverse inequality, it is enough to prove that $\frac{U(\cdot)}{\varphi(\cdot)}$ is a nonnegative F–concave majorant of $\frac{h(\cdot)}{\varphi(\cdot)}$.

As in the proof of Proposition 3.4, one can show using Lemma 4.3 that $\frac{U(\cdot)}{\varphi(\cdot)}$ can be written as the lower envelope of a family of nonnegative F–concave functions, i.e. itself is nonnegative and F–concave. To show that it also majorizes $\frac{h(\cdot)}{\varphi(\cdot)}$, assume on the contrary that

$$\theta \triangleq \max_{x \in [c,d]} \left(\frac{h(x)}{\varphi(x)} - \frac{U(x)}{\varphi(x)} \right) > 0, \tag{4.14}$$

and arrive at a contradiction, namely $\theta = 0$, as in the proof of Proposition 3.4, by using the fact that $\frac{U(\cdot)}{\varphi(\cdot)} + \theta$ is a nonnegative, F–concave majorant of $\frac{h(\cdot)}{\varphi(\cdot)}$. □

In Remark 4.1, we point out the connection between two optimal stopping problems, our original problem and that of (4.10). It is therefore not surprising that their optimal stopping regions are also related.

If h is continuous on $[c,d]$, then H will be continuous on the closed bounded interval $[F(c), F(d)]$. Therefore the optimal stopping problem of (4.10) has an optimal stopping rule $\sigma^* \triangleq \{t \geq 0 : B_t \in \widetilde{\boldsymbol{\Gamma}}\}$, where

$$\widetilde{\boldsymbol{\Gamma}} \triangleq \{y \in [F(c), F(d)] : W(y) = H(y)\}$$

is the optimal stopping region of the same problem. $\mathbf{\Gamma}$ of (4.13) is related to $\widetilde{\mathbf{\Gamma}}$ as in the following

Corollary 4.1. *Suppose h is continuous on $[c,d]$. Let H and W be as in Proposition 4.3. Then $\mathbf{\Gamma} = F^{-1}(\widetilde{\mathbf{\Gamma}})$.*

Proof. Follows from Proposition 4.3 as in the proof of Proposition 3.2. $\qquad\qquad\square$

Observe that, since H and W are continuous, $\widetilde{\mathbf{C}} \triangleq [F(c), F(d)] \backslash \widetilde{\mathbf{\Gamma}}$ is open relative to $[F(c), F(d)]$. Therefore $\widetilde{\mathbf{C}}$ is a union of countably many disjoint intervals that are open relative to $[F(c), F(d)]$. Let $\left(\widetilde{J}_\alpha\right)_{\alpha \in \widetilde{\Lambda}}$ is the open cover of $\widetilde{\mathbf{C}}$ by its disjoint subintervals. Corollary implies that $\left(F^{-1}(\widetilde{J}_\alpha)\right)_{\alpha \in \widetilde{\Lambda}}$ is the open cover of $\mathbf{C}$ by its disjoint open (relative to $[c,d]$) intervals. In particular, if $\widetilde{J}_\alpha = (l, r) \subseteq [F(c), F(d))]$, and $l, r \in \widetilde{\mathbf{\Gamma}}$, then we have $\left(F^{-1}(l), F^{-1}(r)\right) \subseteq \mathbf{C}$ and $F^{-1}(l) \in \mathbf{\Gamma}$, $F^{-1}(r) \in \mathbf{\Gamma}$ since F is increasing.

We close this chapter with the proof of Proposition 4.1; that of Proposition 4.2 follows along similar lines of Proposition 3.2.

PROOF OF PROPOSITION 4.1. Sufficiency follows from Lemma 4.3 after a similar argument as in the proof of Proposition 3.2.

To prove the necessity, suppose U is nonnegative and $\frac{U}{\varphi}$ is F- concave on $[c,d]$. As in the proof of Proposition 3.1, thanks to the the strong Markov property of X and the optional sampling theorem for nonnegative supermartigales, it is enough to prove that

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\rho \wedge t)} U(X_{\rho \wedge t})], \quad x \in [c,d], \ t \geq 0, \tag{4.15}$$

where $\rho \triangleq \inf\{t \geq 0 : X_t \notin (c,d)\}$.

Observe that, if $x = c$ or $x = d$, then $\mathbb{P}_x\{\rho = 0\} = 1$, and, $\mathbb{E}_x[e^{-\beta(\rho \wedge t)} U(X_{\rho \wedge t})] = U(x)$, for every $t \geq 0$, i.e. (4.15) is true.

Next fix any $x \in (c, d)$. Since $\frac{U}{\varphi}$ is F–concave on $[c, d]$, Proposition A.6(ii) shows that there exists some function $L : [c, d] \to \mathbb{R}$ in the form of

$$L(y) \triangleq c_1 F(y) + c_2, \quad \forall y \in [c, d],$$

where c_1 and c_2 are constants such that

$$L(x) = \frac{U(x)}{\varphi(x)}, \quad \text{and,} \quad L(y) \geq \frac{U(y)}{\varphi(y)}, \ \forall y \in [c, d].$$

Now observe that

$$
\begin{aligned}
\mathbb{E}_x[e^{-\beta(\rho \wedge t)} U(X_{\rho \wedge t})] &= \mathbb{E}_x \left[e^{-\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t}) \frac{U(X_{\rho \wedge t})}{\varphi(X_{\rho \wedge t})} \right] \leq \mathbb{E}_x[e^{\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t}) L(X_{\rho \wedge t})] \\
&= \mathbb{E}_x \left[e^{-\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t}) \big(c_1 F(X_{\rho \wedge t}) + c_2 \big) \right] \\
&= c_1 \mathbb{E}_x[e^{-\beta(\rho \wedge t)} \psi(X_{\rho \wedge t})] + c_2 \mathbb{E}_x[e^{-\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t})], \quad \forall t \geq 0.
\end{aligned}
$$

Because $\psi(\cdot)$ and $\varphi(\cdot)$ are $C^2[c, d]$, we can apply Ito's Rule to $e^{-\beta t} \psi(X_t)$ and $e^{-\beta t} \varphi(X_t)$. Stochastic integrals become square-integrable martingales since their quadratic variation processes are integrable. Because $\mathcal{A}u = \beta u$, $u = \psi, \varphi$, we obtain

$$\mathbb{E}_x[e^{-\beta(\rho \wedge t)} \psi(X_{\rho \wedge t})] = \psi(x) + \mathbb{E}_x \left[\int_0^{\rho \wedge t} e^{-\beta s} (\mathcal{A} - \beta) \psi(X_s) ds \right] = \psi(x),$$

for every $t \geq 0$. Similarly, $\mathbb{E}_x[e^{-\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t})] = \varphi(x)$, for every $t \geq 0$. Therefore, we find

$$
\begin{aligned}
\mathbb{E}_x[e^{-\beta(\rho \wedge t)} U(X_{\rho \wedge t})] &\leq c_1 \mathbb{E}_x[e^{-\beta(\rho \wedge t)} \psi(X_{\rho \wedge t})] + c_2 \mathbb{E}_x[e^{-\beta(\rho \wedge t)} \varphi(X_{\rho \wedge t})] \\
&= c_1 \psi(x) + c_2 \varphi(x) = \varphi(x) \big(c_1 F(x) + c_2 \big) \\
&= \varphi(x) L(x) = \varphi(x) \frac{U(x)}{\varphi(x)} = U(x).
\end{aligned}
$$

This proves (4.15). $\qquad\square$

Chapter 5

Boundaries and Optimal Stopping

In Chapter 3 and Chapter 4, we assumed that the process is allowed to diffuse in a closed and bounded interval and is stopped when it reaches the boundaries. There are many other interesting cases, where the state space may not be a compact subinterval of $\mathbb{R}$, or the behavior of the process is different near the boundaries.

It is always possible to prove that the value function $V(\cdot)$ must satisfy the properties of Proposition 3.2 or Proposition 4.2. Additional necessary conditions on $V(\cdot)$ appear, if one or more boundaries are regular reflecting (for example, $V(\cdot)$ of Chapter 3 should be non–increasing if c is reflecting, and non–decreasing if d is reflecting).

The challenge is to show that $V(\cdot)$ is the smallest function with these necessary conditions. Proposition 3.1 and Proposition 4.1 meet this challenge when the boundaries are absorbing. Their proofs illustrate the key tools. Observe that the local martingales, $S(X_t)$ and the constant 1 of Chapter 3, and $e^{-\beta t}\psi(X_t)$ and $e^{-\beta t}\varphi(X_t)$ of Chapter 4, are fundamental in the proofs of sufficiency.

One can almost always show that the concavity of the appropriate quotient of some nonnegative function $U(\cdot)$ with respect to a quotient of monotone fundamental solutions of $\mathcal{A}u = \beta u$, $\beta \geq 0$, implies that $U(\cdot)$ is β–excessive. The main tools in this effort are Itô's rule, the localization of local martingales, the lower semi–

continuity of $U(\cdot)$ (usually implied by concavity of some sort), and Fatou's Lemma. Different boundary conditions may necessitate additional care to complete the proof of superharmonicity.

We shall not attempt to formulate a general theorem that covers all cases. In this chapter, we state and prove the key propositions for a diffusion process with absorbing and/or natural boundaries. We shall illustrate how the propositions look like, and what additional tools we may need, to overcome potential difficulties with the boundaries.

5.1 Left–boundary is absorbing, and right–boundary is natural.

Suppose the right boundary $b \leq \infty$ of the state–space $\mathcal{I}$ of X is natural. Let $c \in \text{int}(\mathcal{I})$. Note that the process X, starting in (c, b), reaches c in finite time with positive probability. Consider the stopped process X_t, which starts in $[c, b)$, and is stopped when it reaches c.

Finally, let $\psi(\cdot)$ and $\varphi(\cdot)$ be the increasing and decreasing fundamental solutions of $\mathcal{A}u = \beta u$ in $\mathcal{I}$, for some constant $\beta > 0$ (cf. Chapter 2). Since $c \in \text{int}(\mathcal{I})$, we have $0 < \psi(c) < \infty$, $0 < \varphi(c) < \infty$. Because b is natural, we have $\psi(b-) = \infty$ and $\varphi(b-) = 0$.

Let the reward function $h : [c, b) \to \mathbb{R}$ be bounded on every compact subset of $[c, b)$. Finally, define

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta \tau} h(X_\tau)], \quad x \in [c, b).$$

Let $(b_n)_{n \geq 1} \subset [c, b)$ be an increasing sequence such that $b_n \to b$ as $n \to \infty$. Define the stopping times

$$\sigma_n \triangleq \inf\{t \geq 0 : X_t \notin (c, b_n)\}, \ n \geq 1; \ \text{ and } \ \sigma \triangleq \inf\{t \geq 0 : X_t \notin (c, b)\}.$$

Note that $\sigma_n \uparrow \sigma$ as $n \to \infty$. Since b is a natural boundary, we in fact have $\sigma = \inf\{t \geq 0 : X_t = c\}$ almost surely. We can now state and prove the key

Proposition 5.1. *A function $U : [c, b) \to \mathbb{R}$ is nonnegative, and $\frac{U(\cdot)}{\psi(\cdot)}$ is G–concave on $[c, b)$ if and only if*

$$U(x) \geq \mathbb{E}_x[e^{-\beta\tau}U(X_\tau)], \tag{5.1}$$

for every $x \in [c, b)$ and every stopping time τ of X.

Proof. Sufficiency follows from (5.1) and Lemma 4.3 when we let τ be 0, ∞, and $\tau_l \wedge \tau_r$, for every choice of $x \in [l, r] \subset [c, b)$.

For the necessity, we only have to show that

$$U(x) \geq \mathbb{E}_x[e^{-\beta t}U(X_t)], \quad x \in [c, b), \ t \geq 0. \tag{5.2}$$

Indeed, this and strong Markov property of X imply that $e^{-\beta t}U(X_t)$ is a nonnegative supermartingale, and (5.1) follows from the Optional Sampling Theorem for nonnegative supermartingales. As in the proof of Proposition 4.1, we first prove a simpler version of (5.2), namely

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\sigma \wedge t)}U(X_{\sigma \wedge t})], \quad x \in [c, b), \ t \geq 0. \tag{5.3}$$

The main reason was that the behavior of the process by time σ of reaching the boundaries is completely determined by the infinitesimal generator $\mathcal{A}$ of the process. We can therefore use Ito's rule without worrying about what happens after the process reaches the boundaries.

Let $(b_n)_{n \geq 1}$ and $(\sigma_n)_{n \geq 1}$ be as defined at page 32. We shall take one step further and start with showing

$$U(x) \geq \mathbb{E}_x[e^{-\beta(\sigma_n \wedge t)}U(X_{\sigma_n \wedge t})], \quad x \in [c, b), \ t \geq 0, \ n \geq 1. \tag{5.4}$$

Fix $n \geq 1$. If $x \notin (c, b_n)$, then $\mathbb{P}_x\{\sigma_n = 0\} = 1$. Therefore $\mathbb{E}_x[e^{-\beta(\sigma_n \wedge t)}U(X_{\sigma_n \wedge t})] = U(x)$, i.e. (5.4) holds for $x \notin (c, b_n)$.

Next, fix $x_0 \in (c, b)$. Observe that $X_{\sigma_n \wedge t}$ lives in the closed bounded interval $[c, b_n]$ contained in the interior of $\mathcal{I}$. Furthermore, c and b_n are absorbing for $X_{\sigma_n \wedge t}$. Using a similar argument as in the proof of Proposition 4.1, one can complete the proof of (5.4).

Since $G(\cdot)$ is continuous on $[c, b)$, and $\frac{U(\cdot)}{\psi(\cdot)}$ is G–concave on $[c, b)$, Proposition A.1 implies that U is lower semi–continuous on $[c, b)$, i.e. $\liminf_{y \to x} U(y) \geq U(x)$, for every $x \in [c, b)$. Because $\sigma_n \wedge t \to \sigma \wedge t$ and $X_{\sigma_n \wedge t} \to X_{\sigma \wedge t}$, as $n \to \infty$, we have

$$\mathbb{E}_x[e^{-\beta(\sigma \wedge t)} U(X_{\sigma \wedge t})] \leq \mathbb{E}_x\Big[\liminf_{n \to \infty} e^{-\beta(\sigma_n \wedge t)} U(X_{\sigma_n \wedge t})\Big]$$

$$\leq \liminf_{n \to \infty} \mathbb{E}_x\big[e^{-\beta(\sigma_n \wedge t)} U(X_{\sigma_n \wedge t})\big] \leq U(x),$$

where we use lower semi–continuity for the first inequality, nonnegativity of U and Fatou's Lemma for the second inequality, and (5.4) for the third inequality. This proves (5.3).

Finally, since c is absorbing, and $\sigma \equiv \inf\{t \geq 0 : X_t = c\}$, we have $X_t = X_\sigma = c$ on $t \geq \sigma$. Therefore (5.2) follows from (5.3) as in

$$\mathbb{E}_x[e^{-\beta t} U(X_t)] = \mathbb{E}_x[e^{-\beta t} U(X_{\sigma \wedge t})] \leq \mathbb{E}_x[e^{-\beta(\sigma \wedge t)} U(X_{\sigma \wedge t})] \leq U(x), \quad x \in [c, b), \ t \geq 0.$$

This completes the proof.   □

We shall first investigate when the value function is real–valued. It turns out that this is determined by the quantity

$$\ell_b \triangleq \limsup_{x \to b} \frac{h^+(x)}{\psi(x)} \in [0, +\infty], \tag{5.5}$$

where $h^+(\cdot) \triangleq \max\{0, h(\cdot)\}$ on $[c, b)$.

We shall first show that $V(x) = +\infty$ for every $x \in (c, b)$, if $\ell_b = +\infty$. To this end, fix any $x \in (c, b)$. Let $(r_n)_{n \in \mathbb{N}} \subset (x, b)$ be any strictly increasing sequence with limit b. Define the stopping times $\tau_{r_n} \triangleq \inf \{t \geq 0 : X_t \geq r_n\}$, $n \geq 1$. Lemma 4.3 implies

$$V(x) \geq \mathbb{E}_x[e^{-\beta \tau_{r_n}} h(X_{\tau_{r_n}})] = \psi(x)\frac{h(r_n)}{\psi(r_n)} \cdot \frac{G(x) - G(c)}{G(r_n) - G(c)}, \quad n \geq 1.$$

On the other hand, since $\tau \equiv +\infty$ is also a stopping time, we also have $V \geq 0$. Therefore

$$\frac{V(x)}{\psi(x)} \geq 0 \vee \left(\frac{h(r_n)}{\psi(r_n)} \cdot \frac{G(x) - G(c)}{G(r_n) - G(c)} \right) = \frac{h^+(r_n)}{\psi(r_n)} \cdot \frac{G(x) - G(c)}{G(r_n) - G(c)}, \quad n \geq 1. \qquad (5.6)$$

Remember that G is strictly increasing and negative (i.e. bounded from above). Therefore $G(b-)$ exists, and $-\infty < G(c) < G(b-) \leq 0$. Furthermore since $x > c$, $G(x) - G(c) > 0$. By taking limit supremum of both sides in (5.6) as $n \to +\infty$, we find

$$\frac{V(x)}{\psi(x)} \geq \limsup_{n \to +\infty} \frac{h^+(r_n)}{\psi(r_n)} \cdot \frac{G(x) - G(c)}{G(r_n) - G(c)} = \ell_b \cdot \frac{G(x) - G(c)}{G(b-) - G(c)} = +\infty.$$

Since $x \in (c, b)$ was arbitrary, this proves

$$V(x) = +\infty \text{ for all } x \in (c, b), \text{ if } \ell_b \text{ of } (5.5) \text{ is equal to } +\infty. \qquad (5.7)$$

Suppose now that ℓ_b is finite. We shall now show that $\mathbb{E}_x[e^{-\beta \tau} h(X_\tau)]$ is well–defined for every stopping time τ, and V is finite on $[c, b)$. Since $\ell_b < \infty$, there exists some $b_0 \in (c, b)$ such that $h^+(x) < (1 + \ell_b)\psi(x)$, for every $x \in (b_0, b)$. Since h is bounded on the closed and bounded interval $[c, b_0]$, we conclude that there exists some finite constant $K > 0$ such that

$$h^+(x) \leq K\psi(x), \quad \text{for all } x \in [c, b). \qquad (5.8)$$

When we let $U \triangleq \psi$ in Proposition 5.1, then U is nonnegative and real–valued on $[c, b)$. Furthermore $\frac{U}{\psi} \equiv 1$ is G–concave on $[c, b)$. Therefore we conclude that

$$\psi(x) \geq \mathbb{E}_x[e^{-\beta \tau} \psi(X_\tau)], \quad \forall\, x \in [c, b), \text{ and, every stopping time } \tau. \qquad (5.9)$$

This and (5.8) lead us to

$$K\psi(x) \geq \mathbb{E}_x[e^{-\beta \tau} K\psi(X_\tau)] \geq \mathbb{E}_x[e^{-\beta \tau} h^+(X_\tau)],$$

for every $x \in [c, b)$ and every stopping time τ. Thus $\mathbb{E}_x[e^{-\beta \tau} h(X_\tau)]$ is well–defined (i.e. expectation exists) for every stopping time τ. Since $K\psi(x) \geq \mathbb{E}_x[e^{-\beta \tau} h^+(X_\tau)] \geq$

$\mathbb{E}_x\big[e^{-\beta\tau}h(X_\tau)\big]$, for every $x \in [c, b)$ and stopping time τ, we also have

$$0 \leq V(x) \leq K\psi(x), \quad \text{or,} \quad 0 \leq \frac{V(x)}{\psi(x)} \leq K, \quad \forall\, x \in [c, b), \tag{5.10}$$

i.e. $V(x)$ is finite for every $x \in [c, b)$. We have proved the following result:

Proposition 5.2. *We have either $V \equiv +\infty$ in (c, d), or $V(x) < +\infty$ for all $x \in [c, b)$. Moreover, $V(x) < +\infty$ for every $x \in [c, b)$ if and only if ℓ_b of (5.5) is finite.*

In the remaining part of this Section, we shall assume that

$$\text{the quantity } \ell_b \text{ of (5.5) is finite,} \tag{5.11}$$

so that $V(\cdot)$ is real–valued. We shall investigate the properties of V, and describe how to find it. The main result is as follows; its proof is almost identical to the proof of Proposition 4.2, with the obvious changes, such as we use Proposition 5.1 instead of Proposition 4.1.

Proposition 5.3. *$V(\cdot)$ is the smallest nonnegative majorant of $h(\cdot)$ on $[c, b)$ such that $V(\cdot)/\psi(\cdot)$ is G–concave on $[c, b)$.*

We shall continue our discussion by first relating ℓ_b of (5.5) to $V(\cdot)$ as in Proposition 5.4. Since $\frac{V(\cdot)}{\psi(\cdot)}$ is G–concave, Proposition A.2 shows that $\lim_{x \to b} \frac{V(x)}{\psi(x)}$ exists, and (5.10) implies that this limit is finite. Since $V(\cdot)$ moreover majorizes $\max\{0, h(\cdot)\}$, we have

$$\ell_b = \limsup_{x \to b} \frac{h^+(x)}{\psi(x)} \leq \lim_{x \to b} \frac{V(x)}{\psi(x)} < +\infty. \tag{5.12}$$

Proposition 5.4. *If $h : [c, b) \to \mathbb{R}$ is bounded on compact subintervals of $[c, b)$, and (5.11) holds, then*

$$\lim_{x \to b} \frac{V(x)}{\psi(x)} = \ell_b.$$

Proof. Fix any arbitrarily small $\varepsilon > 0$. (5.11) implies that there exists some $l \in (c, b)$ such that

$$y \in [l, b) \implies h(y) \leq h^+(y) \leq (\ell_b + \varepsilon)\psi(y). \tag{5.13}$$

For every $x \in (l,b)$ and stopping time τ, we have

$$\{X_\tau \in [c,l)\} \subseteq \{\tau_l < \tau\}, \quad \text{on } \{X_0 = x\}. \tag{5.14}$$

Note also that the strong Markov property of X and (5.9) imply that $e^{-\beta t}\psi(X_t)$ is a nonnegative supermartingale. Therefore, we have

$$\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)] = \mathbb{E}_x[e^{-\beta\tau}h(X_\tau)1_{\{X_\tau \in [c,l)\}}] + \mathbb{E}_x[e^{-\beta\tau}h(X_\tau)1_{\{X_\tau \in (l,b)\}}]$$

$$\leq K\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{X_\tau \in [c,l)\}}] +$$

$$(\ell_b + \varepsilon)\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{X_\tau \in (l,b)\}}]$$

$$\text{(by (5.10) and (5.13))}$$

$$\leq K\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)1_{\{\tau_l < \tau\}}] + (\ell_b + \varepsilon)\mathbb{E}_x[e^{-\beta\tau}\psi(X_\tau)] \quad \text{(by (5.14))}$$

$$\leq K\mathbb{E}_x[e^{-\beta\tau_l}\psi(X_{\tau_l})1_{\{\tau_l < \tau\}}] + (\ell_b + \varepsilon)\psi(x)$$

$$\text{(by Optional Sampling Theorem for nonnegative}$$

$$\text{supermartingales)}$$

$$\leq K\mathbb{E}_x[e^{-\beta\tau_l}\psi(X_{\tau_l})1_{\{\tau_l < \infty\}}] + (\ell_b + \varepsilon)\psi(x)$$

$$= K\psi(l)\mathbb{E}_x[e^{-\beta\tau_l}] + (\ell_b + \varepsilon)\psi(x)$$

$$\leq K\psi(x)\mathbb{E}_x[e^{-\beta\tau_l}] + (\ell_b + \varepsilon)\psi(x) \quad \text{(ψ is increasing)}$$

$$= K\psi(x)\frac{\varphi(x)}{\varphi(l)} + (\ell_b + \varepsilon)\psi(x) \quad \text{(cf. Chapter 2)}$$

Note that right–hand side no longer depends on the stopping time τ. Therefore, we can take supremum of lefthand side over all stopping times, and then divide both sides by $\psi(x)$ to get

$$\frac{V(x)}{\psi(x)} \leq \frac{K}{\varphi(l)}\varphi(x) + \ell_b + \varepsilon, \quad \text{for every } x \in (l,b).$$

Now remember that $\varphi(b-) = 0$ and $\lim_{x \to b} \frac{V(x)}{\psi(x)}$ exist. By taking limits of both sides as x tends to b, we obtain

$$\lim_{x \to b} \frac{V(x)}{\psi(x)} \leq \frac{K}{\varphi(l)}\varphi(b-) + \ell_b + \varepsilon = \ell_b + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, this implies $\lim_{x \to b} \frac{V(x)}{\psi(x)} \leq \ell_b$. Finally, (5.12) completes the proof.   □

One can now easily show as in Chapter 4 the following

Proposition 5.5. *Let* $\widetilde{W} : [G(c), G(b-)) \to \mathbb{R}$ *be the smallest nonnegative concave majorant of* $\widetilde{H} : [G(c), G(b-)) \to \mathbb{R}$, *defined by,*

$$\widetilde{H}(y) \triangleq \frac{h\big(G^{-1}(y)\big)}{\psi\big(G^{-1}(y)\big)}, \quad y \in [G(c), G(b-)).$$

Then $V(x) = \psi(x)\widetilde{W}\big(G(x)\big)$, *for every* $x \in [c, b)$.

We can in fact show more in this case. Note that $G(b-) = -\frac{\varphi(b-)}{\psi(b-)} = 0$ and $-\infty < G(c) < 0$. Since we assume (5.11), we have

$$\limsup_{y \uparrow 0} \widetilde{H}^+(y) = \limsup_{x \uparrow b} \frac{h^+(x)}{\psi(x)} = \ell_b.$$

It is interesting to know what happens if we extend $\widetilde{H}$ to $[G(c), 0]$ by defining $\widetilde{H}(0) \triangleq \ell_b$. $\widetilde{W}$ should be closely related to the smallest nonnegative concave majorant of the extension of $\widetilde{H}$ onto $[G(c), 0]$. In fact, we have

Proposition 5.6. *Let* $W : [G(c), 0] \to \mathbb{R}$ *be the smallest nonnegative majorant of the function* $H : [G(c), 0] \to \mathbb{R}$, *given by*

$$H(y) \triangleq \begin{cases} \dfrac{h\big(G^{-1}(y)\big)}{\psi\big(G^{-1}(y)\big)}, & if \ \ y \in [G(c), 0), \\[2mm] \ell_b, & if \ \ y = 0. \end{cases} \tag{5.15}$$

Then $V(x) = \psi(x)W\big(G(x)\big)$, *for every* $x \in [c, b)$. *Furthermore,* $W(0) = \ell_b$, *and* W *is continuous at* 0.

Proof. Let $\widetilde{W}$ and $\widetilde{H}$ be as defined in Proposition 5.5 (remember $G(b-) = 0$). Since $\widetilde{H} = H$ on $[G(c), 0)$, the restriction of W to $[G(c), 0)$ is a nonnegative concave majorant of $\widetilde{H}$. Therefore, we have $W \geq \widetilde{W}$ on $[G(c), 0)$.

We shall next prove the reverse inequality. Let $\widehat{W} : [G(c), 0] \to \mathbb{R}$ be given by

$$\widehat{W}(y) \triangleq \begin{cases} \widetilde{W}(y), & \text{if } y \in [G(c), 0), \\ \\ \ell_b, & \text{if } y = 0. \end{cases}$$

Obviously, $\widehat{W}$ is nonnegative. Moreover $\widehat{W}(0) = \ell_b = H(0)$ and; for every $y \in [G(c), 0)$, $\widehat{W}(y) = \widetilde{W}(y) \geq \widetilde{H}(y) = H(y)$. Hence $\widehat{W}$ majorizes H on $[G(c), 0]$.

Next, we shall prove that $\widehat{W}$ is concave on $[G(c), 0]$. Since $\widetilde{W}$ is concave on $[G(c), 0)$, and, $\widehat{W}$ and $\widetilde{W}$ coincide in $[G(c), 0)$, $\widehat{W}$ is concave in $[G(c), 0)$. Therefore, we only need to prove that

$$\widehat{W}(y) \geq \widehat{W}(l) \cdot \frac{0 - y}{0 - l} + \widehat{W}(0) \cdot \frac{y - l}{0 - l}, \quad \text{for every } y \in (l, 0) \subset [G(c), 0].$$

First observe that $\lim_{y \uparrow 0} \widetilde{W}(y) = \lim_{x \to b} \widetilde{W}\big(G(x)\big) = \lim_{x \to b} \frac{V(x)}{\psi(x)} = \ell_b$ using Proposition 5.4 and 5.5. Therefore, for any $y \in (l, 0) \subset [G(c), 0]$, we have

$$\widehat{W}(l) \cdot \frac{0 - y}{0 - l} + \widehat{W}(0) \cdot \frac{y - l}{0 - l} = \widetilde{W}(l) \cdot \frac{y}{l} + \ell_b \cdot \frac{y - l}{0 - l} =$$
$$= \lim_{z \uparrow 0} \left[\widetilde{W}(l) \cdot \frac{z - y}{z - l} + \widetilde{W}(z) \cdot \frac{y - l}{z - l} \right] \leq \widetilde{W}(y) = \widehat{W}(y)$$

where we used the facts that $\widetilde{W}$ is nonnegative and concave in $[G(c), 0)$.

Thus, we proved that $\widehat{W}$ is a nonnegative concave majorant of H on $[G(c), 0]$. Since W is the smallest of such functions, we have $\widehat{W} \geq W$ on $[G(c), 0]$. This however implies that (i) $\widetilde{W} \geq W$ in $[G(c), 0)$, and (ii) $\ell_b = H(0) \leq W(0) \leq \widehat{W}(0) = \ell_b$.

We conclude that $W = \widetilde{W}$ on $[G(c), 0)$ and $W(0) = \ell_b$. From Proposition 5.5, we obtain $V(x) = \psi(x)\widetilde{W}\big(G(x)\big) = \psi(x)W\big(G(x)\big)$, for every $x \in [c, b)$. Finally, this and Proposition 5.4 imply

$$\lim_{y \uparrow 0} W(y) = \lim_{x \to b} W\big(G(x)\big) = \lim_{x \to b} \frac{V(x)}{\psi(x)} = \ell_b = W(0),$$

i.e. W is indeed continuous at 0. □

Since G is continuous in $[c, b)$, and $\frac{V}{\psi}$ is G–concave, $\frac{V}{\psi}$ is continuous in (c, b), and

$$\frac{V(c)}{\psi(c)} \leq \liminf_{x \downarrow c} \frac{V(x)}{\psi(x)}.$$

However ψ itself is continuous in $[c, b)$. Therefore, V is continuous in (c, b) and

$$V(c) \leq \liminf_{x \downarrow c} V(x).$$

Proposition 5.7. *If $h : [c, b) \to \mathbb{R}$ is continuous, and (5.11) is satisfied, then V is continuous on $[c, b)$.*

Proof. Note that h is bounded on every compact subinterval of $[c, b)$. Therefore, as discussed above, V is continuous on (c, b) and $V(c) \leq \liminf_{x \downarrow c} V(x)$. We only have to establish

$$V(c) \geq \limsup_{x \downarrow c} V(x).$$

Define $\widetilde{h} : [c, b) \to \mathbb{R}$ by $\widetilde{h}(x) \triangleq \max_{y \in [c, x]} h(y)$, for every $y \in [c, b)$. Since h is uniformly continuous on compact subintervals on $[c, b)$, $\widetilde{h}$ is continuous.

Another crucial observation is that the strong Markov property and (5.9) imply that $e^{-\beta t} \psi(X_t)$ is a nonnegative supermartingale.

Fix some $r \in (c, b)$. For every $x \in [c, r)$ and stopping time τ, we have

$$\{X_\tau \in (r, b)\} \subset \{\tau_r < \tau\} \subset \{\tau_r < \infty\} = \{\tau_r < \tau_c\}, \quad \text{on } \{X_0 = x\}, \qquad (5.16)$$

since c is absorbing. Now observe that

$$\mathbb{E}_x[e^{-\beta \tau} h(X_\tau)] = \mathbb{E}_x\big[e^{-\beta \tau} h(X_\tau) 1_{\{X_\tau \in [c, r]\}}\big] + \mathbb{E}_x\big[e^{-\beta \tau} h(X_\tau) 1_{\{X_\tau \in (r, b)\}}\big]$$

$$\leq \widetilde{h}(r) + K\mathbb{E}_x\big[e^{-\beta \tau} \psi(X_\tau) 1_{\{X_\tau \in (r, b)\}}\big] \qquad \text{(By (5.10))}$$

$$\leq \widetilde{h}(r) + K\mathbb{E}_x\big[e^{-\beta \tau} \psi(X_\tau) 1_{\{\tau_r < \tau\}}\big] \qquad \text{(By (5.16))}$$

$$\leq \widetilde{h}(r) + K\mathbb{E}_x\big[e^{-\beta \tau_r} \psi(X_{\tau_r}) 1_{\{\tau_r < \tau\}}\big]$$

(By the Optional Sampling Theorem for nonnega-

tive supermartingales)

$$\leq 0 \vee \widetilde{h}(r) + K\mathbb{E}_x\big[e^{-\beta\tau_r}\psi(X_{\tau_r})1_{\{\tau_r<\tau_c\}}\big] \qquad\qquad \text{(By (5.16))}$$

$$\leq 0 \vee \widetilde{h}(r) + K\psi(r)\mathbb{P}_x\{\tau_r < \tau_c\}$$

$$= 0 \vee \widetilde{h}(r) + K\psi(r)\frac{S(x) - S(c)}{S(r) - S(c)}.$$

Observe that right–hand side no longer depends on the stopping time τ. Therefore

we can take supremum of lefthand side over all stopping times τ, and get

$$V(x) \leq 0 \vee \widetilde{h}(r) + K\psi(r)\frac{S(x) - S(c)}{S(r) - S(c)}, \quad \forall\, x \in [c, r).$$

Take the limit supremum of both sides as x tends to c. Since S is continuous in $[c, b)$,

we obtain

$$\limsup_{x\downarrow c} V(x) \leq 0 \vee \widetilde{h}(r), \quad \forall\, r \in (c, b).$$

Finally, take limit as r tend to c. Since $x \to 0 \vee x$ is a continuous real–valued function,

and $\widetilde{h}$ is continuous in $[c, b)$, we conclude

$$\limsup_{x\downarrow c} V(x) \leq \lim_{r\downarrow c} 0 \vee \widetilde{h}(r) = 0 \vee \widetilde{h}(c) = 0 \vee h(c).$$

However, c is absorbing. Therefore $\mathbb{P}_c\{X_t = c,\ \forall t \geq 0\} = 1$, and $V(c) = 0 \vee h(c)$

("never stop" versus "stop immediately"; other stopping times give values between 0

and $h(c)$ because of the discounting factor). This completes the proof. $\square$

In the remaining part of the section, we shall investigate when we have an optimal

stopping time. Proposition 5.8 shows that the existence of an optimal stopping time is

guaranteed when ℓ_b of (5.5) equals zero. Lemma 5.2 gives the necessary and sufficient

condition for the existence of an optimal stopping time when ℓ_b is positive. Finally,

no optimal stopping time exists when ℓ_b equals $+\infty$, since the value function equals

$+\infty$ everywhere. As usual, we define

$$\boldsymbol{\Gamma} \triangleq \{x \in [c, b) : V(x) = h(x)\}, \quad \text{and} \quad \tau^* \triangleq \inf\{t \geq 0 : X_t \in \boldsymbol{\Gamma}\}. \qquad (5.17)$$

Proposition 5.8. *Suppose $h : [c, b) \to \mathbb{R}$ is continuous, and the quantity ℓ_b of (5.5)*
equals zero. Then τ^* of (5.17) is an optimal stopping time.

We shall prove the Proposition by using the results of Appendix B, where we
studied the properties of the smallest nonnegative concave majorant of continuous
functions. If $h(\cdot)$ is continuous, then the continuity of $V(\cdot)$ and the limiting behavior
of $\frac{V(\cdot)}{\psi(\cdot)}$ near b follow from Corollary B.1. Proposition B.2 also leads to the

Proof of Proposition 5.8. Using the notation of Appendix B, here we have $I = [c, b)$.
Since $c \in I$, by the continuity of h on I, $\ell_c = \frac{h^+(c)}{\varphi(c)}$. By Corollary B.1 we also have
$\frac{V(c)}{\varphi(c)} = \ell_c$.

Let $U(x) = \mathbb{E}_x[e^{-\beta \tau^*} h(X_{\tau^*})]$, $x \in [c, b)$; $\mathbf{\Gamma} \triangleq \{x \in [c, b) : V(x) = h(x)\}$ and
$\mathbf{C} \triangleq [c, b) \backslash \mathbf{\Gamma}$. Since $\mathbb{P}_x\{\tau^* = 0\} = 1$ for $x \in \mathbf{\Gamma}$, $V = U$ on $\mathbf{\Gamma}$.

We need to show equality of V and U in $\mathbf{C}$. Since $\mathbf{C}$ is open relative to $[c, b)$,
it is union of countably many disjoint open (relative to $[c, b)$) subintervals of $[c, b)$.
Therefore it enough to show $V = U$ in each of those subintervals covering $\mathbf{C}$.

Suppose $(l, r) \subseteq \mathbf{C}$ for some $l, r \in \mathbf{\Gamma}$ (*Type 1 Interval*). Since $\mathbb{P}_x\{\tau^* = \tau_l \wedge \tau_r\} = 1$
for $x \in (l, r)$, Lemma 4.3 implies

$$\frac{U(x)}{\psi(x)} = \frac{\mathbb{E}_x[e^{-\beta \tau_l \wedge \tau_r} h(X_{\tau_l \wedge \tau_r})]}{\psi(x)} = \frac{h(l)}{\psi(l)} \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{h(r)}{\psi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}, \quad \forall \, x \in (l, r).$$

Observe that $V = U$ in (l, r) by (i) of Proposition B.2.

Now suppose $[c, r) \subseteq \mathbf{C}$ for some $r \in \mathbf{\Gamma}$ (*Type 2 Interval*). Because $\mathbb{P}_x\{\tau^* = \tau_r\} = 1$
for $x \in [c, r)$, Lemma 4.3 implies

$$\frac{U(x)}{\varphi(x)} = \frac{\mathbb{E}_x[e^{-\beta \tau_r} h(X_{\tau_r})]}{\varphi(x)} = \frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(c)}{F(r) - F(c)}, \quad x \in [a, r).$$

Since $c \in \mathbf{C}$, $\frac{h^+(c)}{\varphi(c)} = \ell_c = \frac{V(c)}{\varphi(c)} > \frac{h(c)}{\varphi(c)}$. This implies that $h(c) < 0$ and $\ell_c = \frac{h^+(c)}{\varphi(c)} = 0$.
Therefore Proposition B.2(ii) implies that U and V coincide in $[a, r)$.

Finally, suppose $(l, b) \subseteq \mathbf{C}$ for some $l \in \mathbf{\Gamma}$ (*Type 3 Interval*). Since $\mathbb{P}_x\{\tau^* = \tau_l\} = 1$

for $x \in (l, b)$, Lemma 4.3 implies

$$\frac{U(x)}{\psi(x)} = \frac{\mathbb{E}_x[e^{-\beta \tau_l} h(X_{\tau_l})]}{\psi(x)} = \frac{h(l)}{\psi(l)} \cdot \frac{G(b-) - G(x)}{G(b-) - G(l)}, \quad x \in (l, b). \tag{5.18}$$

Since by hypothesis $\ell_b = 0$, Proposition B.2(iii) implies $V = U$ on (l, b) as well. Thus we exhausted all possible forms of subintervals of $[c, b)$ that can exclusively cover $\mathbf{C}$ and showed that $V = U$ in the subintervals in each case. Hence $V = U$ also in $\mathbf{C}$. $\square$

Lemma 5.1. *Suppose W and H be functions defined on $[G(c), 0]$ as in Proposition 5.6. Let*

$$\widetilde{\mathbf{\Gamma}} \triangleq \{y \in [G(c), 0) : W(y) = H(y)\}.$$

Then $\mathbf{\Gamma} = G^{-1}(\widetilde{\mathbf{\Gamma}})$.

Proof. Follows directly from Proposition 5.6. $\square$

Suppose h is continuous on $[c, b)$. Then H of (5.15) is continuous on the closed and bounded interval $[G(c), 0]$. Let B_t be a standard Brownian motion in $[G(c), 0]$ which is stopped whenever it reaches $G(c)$ or 0. Remember that W is, by definition, the smallest nonnegative concave majorant of H on $[G(c), 0]$. Since the scale function of B is the identity function, Proposition 3.2 relates W and H as in

$$W(y) = \sup_{\tau \geq 0} \mathbb{E}_y[H(B_\tau)], \quad \forall\, y \in [G(c), 0]. \tag{5.19}$$

Proposition 3.5 shows that $\widetilde{\mathbf{\Gamma}}$ of Lemma 5.1 is the optimal stopping region of (5.19). An alternative proof of Proposition 5.8 can also be given by using the connection between the two optimal stopping problems and the results of Chapter 3.

It is worth mentioning that the transformation of a discounted optimal stopping problem of any diffusion process living in a non–compact interval into an optimal stopping problem of standard Brownian motion restricted to a compact interval proves itself to be very useful in calculations (cf. Examples in Chapter 6).

Let $\mathbf{C} \triangleq [c, b) \backslash \mathbf{\Gamma}$ as in the proof of Proposition 5.8, and $\widetilde{\mathbf{C}} \triangleq [G(c), 0) \backslash \widetilde{\mathbf{\Gamma}}$. Since h is continuous on $[c, b)$, then H and W are continuous on $[G(c), 0)$. Therefore $\mathbf{C}$ and

$\widetilde{\mathbf{C}}$ are open relative to $[c,b)$ and $[G(c),0)$, respectively. Then both sets are covered by countable families of disjoint open (relative to $[c,b)$ and $[G(c),0)$, respectively) subintervals of $[c,b)$ and $[G(c),0)$, respectively. Denote a covering of $\widetilde{\mathbf{C}}$ by $(\widetilde{J}_\alpha)_{\alpha\in\widetilde{\Lambda}}$.

Lemma 5.1 implies that $\mathbf{C} = G^{-1}(\widetilde{\mathbf{C}})$. Therefore $\left(J_\alpha \triangleq G^{-1}(\widetilde{J}_\alpha)\right)_{\alpha\in\widetilde{\Lambda}}$ is a covering of $\mathbf{C}$ with the properties described above. We shall especially use the following consequences as we study examples in the coming chapters:

- if $(l,r) \subset \widetilde{\mathbf{C}}$, $l < r$, for some $l,r \in \widetilde{\Gamma}$, then $G^{-1}(l) \in \Gamma$, $G^{-1}(r) \in \Gamma$ and $(G^{-1}(l), G^{-1}(r)) \subset \mathbf{C}$.

- If $[G(c),r) \subset \widetilde{\mathbf{C}}$, $G(c) < r$, for some $r \in \widetilde{\Gamma}$, then $c < G^{-1}(r) \in \Gamma$ and $[c, G^{-1}(r)) \subset \mathbf{C}$.

- Finally, if $(l,0) \subset \widetilde{\mathbf{C}}$ for some $l \in \widetilde{\Gamma}$, then $G^{-1}(l) \in \Gamma$ and $(G^{-1}(l), b) \subset \mathbf{C}$.

Do we have an optimal stopping time when $\ell_b > 0$? The answer depends on the shape of Γ.

First of all, if there exists any optimal stopping time, then τ^* of (5.17) must also be an optimal stopping time (Oksendal [10]). Therefore, it is enough to investigate when τ^* becomes an optimal stopping time. Let

$$U(x) \triangleq \mathbb{E}_x[e^{-\beta\tau^*}h(X_{\tau^*})], \qquad x \in [c,b),$$

as in the proof of Proposition 5.8. We always have $V \geq U$, and equality holds (i.e., τ^* is an optimal stopping time) if and only if $\frac{U(\cdot)}{\psi(\cdot)}$ is a nonnegative G–concave majorant of $\frac{h(\cdot)}{\psi(\cdot)}$ on $[c,b)$, thanks to Proposition 5.3.

If $\ell_b > 0$, then it can be shown that $\frac{U(\cdot)}{\psi(\cdot)}$ is still a nonnegative G–concave function on $[c,b)$. However, it is not always true that $\frac{U(\cdot)}{\psi(\cdot)}$ majorizes $\frac{h(\cdot)}{\psi(\cdot)}$ on $[c,b)$, when $\ell_b > 0$.

Suppose, for example, that there exists some $l \in [c,b)$ such that $l \in \Gamma$ and $(l,b) \subseteq \mathbf{C}$ (i.e. (l,b) is a Type–3 Interval studied in the proof of Proposition 5.8).

(5.18) gives $\frac{U}{\psi}$ in (l, b). Since $G(b-) = 0$, by taking limit in (5.18) as $x \to b$, we find that

$$\lim_{x \to b} \frac{U(x)}{\psi(x)} = 0. \tag{5.20}$$

Therefore, there is some $\widetilde{l} \in [c, b)$ such that $\frac{U(x)}{\psi(x)} < (1/2)\ell_b$ for all $x \in (\widetilde{l}, b)$.

Since by hypothesis $+\infty > \limsup_{x \to b} \frac{h^+(x)}{\psi(x)} = \ell_b > 0$, the definition of limit supremum implies that for every $l \in [c, b)$ and for every $0 < \varepsilon < \ell_b$, there exist some $x \in (l, b)$ such that

$$\frac{h^+(x)}{\psi(x)} > \ell_b - \varepsilon > 0, \quad \text{i.e.} \quad \frac{h^+(x)}{\psi(x)} = \frac{h(x)}{\psi(x)} > \ell_b - \varepsilon. \tag{5.21}$$

In particular, by choosing $l = \widetilde{l}$ and $\varepsilon = (1/2)\ell_b$, we realize that there exists some $x \in (\widetilde{l}, b)$ such that

$$\frac{h(x)}{\psi(x)} > \ell_b - \frac{1}{2}\ell_b = \frac{1}{2}\ell_b > \frac{U(x)}{\psi(x)}.$$

Hence U fails to majorize h everywhere on $[c, b)$ (See Figure 5.1). Therefore $U \neq V$ if $\ell_b > 0$ and $(l, b) \subseteq \mathbf{C}$ for some $l \in [c, b)$.
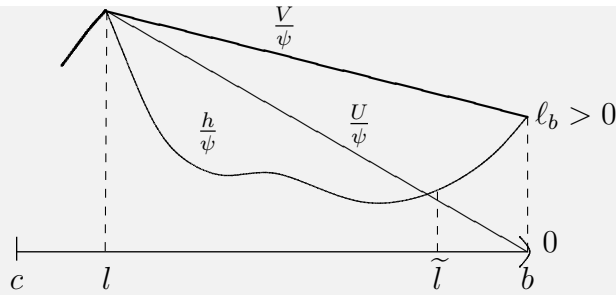
Figure 5.1: If $\ell_b > 0$ and $(l, b) \subseteq \mathbf{C}$ for some $l \in [c, b)$, then $U(x) \triangleq \mathbb{E}_x[e^{-\beta \tau^*} h(X_{\tau^*})]$, $x \in [c, b)$, no longer majorizes h even though it is nonnegative, and U/ψ is still G–concave. Therefore $U \neq V$, i.e. τ^* of (5.17) can no longer be an optimal stopping time (for the sake of simplicity, we assumed $\lim_{x \to b} \frac{h(x)}{\psi(x)}$ exists as we draw the figure).

We shall conclude our informal discussion with a more precise statement, namely

Lemma 5.2. *Suppose $\ell_b > 0$ is finite and, h is continuous. Then τ^* of (5.17) is an optimal stopping time if and only if there is no $l \in [c, b)$ such that $(l, b) \subseteq \mathbf{C}$.*[1]

Proof. Let U, $\mathbf{\Gamma}$ and $\mathbf{C}$ be as in the *Proof of Proposition 5.8*. We have $U = V$ on $\mathbf{\Gamma}$ and in every Type 1– and Type 2 Intervals that exclusively cover $\mathbf{C}$ as before. Therefore τ^* is an optimal stopping time if $\mathbf{C}$ does not contain any Type 3 Interval.

If on the other hand there is a Type 3 Interval $(l, b) \subseteq \mathbf{C}$ for some $l \in [c, b)$, then

$$\frac{U(x)}{\psi(x)} = \frac{h(l)}{\psi(l)} \cdot \frac{G(b-) - G(x)}{G(b-) - G(l)}$$
$$< \frac{h(l)}{\psi(l)} \cdot \frac{G(b-) - G(x)}{G(b-) - G(l)} + \ell_b \cdot \frac{G(x) - G(l)}{G(b-) - G(l)} = \frac{V(x)}{\psi(x)}, \quad x \in (l, b),$$

by Proposition B.2 since $\ell_b > 0$, i.e. $U \neq V$ in a nonempty subset of $\mathbf{C}$. Therefore τ^* cannot be an optimal stopping time. $\square$

Even if we may not have an optimal stopping time, we *always* have $\varepsilon(x)$–optimal stopping times. We shall introduce for every $\varepsilon > 0$

$$\mathbf{\Gamma}^b_\varepsilon \triangleq \left\{ x \in [c, b) \,\middle|\, \frac{h(x)}{\psi(x)} \geq \frac{V(x)}{\psi(x)} - \varepsilon \right\} \quad \text{and} \quad \tau^b_\varepsilon \triangleq \inf \left\{ t \geq 0 : X_t \in \mathbf{\Gamma}^b_\varepsilon \right\}. \quad (5.22)$$

Proposition 5.9. *Suppose h is continuous, and (5.11) holds. For every $\varepsilon > 0$,*

$$V^b_\varepsilon(x) \triangleq \mathbb{E}_x[e^{-\beta \tau^b_\varepsilon} h(X_{\tau^b_\varepsilon})] \geq V(x) - \varepsilon \psi(x), \qquad x \in [c, b).$$

Proof. Since V, h and ψ are continuous, $\mathbf{\Gamma}^b_\varepsilon$ is closed (relative to $[c, b)$). Therefore, $X_{\tau^b_\varepsilon} \in \mathbf{\Gamma}^b_\varepsilon$ on $\{\tau^b_\varepsilon < \infty\}$, and

$$V^b_\varepsilon(x) \geq \mathbb{E}_x[e^{-\beta \tau^b_\varepsilon} V(X_{\tau^b_\varepsilon})] - \varepsilon \mathbb{E}_x[e^{-\beta \tau^b_\varepsilon} \psi(X_{\tau^b_\varepsilon})] \geq \mathbb{E}_x[e^{-\beta \tau^b_\varepsilon} V(X_{\tau^b_\varepsilon})] - \varepsilon \psi(x), \quad x \in [c, b),$$

---

[1]This condition is not the same as "for some $l \in [c, b)$, $(l, b) \subseteq \mathbf{\Gamma}$". It is *more* than that: Suppose there exists a strictly increasing sequence $b_n \to b$ such that $(b_{n_k}, b_{n_k+1}) \subseteq \mathbf{C}$ and $b_{n_k}, b_{n_k+1} \in \mathbf{\Gamma}$ for some subsequence (b_{n_k}). The original condition in Lemma 5.2 still holds whereas there is no $l \in [c, b)$ such that $(l, b) \subseteq \mathbf{\Gamma}$.

where the last inequality follows from by Proposition 5.1. It is therefore enough to prove that

$$U(x) \triangleq \mathbb{E}_x[e^{-\beta \tau_\varepsilon^b} V(X_{\tau_\varepsilon^b})] \geq V(x), \qquad x \in [c, b).$$

We claim that $\frac{U}{\psi}$ is a nonnegative majorant G–concave majorant of $\frac{h}{\psi}$ on $[c, b)$ (this will then imply $U \geq V$ by Proposition 5.3).

Since V is always nonnegative, U is also nonnegative. In order to prove that $\frac{U}{\psi}$ is G–concave on $[c, b)$, we shall show that

$$U(x) \geq \mathbb{E}_x[e^{-\beta \tau} U(X_\tau)], \qquad x \in [c, b),$$

for every stopping time $\tau \geq 0$, and then use Proposition 5.1. Let τ be any stopping time. Then

$$\mathbb{E}_x[e^{-\beta \tau} U(X_\tau)] = \mathbb{E}_x\left[e^{-\beta \tau} \mathbb{E}_{X_\tau}\left[e^{-\beta \tau_\varepsilon^b} V(X_{\tau_\varepsilon^b})\right]\right] = \mathbb{E}_x\left[e^{-\beta \tau} \theta_\tau\left(e^{-\beta \tau_\varepsilon^b} V(X_{\tau_\varepsilon^b})\right)\right]$$

$$= \mathbb{E}_x\left[e^{-\beta(\tau + \tau_\varepsilon^b \circ \theta_\tau)} V(X_{\tau + \tau_\varepsilon^b \circ \theta_\tau})\right] \leq \mathbb{E}_x\left[e^{-\beta \tau_\varepsilon^b} V(X_{\tau_\varepsilon^b})\right] = U(x), \quad x \in [c, b).$$

Second equality follows from strong Markov property of diffusion processes. In order to understand the inequality, first remember that $\frac{V}{\psi}$ is G–concave. Therefore Proposition 5.1 and strong Markov property of X imply together that $e^{-\beta t} V(X_t)$ is a nonnegative supermartingale. Since $\tau + \tau_\varepsilon^b \circ \theta_\tau$ is also a stopping time and is greater than or equal to τ_ε^b, the inequality above immediately follows from Optional Sampling Theorem for nonnegative supermartingales.

It remains to prove that $\frac{U}{\psi}$ majorizes $\frac{h}{\psi}$ on $[c, b)$. Assume on the contrary that

$$\theta \triangleq \sup_{x \in [c,b)} \left(\frac{h(x)}{\psi(x)} - \frac{U(x)}{\psi(x)}\right) > 0.$$

Since $\frac{h}{\psi}$ is bounded (see (5.8) at page 35), and $U \geq 0$, θ is finite. If we let $\widetilde{U}(x) \triangleq U(x) + \theta \psi(x)$, $x \in [c, b)$, then $\frac{\widetilde{U}}{\psi}$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$ on $[c, b)$. Therefore $\widetilde{U} \geq V$ on $[c, b)$.

Since θ is finite, for every $0 < \delta < \varepsilon$ there exists some $x_\delta \in [c, b)$ such that $\frac{h(x_\delta)}{\psi(x_\delta)} - \frac{U(x_\delta)}{\psi(x_\delta)} \geq \theta - \delta$. Now observe that

$$\frac{h(x_\delta)}{\psi(x_\delta)} \geq \frac{U(x_\delta)}{\psi(x_\delta)} + \theta - \delta = \frac{\widetilde{U}(x_\delta)}{\psi(x_\delta)} - \delta \geq \frac{V(x_\delta)}{\psi(x_\delta)} - \delta \geq \frac{V(x_\delta)}{\psi(x_\delta)} - \varepsilon.$$

Therefore $x_\delta \in \mathbf{\Gamma}_\varepsilon^b$. Therefore $\mathbb{P}_{x_\delta}(\tau_\varepsilon^b = 0) = 1$ and $U(x_\delta) = V(x_\delta) \geq h(x_\delta)$. Thus

$$0 \geq \frac{h(x_\delta)}{\psi(x_\delta)} - \frac{U(x_\delta)}{\psi(x_\delta)} \geq \theta - \delta,$$

i.e. $\theta < \delta$ for every $0 < \delta < \varepsilon$. Therefore $\theta \leq 0$. Contradiction. Hence U majorizes h on $[c, b)$. This completes the proof as pointed out before. $\square$

5.2 Left–boundary is natural, and right–boundary is absorbing

We shall now suppose that the left–boundary a of $\mathcal{I}$ is natural, and consider the stopped process X which starts in $(a, d]$, $d \in \text{int}(\mathcal{I})$, and is absorbed at d, whenever it reaches. Because a is natural, we have $\psi(a+) = 0$ and $\varphi(a+) = \infty$. Since d is an interior point of $\mathcal{I}$, process reaches d in finite time with positive probability. Therefore $0 < \psi(x), \varphi(x) < \infty$, $x \in (a, d]$.

Let the reward function $h : (a, d] \to \mathbb{R}$ be bounded on every compact subset of $(a, d]$. We shall look at

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta \tau} h(X_\tau)], \quad x \in (a, d].$$

We shall state the results without proofs since they are almost identical to those in Section 5.1 with obvious changes. The key result is

Proposition 5.10. $U : (a, d] \to \mathbb{R}$ *is nonnegative and* $\frac{U}{\varphi}$ *is F–concave on* $(a, d]$ *if and only if*

$$U(x) \geq \mathbb{E}_x[e^{-\beta \tau} U(X_\tau)],$$

for every $x \in (a, d]$ *and every stopping time* τ *of* X.

In order to address the existence problem of the value function V, we introduce[2]

$$\ell_a \triangleq \limsup_{x \to a} \frac{h^+(x)}{\varphi(x)}.$$ (5.23)

Using Proposition 5.10, we can show that $\mathbb{E}_x[e^{-\beta\tau}h(X_\tau)]$ is well–defined for every stopping time τ, and V is finite on $(a, d]$ if and only $\ell_a < \infty$. In fact,

Proposition 5.11. *We have either $V \equiv +\infty$ in (a, d), or $V(x) < +\infty$ for all $x \in (a, d]$. Moreover, $V(x) < +\infty$ for every $x \in (a, d]$ if and only if ℓ_a is finite.*

In the remaining part of this Section, we shall assume that

$$\text{the quantity } \ell_a \text{ of (5.23) is finite.}$$ (5.24)

Therefore V always exist (i.e. is finite everywhere) by Proposition 5.11. We shall investigate the properties of V, and describe how to find it.

Since $\varphi > 0$ on $(a, d]$, F is well–defined and continuous on $(a, d]$. Observe also that $F(a+) = 0$. The main result is

Proposition 5.12. *V is the smallest nonnegative majorant of h on $(a, d]$ such that $\frac{V}{\varphi}$ is F–concave on $(a, d]$, in the sense that, if $\frac{U}{\varphi}$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on $(a, d]$ for some $U : (a, d] \to \mathbb{R}$, then $U \geq V$.*

Since $\frac{V}{\varphi}$ is G–concave, Proposition A.2 implies that the limit $\lim_{x \to a} \frac{V(x)}{\varphi(x)}$ exists. In fact, we can show that

$$\lim_{x \to a} \frac{V(x)}{\varphi(x)} = \ell_a.$$

---

[2]Since the left–boundary is now open, we would like to control growth of h near a. Since a is left–boundary of state space, the role of ψ will now be taken by φ. It is not hard to expect us to replace every ψ and G above with φ and F, respectively, in the remaining part of the chapter. Note also that the duality in $(\psi, G) \leftrightarrow (\varphi, F)$ is equivalent to the duality in $(\psi, -\varphi \equiv \psi \cdot G) \leftrightarrow (\varphi, \psi \equiv \varphi \cdot F)$.

Proposition 5.13. *Let $W : [0, F(d)] \to \mathbb{R}$ be the smallest nonnegative concave majorant of $H : [0, F(d)] \to \mathbb{R}$, given by*

$$H(y) \triangleq \begin{cases} \dfrac{h\big(F^{-1}(y)\big)}{\varphi\big(F^{-1}(y)\big)}, & if \ \ y \in (0, F(d)], \\[3mm] \ell_a, & if \ \ y = 0. \end{cases}$$

Then $V(x) = \varphi(x)W\big(F(x)\big)$, for every $x \in (a, d]$. Furthermore, $W(0) = \ell_a$, and W is continuous at 0.

Proposition 5.14. *If $h : (a, d] \to \mathbb{R}$ is continuous, and (5.24) holds, then V is continuous on $(a, d]$.*

Proposition 5.15. *Suppose $h : (a, d] \to \mathbb{R}$ is continuous, and $\ell_a = 0$. Then τ^* of (5.17) is an optimal stopping time.*

Lemma 5.3. *Suppose W and H be functions defined on $[0, F(d)]$ as in Proposition 5.13. Let*

$$\widetilde{\Gamma} \triangleq \{y \in (0, F(d)] : W(y) = H(y)\}.$$

Then $\Gamma = F^{-1}\big(\widetilde{\Gamma}\big)$, where Γ is defined as in (5.17).

Lemma 5.4. *Suppose $\ell_a > 0$ is finite and h is continuous. Then τ^* of (5.17) is an optimal stopping time if and only if there is no $r \in (a, d]$ such that $(a, r) \subseteq \mathbf{C}$.*

Even if we do not always have an optimal stopping time, we have $\varepsilon(x)$–optimal stopping times. We shall introduce for every $\varepsilon > 0$

$$\Gamma_\varepsilon^a \triangleq \left\{x \in (a, d] \,\middle|\, \frac{h(x)}{\varphi(x)} \geq \frac{V(x)}{\varphi(x)} - \varepsilon\right\} \quad \text{and} \quad \tau_\varepsilon^a \triangleq \inf \{t \geq 0 : X_t \in \Gamma_\varepsilon^a\}. \quad (5.25)$$

Proposition 5.16. *Suppose h is continuous, and (5.24) holds. For every $\varepsilon > 0$,*

$$V_\varepsilon^a(x) \triangleq \mathbb{E}_x[e^{-\beta\tau_\varepsilon^a} h(X_{\tau_\varepsilon^a})] \geq V(x) - \varepsilon\varphi(x), \qquad x \in (a, d].$$

5.3 Both boundaries are natural.

Suppose that both a and b are natural. Consider the process X in $\mathcal{I} = (a, b)$. We have $\psi(a+) = \varphi(b-) = 0$, $\psi(b-) = \varphi(a+) = +\infty$, and $0 < \psi, \varphi < \infty$, in (a, b).

Let the reward function $h : (a, b) \to \mathbb{R}$ be bounded on every compact subset of (a, b). Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta \tau} h(X_\tau)], \quad x \in (a, b).$$

The key result is

Proposition 5.17. *$U : (a, b) \to \mathbb{R}$ is nonnegative and $\frac{U}{\varphi}$ is F–concave on (a, b) (equivalently, $\frac{U}{\psi}$ is G–concave on (a, b)) if and only if*

$$U(x) \geq \mathbb{E}_x[e^{-\beta \tau} U(X_\tau)],$$

for every $x \in (a, b)$ and every stopping time τ of X.

The proof of the Proposition 5.17 is similar to that of Proposition 5.1 after we redefine the sequence of stopping times $(\sigma_n)_{n \geq 1}$. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of real numbers such that $a < a_{n+1} < a_n < \cdots < a_1 < b_1 < \cdots < b_n < b_{n+1} < b$. Define $\sigma_n \triangleq \inf\{t \geq 0 : X_t \notin (a_n, b_n)\}$. Observe that ψ and φ are $C^2[a_n, b_n]$ for every $n \geq 1$.

Introduce

$$\ell_a \triangleq \limsup_{x \to a} \frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad \ell_b \triangleq \limsup_{x \to b} \frac{h^+(x)}{\psi(x)} \tag{5.26}$$

Using Proposition 5.17, we can show that $\mathbb{E}_x[e^{-\beta \tau} h(X_\tau)]$ is well–defined for every stopping time τ, and V is finite on (a, b) if both ℓ_a and ℓ_b are finite. In fact,

Proposition 5.18. *We have either $V \equiv +\infty$ in (a, b), or $V(x) < +\infty$ for all $x \in (a, b)$. Moreover, $V(x) < +\infty$ for every $x \in (a, b)$ if and only if both ℓ_a and ℓ_b are finite.*

In the remaining part of this Section, we shall assume that

$$\text{the quantities } \ell_a \text{ and } \ell_b \text{ of (5.26) are finite.} \tag{5.27}$$

Therefore V always exist (i.e. is finite everywhere) by Proposition 5.18. We shall investigate the properties of V, and describe how to find it.

Since ψ and φ never vanish, and are continuous on (a, b), F and G are well–defined and continuous on (a, b). Observe also that $F(a+) = G(b-) = 0$ and $F(b-) = -G(a+) = +\infty$ The main result is

Proposition 5.19. *V is the smallest nonnegative majorant of h on (a, b) such that $\frac{V}{\varphi}$ is F–concave (equivalently, $\frac{V}{\psi}$ is G–concave) on (a, b), in the sense that, if $\frac{U}{\varphi}$ ($\frac{U}{\psi}$, respectively) is a nonnegative F–concave (G–concave, respectively) majorant of $\frac{h}{\varphi}$ ($\frac{h}{\psi}$, respectively) on (a, b) for some $U : (a, b) \to \mathbb{R}$, then $U \geq V$.*

Because $\frac{V}{\varphi}$ is F–concave, and $\frac{V}{\psi}$ is G–concave, Proposition A.2 implies that limits $\lim_{x \to a} \frac{V(x)}{\varphi(x)}$ and $\lim_{x \to b} \frac{V(x)}{\psi(x)}$ exist. In fact, we have

$$\lim_{x \to a} \frac{V(x)}{\varphi(x)} = \ell_a \quad \text{and} \quad \lim_{x \to b} \frac{V(x)}{\psi(x)} = \ell_b.$$

Proposition 5.20. *Let $W : [0, +\infty) \to \mathbb{R}$ be the smallest nonnegative concave majorant of $H : [0, +\infty) \to \mathbb{R}$, given by*

$$H(y) \triangleq \begin{cases} \dfrac{h\big(F^{-1}(y)\big)}{\varphi\big(F^{-1}(y)\big)}, & \text{if } y \in (0, +\infty), \\[3mm] \ell_a, & \text{if } y = 0. \end{cases}$$

Then $V(x) = \varphi(x)W\big(F(x)\big)$, for every $x \in (a, b)$. Furthermore, $W(0) = \ell_a$, and W is continuous at 0.

Similarly, if $\widetilde{W} : (-\infty, 0] \to \mathbb{R}$ is the smallest nonnegative concave majorant of $\widetilde{H} : (-\infty, 0] :\to \mathbb{R}$, given by

$$\widetilde{H}(y) \triangleq \begin{cases} \dfrac{h\big(G^{-1}(y)\big)}{\psi\big(G^{-1}(y)\big)}, & \text{if } y \in (-\infty, 0), \\[3mm] \ell_b, & \text{if } y = 0. \end{cases}$$

Then $V(x) = \psi(x)\widetilde{W}\big(G(x)\big)$, for every $x \in (a, b)$. Furthermore, $\widetilde{W}(0) = \ell_b$, and $\widetilde{W}$ is continuous at 0.

The proof is similar to that of Proposition 5.6. Since $\frac{V}{\varphi}$ is F–concave, and F is continuous on (a, b), V is continuous on (a, b) by Proposition A.1.

Proposition 5.21. *Suppose $h : (a, b) \to \mathbb{R}$ is continuous, and $\ell_a = \ell_b = 0$. Then τ^* of (5.17) is an optimal stopping time.*

Proof. We can calculate U explicitly as in the proof of Proposition 5.8, and show that it agrees with Proposition B.2 in Appendix B. $\qquad\square$

Lemma 5.5. *Suppose W and H ($\widetilde{W}$ and $\widetilde{H}$, respectively) be the functions defined on $[0, +\infty)$ (on $(-\infty, 0]$, respectively) as in Proposition 5.20. Let*

$$\widehat{\mathbf{\Gamma}} \triangleq \{y \in (0, +\infty) : W(y) = H(y)\}, \quad (\widetilde{\mathbf{\Gamma}} \triangleq \{y \in (-\infty, 0) : \widetilde{W}(y) = \widetilde{H}(y)\}, \text{ respectively})$$

Then $\mathbf{\Gamma} = F^{-1}\big(\widehat{\mathbf{\Gamma}}\big)$ ($\mathbf{\Gamma} = G^{-1}\big(\widetilde{\mathbf{\Gamma}}\big)$, respectively), where $\mathbf{\Gamma}$ is defined as in (5.17).

Lemma 5.6. *If ℓ_a or ℓ_b is strictly positive (provided that they are still finite), and h is continuous, then τ^* of (5.17) is still an optimal stopping time if and only if[3].*

$$\left\{ \begin{array}{c} \text{there is no } r \in (a, b) \\ \text{such that } (a, r) \subset \mathbf{C} \\ \text{if } \ell_a > 0 \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} \text{there is no } l \in (a, b) \\ \text{such that } (l, b) \subset \mathbf{C} \\ \text{if } \ell_b > 0 \end{array} \right\}.$$

Proof. Similar to the proof of Lemma 5.2. $\qquad\square$

No matter whether ℓ_a and ℓ_b are are zero or not, we always have $\varepsilon(x)$–optimal stopping times, as long as both are finite. Introduce for every $\varepsilon > 0$

$$\mathbf{\Gamma}_\varepsilon \triangleq \left\{ x \in (a, b) \,\middle|\, \frac{h(x)}{\psi(x) + \varphi(x)} \geq \frac{V(x)}{\psi(x) + \varphi(x)} - \varepsilon \right\} \quad \text{and} \quad \tau_\varepsilon^* \triangleq \inf\ \{t \geq 0 : X_t \in \mathbf{\Gamma}\}.$$

---

[3]See the footnote at page 45

Proposition 5.22. *Suppose h is continuous, and (5.27) hold. Then*

$$V_\varepsilon(x) \triangleq \mathbb{E}_x[e^{-\beta \tau_\varepsilon^*} h(X_{\tau_\varepsilon^*})] \geq V(x) - \varepsilon[\psi(x) + \varphi(x)], \qquad x \in (a,b).$$

Proof. By the continuity of all the functions involved, $\mathbf{\Gamma}_\varepsilon$ is closed. Therefore $X_{\tau_\varepsilon^*} \in \mathbf{\Gamma}_\varepsilon$ on $\{\tau_\varepsilon^* < \infty\}$, and

$$V_\varepsilon(x) \geq \mathbb{E}_x[e^{-\beta \tau_\varepsilon^*} V(X_{\tau_\varepsilon^*})] - \varepsilon \mathbb{E}_x[e^{-\beta \tau_\varepsilon^*}(\psi + \varphi)(X_{\tau_\varepsilon^*})]$$

$$\geq \mathbb{E}_x[e^{-\beta \tau_\varepsilon^*} V(X_{\tau_\varepsilon^*})] - \varepsilon[\psi(x) + \varphi(x)], \quad x \in (a,b).$$

Second inequality follows from Proposition 5.19 since $(\psi + \varphi)/\varphi = F + 1$ is F–concave in (a,b). Note that the proof will be complete if we can show

$$U(x) \triangleq \mathbb{E}_x[e^{-\beta \tau_\varepsilon^*} V(X_{\tau_\varepsilon^*})] \geq V(x), \qquad x \in (a,b).$$

We shall do this by showing that U is a nonnegative majorant of h such that $\frac{U}{\varphi}$ is F–concave on (a,b) (indeed, the conclusion then follows immediately from Proposition 5.19).

U is nonnegative since V is nonnegative. Using strong Markov property and Optional Sampling Theorem for nonnegative supermartingales, we can show as in the proof of Proposition 5.9 that

$$U(x) \geq \mathbb{E}_x[e^{-\beta \tau} V(X_\tau)], \qquad x \in (a,b),$$

for every stopping time τ. Therefore $\frac{U}{\psi}$ is G–concave by Proposition 5.17.

It remains to show that U majorizes h in (a,b). Assume on the contrary $h > U$ somewhere in (a,b). Then

$$\theta \triangleq \sup_{x \in (a,b)} \frac{h(x) - U(x)}{\psi(x) + \varphi(x)} > 0.$$

Note that θ is finite. Because U is nonnegative, (5.27) implies that for every large enough $[l,r] \subset (a,b)$, we have

$$\frac{h(x) - U(x)}{\psi(x) + \varphi(x)} \leq \max\left\{\frac{h^+(x)}{\psi(x)}, \frac{h^+(x)}{\varphi(x)}\right\} \leq 1 + \max\{\ell_a, \ell_b\}, \qquad x \notin [l,r].$$

Therefore $\widetilde{U}(x) \triangleq U(x) + \theta[\psi(x) + \varphi(x)]$ is a nonnegative majorant of h such that $\frac{\widetilde{U}}{\varphi}$ is F–concave in (a, b). Therefore $\widetilde{U} \geq V$ in (a, b).

Since θ is finite, for every $0 < \delta < \varepsilon$ there exists some $x_\delta \in (a, b)$ such that $\frac{h(x_\delta) - U(x_\delta)}{\psi(x_\delta) + \varphi(x_\delta)} > \theta - \delta$. Therefore

$$h(x_\delta) > U(x_\delta) + (\theta - \delta)[\psi(x_\delta) + \varphi(x_\delta)] = \widetilde{U}(x_\delta) - \delta[\psi(x_\delta) + \varphi(x_\delta)]$$

$$\geq V(x_\delta) - \delta[\psi(x_\delta) + \varphi(x_\delta)] \geq V(x_\delta) - \varepsilon[\psi(x_\delta) + \varphi(x_\delta)].$$

Hence $x_\delta \in \mathbf{\Gamma}_\varepsilon$ for every $0 < \delta < \varepsilon$. Therefore $\mathbb{P}_{x_\delta}\{\tau_\varepsilon^* = 0\} = 1$, i.e. $U(x_\delta) = V(x_\delta) \geq h(x_\delta)$ and

$$0 \geq h(x_\delta) - V(x_\delta) = h(x_\delta) - U(x_\delta) \geq (\theta - \delta)[\psi(x_\delta) + \varphi(x_\delta)].$$

Hence $\theta < \delta$ for every $0 < \delta < \varepsilon$. Therefore $\theta \leq 0$. Contradiction. We conclude that U majorizes h. $\qquad\square$

Chapter 6

Examples

In this chapter, we shall illustrate how the results of Chapters 3–5 apply in various optimal stopping problems that were studied in the literature.

We already witnessed that everything boils down, in optimal stopping, to finding the smallest nonnegative concave majorant of the reward function on an interval. Since we are working with one–dimensional processes, this smallest concave majorant can be constructed easily in most interesting cases.

6.1 Pricing an "Up–and–Out" Barrier Put–Option of American Type under Black–Scholes Model (Karatzas and Wang [9])

Karatzas and Wang [9] solve the pricing problem for an "up–and–out" barrier put–option of American type, by solving the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x \big[e^{-r\tau} (q - S_\tau)^+ \, 1_{\{\tau < \tau_d\}} \big], \quad x \in (0, d), \tag{6.1}$$

using variational inequalities. Here S is the stock price process governed under the risk-neutral measure by

$$dS_t = S_t(r dt + \sigma dB_t), \qquad S_0 = x \in (0, d),$$

B is standard Brownian motion, and the risk–free interest rate $r > 0$ and the volatility $\sigma > 0$ are constant. $d > 0$ is the barrier and $q \in (0, d)$ is the strike–price of the option. Moreover $\tau_d \triangleq \inf\{t \geq 0 : S(t) \geq d\}$ is the time when the option becomes "knocked-out". The state space of S is $\mathcal{I} = (0, \infty)$. Since the drift r is positive, 0 is a natural boundary for S, whereas it hits $c \in \text{int}(\mathcal{I})$ with probability one.

We shall offer here a novel solution for (6.1) by using the techniques of Chapter 5. For this purpose, consider the stopped stock–price process, denoted by $\widetilde{S}_t$, which starts in $(0, d]$ and is absorbed when it reaches the barrier d.

It is clear from (6.1) that $V(x) \equiv 0$, $x \geq d$. We therefore need to determine V on $(0, d]$. Note that V does not depend on the behavior of stock–price process after it reaches the barrier d, and we can rewrite

$$V(x) = \sup_{\tau \geq 0} \mathbb{E}_x \big[e^{-r\tau} h(\widetilde{S}_\tau) \big], \quad x \in (0, d].$$

where $h : (0, d] \to \mathbb{R}$ is the reward function given by $h(x) \triangleq (q - x)^+$ (see Figure 6.1(a)). The infinitesimal generator $\mathcal{A}$ of S is

$$\mathcal{A}u(x) \triangleq \frac{\sigma^2}{2} x^2 u''(x) + r x u'(x),$$

acting on smooth functions u. The increasing and decreasing positive solutions, ψ and φ, respectively, of $\mathcal{A}u = ru$ turn out to be

$$\psi(x) = x \quad \text{and} \quad \varphi(x) = x^{-\frac{2r}{\sigma^2}}, \qquad x \in (0, \infty).$$

Observe that $\psi(0+) = 0$, $\varphi(0+) = +\infty$ and both are continuous functions on $(0, d]$. Furthermore h is continuous on $(0, d]$, and

$$\ell_0 \triangleq \limsup_{x \to 0} \frac{h^+(x)}{\varphi(x)} = \lim_{x \to 0} \frac{h(x)}{\varphi(x)} = 0.$$

Proposition 5.11 implies that V is finite. We can therefore invoke Proposition 5.12 to conclude that V is the smallest nonnegative majorant of h such that $\frac{V}{\varphi}$ is $F-$ concave on $(0, d]$. Furthermore, Proposition 5.13 implies that $V(x) = \varphi(x)W(F(x))$, $x \in (0, d]$, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^{1+\frac{2r}{\sigma^2}} \equiv x^\beta, \quad x \in (0, d]; \qquad \beta \triangleq 1 + \frac{2r}{\sigma^2} > 1, \qquad (6.2)$$

and $W : [0, d^\beta](\equiv [F(0+), F(d)]) \to \mathbb{R}$ is the smallest nonnegative concave majorant of $H : [0, d^\beta] \to \mathbb{R}$, defined by

$$H(y) \triangleq \begin{cases} \left(\frac{h}{\varphi}\right) \circ F^{-1}(y), & y \in (0, d^\beta] \\ \ell_0, & y = 0 \end{cases} = \begin{cases} y^{1-\frac{1}{\beta}}\left(q - y^{\frac{1}{\beta}}\right)^+, & y \in (0, d^\beta] \\ 0, & y = 0 \end{cases}.$$

To identify W explicitly, we shall first sketch H. Since h and φ are nonnegative, H is also nonnegative. Note that $H \equiv 0$ on $[q^\beta, d^\beta]$. On $(0, q^\beta)$, $H(x) = y^{1-\frac{1}{\beta}}\left(q - y^{\frac{1}{\beta}}\right)$ is twice–continuously differentiable, and

$$H'(y) = q\left(1 - \frac{1}{\beta}\right)y^{-\frac{1}{\beta}} - 1, \quad H''(y) = q\,\frac{1-\beta}{\beta^2}\,y^{-(1+\frac{1}{\beta})} < 0, \qquad x \in (0, q^\beta),$$

since $\beta > 1$. Hence H is the strictly concave on $[0, q^\beta]$ (See Figure 6.1(b)).

It is clear from Figure 6.1(c) that strict concavity of H on $[0, q^\beta]$ guarantees that there exists unique $z_0 \in (0, q^\beta)$ such that

$$H'(z_0) = \frac{H(d^\beta) - H(z_0)}{d^\beta - z_0} = -\frac{H(z_0)}{d^\beta - z_0}. \qquad (6.3)$$

Therefore the straight line $L_{z_0} : [0, d^\beta] \to \mathbb{R}$,

$$L_{z_0}(y) \triangleq H(z_0) + H'(z_0)(y - z_0), \quad y \in [0, d^\beta],$$

is tangent to H at z_0 and coincides with the chord expanding between $(z_0, H(z_0))$ and $(d^\beta, H(d^\beta) \equiv 0)$ over the graph of H. Since $H(z_0) > 0$, (6.3) implies that L_{z_0} is decreasing. Therefore $L_{z_0} \geq L_{z_0}(d^\beta) \geq 0$ on $[0, d^\beta]$. Remember also that H is concave
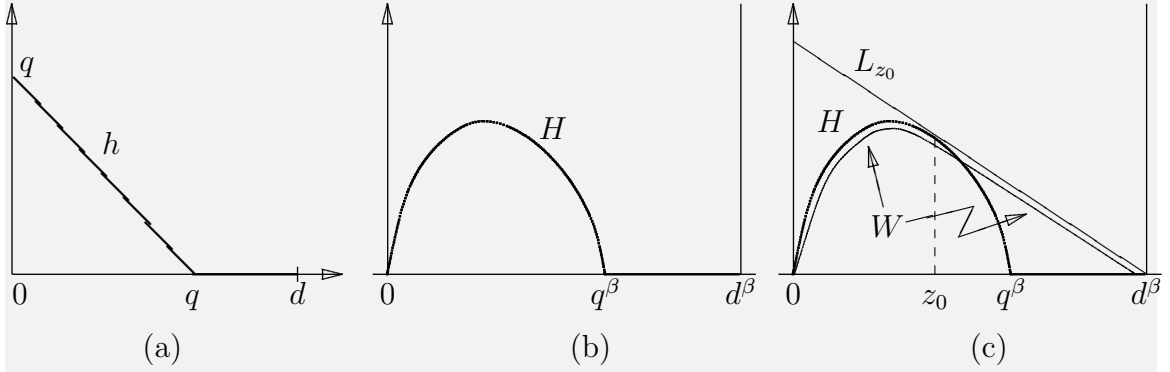
Figure 6.1: (Pricing Barrier Option) Sketches of (a) h, (b) H and (c)H and W.

We find V using Proposition 5.6. We shall therefore need to calculate the smallest nonnegative concave majorant W of H on $[F(0), F(d)] \equiv [0, d^\beta]$.

This problem is equivalent to wrapping an object with cross chapter $H \vee 0 \equiv H$ using the minimum amount of rope. Because H is concave on $[0, q^\beta]$, W tightly wraps H as in (c) from 0 to some $z_0 \in (0, q^\beta)$ where it takes the "short" cut, i.e. the line L_{z_0}, to the right–most end d^β. Since H is strictly concave on $[0, q^\beta]$, z_0 is unique. Since H is smooth on $(0, q^\beta) \ni z_0$, the slope of L_{z_0} must agree with H' at z_0.

Since $V(\cdot) = \varphi(\cdot)W(F(\cdot))$ by Proposition 5.6, $V > h$ if and only if $W(F) > H(F)$. From (c), the optimal continuation region is $\{x \in [0, d] : V(x) > h(x)\} = F^{-1}((z_0, d^\beta)) = \big(F^{-1}(z_0), d\big)$.

on $[0, z_0]$, and every linear function is concave. It is evident from Figure 6.1(c) that the smallest nonnegative concave majorant of H on $[0, d^\beta]$ is given by

$$W(y) = \begin{cases} H(y), & \text{if } y \in [0, z_0] \\ L_{z_0}(y), & \text{if } y \in (z_0, d^\beta] \end{cases} = \begin{cases} H(y), & \text{if } y \in [0, z_0] \\ H(z_0)\dfrac{d^\beta - y}{d^\beta - z_0}, & \text{if } y \in (z_0, d^\beta] \end{cases}$$

where we used the defining relation (6.3) for z_0 to get the second equality. Strict concavity of H on $[0, q^\beta]$ also implies that $\widetilde{\mathbf{C}} \triangleq \{y \in [0, d^\beta] : W(y) > H(y)\} = (z_0, d^\beta)$.

From (6.2), we find $F^{-1}(y) = y^{1/\beta}$, $y \in [0, d^\beta]$. Let $x_0 \triangleq F^{-1}(z_0) = z_0^{1/\beta}$. Then

$x_0 \in (0, d)$. Proposition 5.13 implies that $V(x) = \varphi(x)W(F(x))$, $x \in (0, d]$, i.e.

$$
V(x) =
\begin{cases}
q - x, & 0 \leq x \leq x_0, \\[2mm]
(q - x_0) \cdot \dfrac{x}{x_0} \cdot \dfrac{d^{-\beta} - x^{-\beta}}{d^{-\beta} - x_0^{-\beta}}, & x_0 < x \leq d.
\end{cases}
\tag{6.4}
$$

Since $\ell_0 = 0$, Proposition 5.15 guarantees the existence of an optimal stopping time. The optimal continuation region becomes $\mathbf{C} \triangleq \{x \in (0, d] : V(x) > h(x)\} = F^{-1}(\widetilde{\mathbf{C}}) = F^{-1}((z_0, d^\beta)) = (x_0, d)$. Therefore $\tau^* \triangleq \inf\{t \geq 0 : S_t \notin (x_0, d)\}$ is optimal. Finally, one can show that the relation in (6.3), which uniquely determines z_0 (therefore x_0), takes the form

$$
1 + \beta \frac{x_0}{q} = \beta + \left(\frac{x_0}{d}\right)^\beta,
\tag{6.5}
$$

after some simple algebra using definitions of H, H' and $x_0 \equiv z_0^{1/\beta}$. Compare (6.4) and (6.5) above with (2.18) and (2.19) in Karatzas and Wang [9, pages 263 and 264], respectively.

6.2 Pricing an "Up–and–Out" Barrier Put–Option of American Type under Constant–Elasticity– of–Variance (CEV) Model

We shall look at the same optimal stopping problem of (6.1) by assuming now that the stock price dynamics are described according to CEV model,

$$
dS_t = rS_t dt + \sigma S_t^{1-\alpha} dB_t, \qquad S_0 \in (0, d),
$$

for some $\alpha \in (0, 1)$. The infinitesimal generator for this process is $\mathcal{A} = \frac{1}{2}\sigma^2 x^{2(1-\alpha)} \frac{d^2}{dx^2} + rx \frac{d}{dx}$. The increasing and decreasing solutions of $\mathcal{A}u = ru$ are given by

$$
\psi(x) = x, \quad \varphi(x) = x \cdot \int_x^{+\infty} \frac{1}{z^2} \exp\left\{-\frac{r}{\alpha\sigma^2} z^{2\alpha}\right\} dz, \qquad x \in (0, +\infty),
$$

respectively. Moreover

$$\psi(0+) = 0, \ \varphi(0+) = 1, \quad \text{and} \quad \psi(+\infty) = +\infty, \ \varphi(+\infty) = 0.$$

Therefore 0 is an exit–and–not–entrance boundary, and $+\infty$ is a natural boundary for S. We shall regard 0 as an absorbing boundary (i.e., upon exit at 0, we shall assume that the process remains there forever).

We shall also modify the process such that d becomes an absorbing boundary. Therefore, we have our optimal stopping problem in the canonical form of Chapter 4, with the reward function $h(x) = (q - x)^+$, $x \in [0, d]$.

One can show that the results of Chapter 4 stay valid when the left–boundary of the state space is an exit–and–not–entrance boundary. According to Proposition 4.3, $V(x) = \psi(x)W(G(x))$, $x \in [0, d]$ with

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -\int_x^{+\infty} \frac{1}{u^2} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\} du, \qquad x \in (0, d], \tag{6.6}$$

and $W : (G(0+), G(d)] \equiv (-\infty, G(d)] \rightarrow \mathbb{R}$ is the smallest nonnegative concave majorant of $H : (-\infty, G(d)] \rightarrow \mathbb{R}$, given by

$$H(y) \triangleq \left(\frac{h}{\psi} \circ G^{-1}\right)(y) = \begin{cases} \left[\left(\frac{q}{x} - 1\right) \circ G^{-1}\right](y), & \text{if } -\infty < y < G(q) \\ 0, & \text{if } G(q) \leq y \leq 0 \end{cases}. \tag{6.7}$$

Except for $y = G(q)$, H is twice–differentiable on $(-\infty, G(d))$. Note that

$$\begin{aligned}
\frac{dH}{dy}(y) &= \left(\frac{d}{dx}\frac{h}{\psi}\right)(G^{-1}(y)) \cdot \frac{1}{G'(G^{-1}(y))} = \left[\left(\frac{(h/\psi)'}{G'}\right) \circ G^{-1}\right](y) \\
&= \begin{cases} \left[\left(-q\exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y), & \text{if } -\infty < y < G(q) \\ 0, & \text{if } G(q) < y < 0 \end{cases},
\end{aligned}$$

and with $f(x) \triangleq -q\exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}$, we have

$$\begin{aligned}
\frac{d^2H}{dy^2}(y) &= \frac{d}{dy}\left(f \circ G^{-1}\right)(y) = \left[\left(\frac{f'}{G'}\right) \circ G^{-1}\right](y) \\
&= \begin{cases} \left[\left(-\frac{2rq}{\sigma^2}x^{2\alpha+1}\exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y), & \text{if } -\infty < y < G(q) \\ 0, & \text{if } G(q) < y < 0 \end{cases}.
\end{aligned}$$

Therefore, H is strictly decreasing and strictly concave on $(-\infty, G(q))$. Moreover $H(-\infty) = +\infty$ and $H'(-\infty) = -q$, since $G^{-1}(-\infty) = 0$.

For every $-\infty < y < G(q)$, let $z(y)$ be the point on the y–axis, where the tangent line $L_y(\cdot)$ of $H(\cdot)$ at y intersects the y–axis (cf. Figure 6.2(a)). Then

$$
\begin{aligned}
z(y) &= y - \frac{H(y)}{H'(y)} = G(G^{-1}(y)) - \frac{\left[\left(\frac{q}{x} - 1\right) \circ G^{-1}\right](y))}{\left[\left(-q \exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y)} \\
&= \left[\left(G(x) - \frac{\frac{q}{x} - 1}{-q \exp\left\{\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}}\right) \circ G^{-1}\right](y) \\
&= \left[\left(-\int_x^{+\infty} \frac{1}{u^2} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du + \left(\frac{1}{x} - \frac{1}{q}\right)\exp\left\{-\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y) \\
&= \left[\left(\frac{2r}{\sigma^2}\int_x^{+\infty} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du - \frac{1}{q}\exp\left\{-\frac{r}{\alpha\sigma^2}x^{2\alpha}\right\}\right) \circ G^{-1}\right](y),
\end{aligned}
$$

$$(6.8)$$

where the last equality follows from integration by parts. It is geometrically clear that $y \to z(y) : (-\infty, G(q)) \to (G(q), +\infty)$ is strictly decreasing. Since $G^{-1}(-\infty) = 0$, we have

$$
z(-\infty) = \frac{2r}{\sigma^2}\int_0^{+\infty} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du - \frac{1}{q}
$$

Note that $G(q) < z(-\infty) < +\infty$ if $1/2 < \alpha < 1$, and $z(-\infty) = +\infty$ if $0 < \alpha \leq 1/2$. (This latter conclusion is a little puzzling. At the first glance, one may expect $z(-\infty) < +\infty$ all the time, since H is strictly decreasing and concave, with one zero and $H'(-\infty) = -q < 0$. Especially, one expects that H must have an asymptote in the form of $\ell(y) \triangleq a - qy$ for some real numbers a and b, as y tends to $-\infty$. It is however not difficult to come up with a simple function, which has all of the properties of H. Let $g : (-\infty, 0] \to \mathbb{R}$ be defined by $g(y) \triangleq -y + \sqrt{-y}$. Then $g'(y) = -1 - (1/2)(-y)^{-1/2}$, and g is strictly decreasing and concave. Moreover $g'(-\infty) = -1 < 0$, and $g(0) = 0$. Notice now that

$$
z(y) = y - \frac{g(y)}{g'(y)} = -\frac{y}{1 + 2\sqrt{-y}}, \qquad y \in (-\infty, 0),
$$

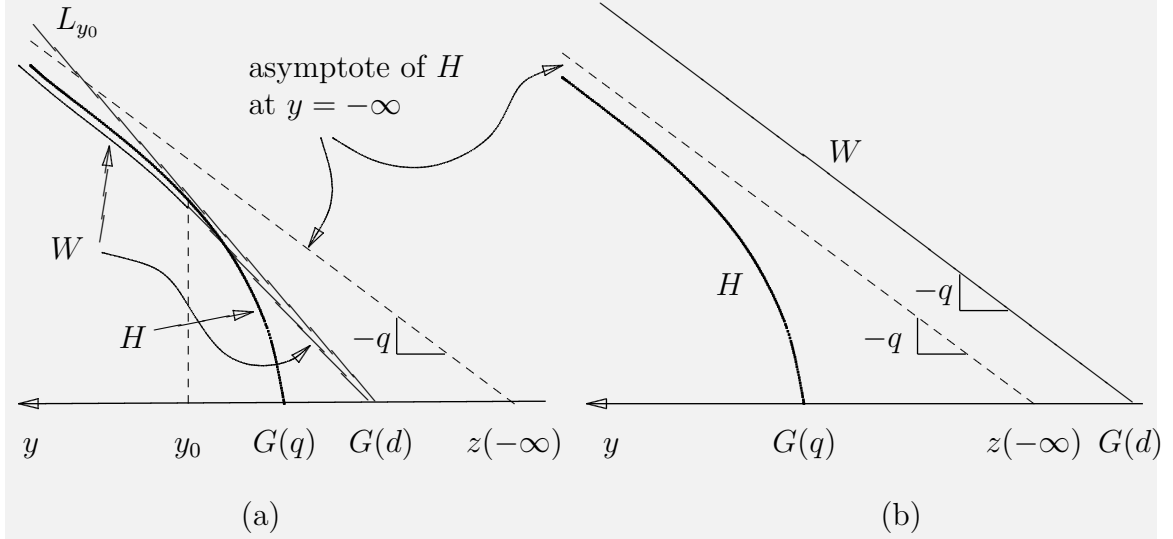and $z(-\infty) = \lim_{y \to -\infty} z(y) = +\infty$).

Figure 6.2: (Pricing Barrier Option under CEV Model) Sketches of functions H and W of Proposition 4.3, when (a) $G(d) < z(-\infty)$ (for this sketch, we assume that $z(-\infty)$ is finite. However, $z(-\infty) = +\infty$ is also possible, in which case H does not have a linear asymtote), and (b) $G(d) > z(-\infty)$.

Case I. Suppose first $G(d) < z(-\infty)$ (especially, when $0 < \alpha \le 1/2$). Then there exists unique $y_0 \in (-\infty, G(q))$ such that $z(y_0) = G(d)$ thanks to the monotinicity and the continuity of $z(\cdot)$. In other words, the tangent line $L_{y_0}(\cdot)$ of $H(\cdot)$ at $y = y_0 < G(q)$ intersects y–axis at $y = G(d)$. It is furthermore clear from Figure 6.2(a) that

$$W(y) = \begin{cases} H(y), & \text{if } -\infty < y \le y_0 \\ H(y_0)\dfrac{G(d) - y}{G(d) - y_0}, & \text{if } y_0 < y \le G(d) \end{cases}$$

is the smallest nonnegative concave majorant of H of (6.7) on $y \in (-\infty, G(d)]$. Define $x_0 \triangleq G^{-1}(y_0)$. According to Proposition 4.3, $V(x) = \psi(x)W(G(x))$, $x \in [0, d]$, i.e.,

$$V(x) = \begin{cases} q - x, & \text{if } 0 \le x \le x_0 \\ (q - x_0) \cdot \dfrac{x}{x_0} \cdot \dfrac{G(d) - G(x)}{G(d) - G(x_0)}, & \text{if } x_0 < x \le d \end{cases}.$$

The optimal continuation region becomes $\mathbf{C} = (x_0, d)$, and $\tau^* \triangleq \inf\{t \ge 0 : S_t \notin (x_0, d)\}$ is an optimal stopping time. The relation $z(G(x_0)) = G(d)$ determines $x_0 \in$

(q, d) uniquely. From (6.6) and integration by parts, we get

$$G(d) = \frac{2r}{\sigma^2} \int_d^{+\infty} u^{2(\alpha-1)} \exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du - \frac{1}{d}\exp\left\{-\frac{r}{\alpha\sigma^2}d^{2\alpha}\right\}, \qquad (6.9)$$

and the defining relation for x_0 can be rewitten, with the help of (6.8) and (6.9), as

$$\frac{2r}{\sigma^2}\int_{x_0}^d u^{2(\alpha-1)}\exp\left\{-\frac{r}{\alpha\sigma^2}u^{2\alpha}\right\}du = \frac{1}{q}\exp\left\{-\frac{r}{\alpha\sigma^2}x_0^{2\alpha}\right\} - \frac{1}{d}\exp\left\{-\frac{r}{\alpha\sigma^2}d^{2\alpha}\right\}.$$

Case II. Suppose now $G(d) > z(-\infty)$ (cf. Figure 6.2(b)). It is then clear that

$$W(y) = -q\big[y - G(d)\big], \qquad -\infty < y \leq G(d)$$

is the smallest nonnegative concave majorant of H of (6.7) on $(-\infty, G(d)]$. According to Proposition 4.3, $V(x) = \psi(x)W(G(x))$, $x \in [0, d]$, i.e.

$$V(x) = -qx\big[G(x) - G(d)\big], \qquad x \in [0, d],$$

where $V(0) = V(0+) = q$. Furthermore, $\tau^* \triangleq \inf\{t \geq 0 : S_t \notin (0, d)\}$ is an optimal stopping time.

6.3 American Capped Call Option on Dividend– Paying Assets (Broadie and Detemple [1])

Let the stock price be driven by

$$dS_t = S_t\big[(r - \delta)dt + \sigma dB_t\big], \quad t \geq 0, \ \ S_0 > 0,$$

with constant $\sigma > 0$, risk–free interest rate $r > 0$ and dividend rate $\delta \geq 0$. Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x\big[e^{-r\tau}(S_\tau \wedge L - K)^+\big], \quad x \in (0, +\infty), \qquad (6.10)$$

with the reward function $h(x) \triangleq (x \wedge L - K)^+$, $x > 0$. The value function $V(\cdot)$ is the arbitrage–free price of the *perpetual* American capped call option with strike price

$K \geq 0$, and the cap $L > K$ on the stock S, which pays dividend at a constant rate δ. We shall reproduce the results of Broadie and Detemple [1] in this subsection.

The infinitesimal generator of X coincides with the second–order differential operator $\mathcal{A} \triangleq (\sigma^2/2)x^2 \frac{d^2}{dx^2} + (r - \delta)x \frac{d}{dx}$. Let $\gamma_1 < 0 < \gamma_2$ be the roots of

$$\frac{1}{2}\sigma^2 x^2 + \left(r - \delta - \frac{\sigma^2}{2}\right) x - r = 0.$$

Then the increasing and decreasing solutions of $\mathcal{A}u = ru$ are given by

$$\psi(x) = x^{\gamma_2}, \quad \text{and} \quad \varphi(x) = x^{\gamma_1}, \qquad x > 0,$$

respectively. Both endpoints of the state–space $\mathcal{I} = (0, +\infty)$ of S are natural. Since

$$\ell_0 \triangleq \limsup_{x \downarrow 0} \frac{h^+(x)}{\varphi(x)} = 0, \quad \text{and} \quad \ell_{+\infty} \triangleq \limsup_{x \to +\infty} \frac{h^+(x)}{\psi(x)} = 0,$$

the value function $V(\cdot)$ of (6.10) is finite, and the stopping time τ^* of (5.17) is optimal. Moreover $V(x) = \varphi(x)W(F(x))$, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^\theta, \quad x > 0, \quad \text{and} \quad \theta \triangleq \gamma_2 - \gamma_1 > 0,$$

and $W : [F(0+), F(+\infty)) \to [0, +\infty)$ is the smallest nonnegative concave majorant of $H : [F(0+), F(+\infty)) \to [0, +\infty)$, given by

$$H(y) \triangleq \left(\frac{h}{\varphi}\right)(F^{-1}(y)) = \begin{cases} 0, & \text{if } 0 \leq y < K^\theta, \\ \left(y^{1/\theta} - K\right) y^{-\gamma_1/\theta}, & \text{if } K^\theta \leq y < L^\theta \\ (L - K)y^{-\gamma_1/\theta}, & \text{if } y \geq L^\theta, \end{cases} \tag{6.11}$$

thanks to Proposition 5.20. We have

$$H'(y) = \begin{cases} 0, & 0 < y < K^\theta \\ \frac{1 - \gamma_1}{\theta} y^{(1-\gamma_2)/\theta} + \frac{\gamma_1}{\theta} K y^{-\gamma_2/\theta}, & K^\theta < y < L^\theta \\ -\frac{\gamma_1}{\theta}(L - K)y^{-\gamma_2/\theta}, & y > L^\theta \end{cases},$$

and

$$H''(y) = \begin{cases} 0, & 0 < y < K^\theta \\ \dfrac{(1-\gamma_1)(1-\gamma_2)}{\theta^2}\, y^{(1-\gamma_2)/\theta-1} - \dfrac{\gamma_1\gamma_2}{\theta^2} K\, y^{-\gamma_2/\theta-1}, & M^\theta < y < L^\theta \\ \dfrac{\gamma_1\gamma_2}{\theta^2}(L-K)y^{-\gamma_1/\theta-1}, & y > L^\theta \end{cases}.$$

The function $H(\cdot)$ is nondecreasing on $[0,+\infty)$ and strictly concave on $[L^\theta, +\infty)$.

By solving the inequality $H''(y) \leq 0$, for $K^\theta \leq y \leq L^\theta$, we find that

$$\text{H}(\cdot) \text{ is } \begin{cases} \text{convex on} & [K^\theta, L^\theta] \cap [0, (r/\delta)^\theta K^\theta] \\ \text{concave on} & [K^\theta, L^\theta] \cap [(r/\delta)^\theta K^\theta, +\infty) \end{cases}.$$

It is easy to check that $H(L^\theta)/L^\theta \geq H'(L^\theta+)$. (See Figure 6.3).

Let $\mathcal{L}_z(y) \triangleq y\, H(z)/z$, for every $y \geq 0$ and $z > 0$. If $(r/\delta)K \geq L$, then

$$\mathcal{L}_{L^\theta}(y) \geq H(y), \quad y \geq 0, \tag{6.12}$$

(cf. Figure 6.3(b)). If $(r/\delta)K < L$, then (6.12) holds if and only if

$$\frac{H(L^\theta)}{L^\theta} < H'(L^\theta-) \quad \Longleftrightarrow \quad \gamma_2 \leq \frac{L}{L-K},$$

(cf. Figure 6.3(d,f)). If $(r/\delta)K < L$ and $\gamma_2 > L/(L-K)$, then the equation $H(z)/z = H'(z)$, $K^\theta < z < L^\theta$ has unique solution, $z_0 \triangleq [\gamma_2/(\gamma_2-1)]^\theta K^\theta > (r/\delta)^\theta K^\theta$, and

$$\mathcal{L}_{z_0}(y) \geq H(y), \quad y \geq 0,$$

(cf. Figure 6.3(c,e)). It is now clear that the smallest nonnegative concave majorant of $H(\cdot)$ is

$$W(y) = \begin{cases} \mathcal{L}_{z_0 \wedge L^\theta}(y), & \text{if } 0 \leq y \leq z_0 \wedge L^\theta \\ H(y), & \text{if } y > z_0 \wedge L^\theta \end{cases}$$

in all cases. Finally

$$V(x) = \varphi(x)W(F(x)) = \begin{cases} (x_0 \wedge L - K)\left(\dfrac{x}{x_0 \wedge L}\right)^{\gamma_2}, & \text{if } 0 < x \leq x_0 \wedge L \\ x \wedge L - K, & \text{if } x > x_0 \wedge L \end{cases},$$

where $x_0 \triangleq F^{-1}(z_0) = K \gamma_2/(\gamma_2 - 1)$. The optimal stopping region is $\mathbf{\Gamma} \triangleq \{x : V(x) = h(x)\} = [x_0 \wedge L, +\infty)$, and the stopping time $\tau^* \triangleq \inf\{t \geq 0 : S_t \in \mathbf{\Gamma}\} = \inf\{t \geq 0 : S_t \geq x_0 \wedge L\}$ is optimal. Finally, it is easy to check that $\gamma_2 = 1$ (therefore $x_0 = +\infty$) if and only if $\delta = 0$.
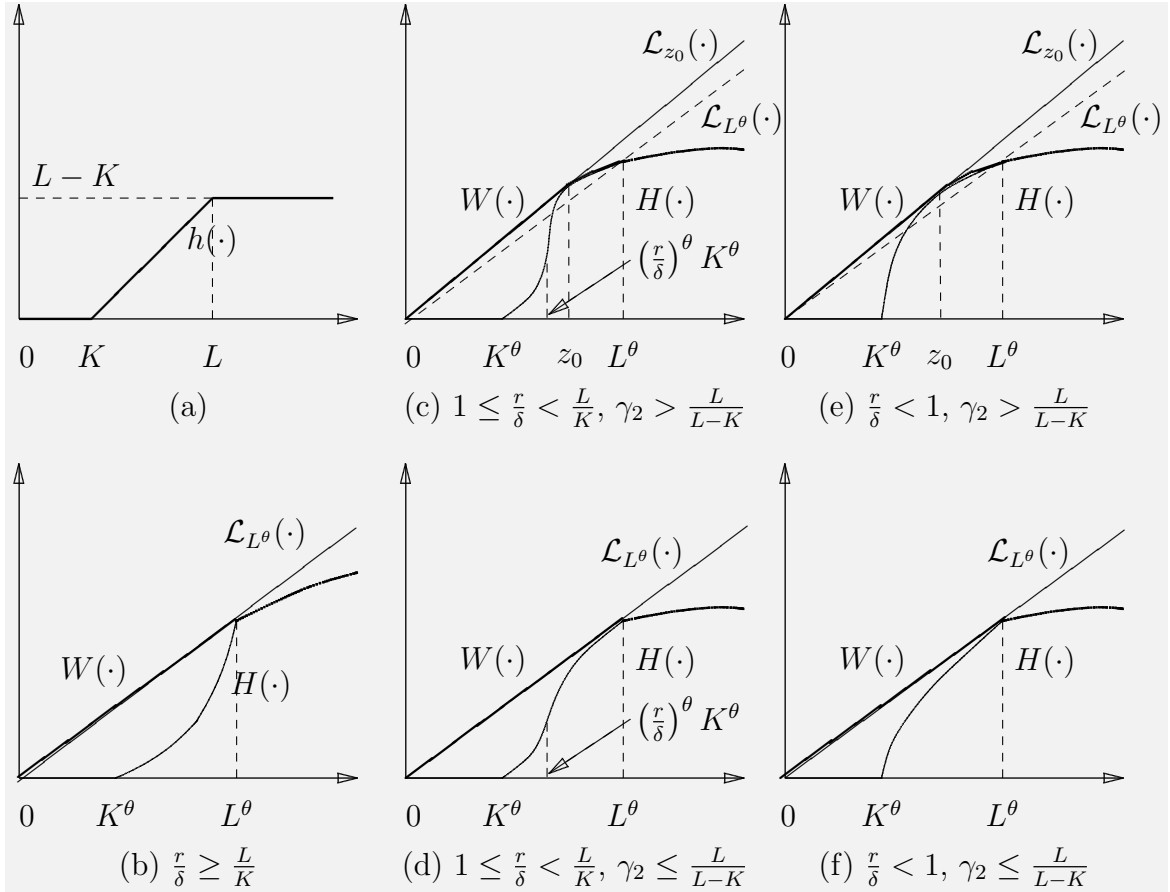
Figure 6.3: (Perpetual American capped call options on dividend–paying assets) Sketches of (a) the reward function $h(\cdot)$, and (b)–(f) the function $H(\cdot)$ of (6.11) and its smallest nonnegative concave majorant $W(\cdot)$.

In cases (b), (d) and (f), the left boundary of the optimal stopping region for the auxiliary optimal stopping problem of (4.10) becomes L^θ, and $W(\cdot)$ does not fit to $H(\cdot)$ smoothly at L^θ. In cases (c) and (e), the left boundary of optimal stopping region, namely z_0, is smaller than L^θ, and $W(\cdot)$ fits to $H(\cdot)$ smoothly at z_0.

6.4 Options for Risk–Averse Investors (Guo and Shepp [4])

Let X be a geometric Brownian Motion with constant drift $\mu \in \mathbb{R}$ and dispersion $\sigma > 0$. Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x\big[e^{-r\tau} \max\{l, X_\tau\}\big], \quad x \in (0, \infty), \tag{6.13}$$

where the reward function is given as $h(x) \triangleq \max\{l, x\}$, $x \in [0, \infty)$, and l and r positive constants.

Guo and Shepp [4] solve this problem using variational inequalities in order to price exotic options of American type. As it is clear from the reward function, the buyer of the option is guaranteed at least l when the option is exercised (an insurance for risk–averse investors). If r is the riskless interest rate, then the price of the option will be obtained when we choose $\mu = r$.

Remember that the dynamics of X is given as

$$dX_t = X_t(\mu dt + \sigma dB_t), \qquad X_t = x \in (0, \infty),$$

where B is standard Brownian motion in $\mathbb{R}$. The infinitesimal generator of X coincides with the second–order differential operator $\mathcal{A} = \frac{\sigma^2}{2}x^2\frac{d^2}{dx^2} + \mu x\frac{d}{dx}$ as it acts on smooth functions. Denote by $\gamma_1, \gamma_0 \triangleq \frac{1}{2}\Big[- \big(\frac{2\mu}{\sigma^2} - 1\big) \mp \sqrt{\big(\frac{2\mu}{\sigma^2} - 1\big)^2 + \frac{8r}{\sigma^2}}\Big]$, with $\gamma_1 < 0 < \gamma_0$, the roots of the second–order polynomial

$$f(x) \triangleq x^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) x - \frac{2r}{\sigma^2}.$$

The positive increasing and decreasing solutions of $\mathcal{A}u = ru$ are then given as

$$\psi(x) = x^{\gamma_0}, \quad \text{and} \quad \varphi(x) = x^{\gamma_1}, \qquad x \in (0, +\infty),$$

respectively. Observe that both end–points, 0 and $+\infty$, of state space of X are natural. Indeed $\psi(0+) = \varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(0+) = +\infty$. We also have

$$\ell_0 \triangleq \limsup_{x \to 0} \frac{h^+(x)}{\varphi(x)} = \lim_{x \to 0} \frac{h(x)}{\varphi(x)} = 0,$$

whereas

$$\ell_\infty \triangleq \limsup_{x \to +\infty} \frac{h^+(x)}{\psi(x)} = \lim_{x \to +\infty} \frac{h(x)}{\psi(x)} = \begin{cases} +\infty, & \text{if } \gamma_0 < 1 \\ 1, & \text{if } \gamma_0 = 1 \\ 0, & \text{if } \gamma_0 > 1 \end{cases}.$$

The value of ℓ_∞ is crucial since it is going to determine both the finiteness of the value function and the existence of an optimal stopping time, as stated in Proposition 5.18 and Proposition 5.21, respecively. Remember that $\gamma_1 < 0 < \gamma_0$ are the roots of $f(x) \triangleq x^2 + \left(\frac{2\mu}{\sigma^2} - 1\right)x - \frac{2r}{\sigma^2}$. Observe that $\gamma_0 < 1$ (i.e. $1 \notin [\gamma_1, \gamma_0]$) if and only if $f(1) > 0$, and $\gamma_0 = 1$ if and only if $f(1) = 0$ (i.e. 1 is the positive root). However

$$f(1) = 1 + \left(\frac{2\mu}{\sigma^2} - 1\right) - \frac{2r}{\sigma^2} = \frac{2}{\sigma^2} \cdot (\mu - r).$$

Therefore

$$\ell_\infty = \begin{cases} +\infty, & \text{if } r < \mu \\ 1, & \text{if } r = \mu \\ 0, & \text{if } r > \mu \end{cases}.$$

Now Proposition 5.18 and Proposition 5.21 imply that

$$\begin{cases} V \equiv +\infty, & \text{if } r < \mu \\ V \text{ is finite, but there is no optimal stopping time,} & \text{if } r = \mu \\ V \text{ is finite, and } \tau^* \text{ of (5.17) is an optimal stopping time,} & \text{if } r > \mu \end{cases}.$$

(Compare with Guo and Shepp[4, Theorem 4 and 5]). There is nothing more to say about the case $r < \mu$. We shall defer the case $r = \mu$ to the next Section (there, we discuss a slightly different and more interesting problem, of essentially the same difficulty as the problem with $r = \mu$).

We shall study the case $r > \mu$ in the remainder part of this Section. So suppose $r > \mu$. V is finite, and it is the smallest nonnegative majorant of h on $(0, \infty)$ such that $\frac{V}{\varphi}$ is F–concave on $(0, \infty)$ by Proposition 5.19.

We shall use Proposition 5.20 in order to identify V. Note that

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^{\gamma_0 - \gamma_1} \equiv x^{\beta}, \quad x \in (0, \infty), \qquad \beta \triangleq \gamma_0 - \gamma_1.$$

Let $W : [0, \infty) \to \mathbb{R}$ be the smallest nonnegative concave majorant of $H : [0, \infty) \to \mathbb{R}$, defined as in Proposition 5.20. Then $V(x) = \varphi(x)W(F(x)) = x^{\gamma_1}W(x^{\beta})$, $x \in (0, \infty)$. Note that

$$H(y) \triangleq \begin{cases} \dfrac{h(F^{-1}(y))}{\varphi(F^{-1}(y))}, & \text{if } y \in (0, +\infty) \\ \ell_0, & \text{if } y = 0 \end{cases} = \begin{cases} H_0(y) \equiv ly^{-\frac{\gamma_1}{\beta}}, & \text{if } 0 \le y < l^{\beta} \\ H_1(y) \equiv y^{\frac{1-\gamma_1}{\beta}}, & \text{if } y \ge l^{\beta} \end{cases},$$

where $H_0(y) \triangleq ly^{-\frac{\gamma_1}{\beta}}$ and $H_1(y) \triangleq y^{\frac{1-\gamma_1}{\beta}}$ for every $y \in (0, \infty)$. See Figure 6.4(a) and Figure 6.4(b) for a sketch of h and H, respectively.

In order to find W, we shall determine convexities and concavities of H. Note that H is twice–continuously differentiable everywhere except at $y = l^{\beta}$. The first two derivatives of H on $(0, \infty) \backslash \{l^{\beta}\}$ are

$$H'(y) = \begin{cases} H_0'(y) = -\dfrac{\gamma_1}{\beta} ly^{-\frac{\gamma_0}{\beta}}, & 0 < y < l^{\beta} \\ H_1'(y) = \dfrac{1 - \gamma_1}{\beta} y^{\frac{1-\gamma_0}{\beta}}, & y > l^{\beta} \end{cases} > 0, \tag{6.14}$$

and

$$H''(y) = \begin{cases} H_0''(y) = \dfrac{\gamma_1\gamma_0}{\beta^2} ly^{-\frac{\gamma_0}{\beta}-1}, & 0 < y < l^{\beta} \\ H_1''(y) = \dfrac{(1 - \gamma_1)(1 - \gamma_0)}{\beta^2} y^{\frac{1-\gamma_0}{\beta}-1}, & y > l^{\beta} \end{cases} < 0. \tag{6.15}$$

Note that H is strictly increasing on $(0, \infty)$, and strictly concave on both of the subintervals $(0, l^{\beta})$ and (l^{β}, ∞). H is concave on $(0, \infty)$ if and only if $H'(l^{\beta}-) \ge H'(l^{\beta}+)$ (which would also imply that $W \equiv H$). However

$$H'(l^{\beta}-) = H_0'(l^{\beta}) = -\frac{\gamma_1}{\beta} l^{1-\gamma_0} < \frac{1 - \gamma_1}{\beta} l^{1-\gamma_0} = H_1'(l^{\beta}) = H'(l^{\beta}+),$$

since $-\gamma_1 > 0$ (See Figure 6.4(b)). Note that H_0 and H_1 are both increasing and concave on $(0, \infty)$. Moreover we always have $H_0 > H_1$ on $[0, l^{\beta})$, and $H_0 < H_1$ on
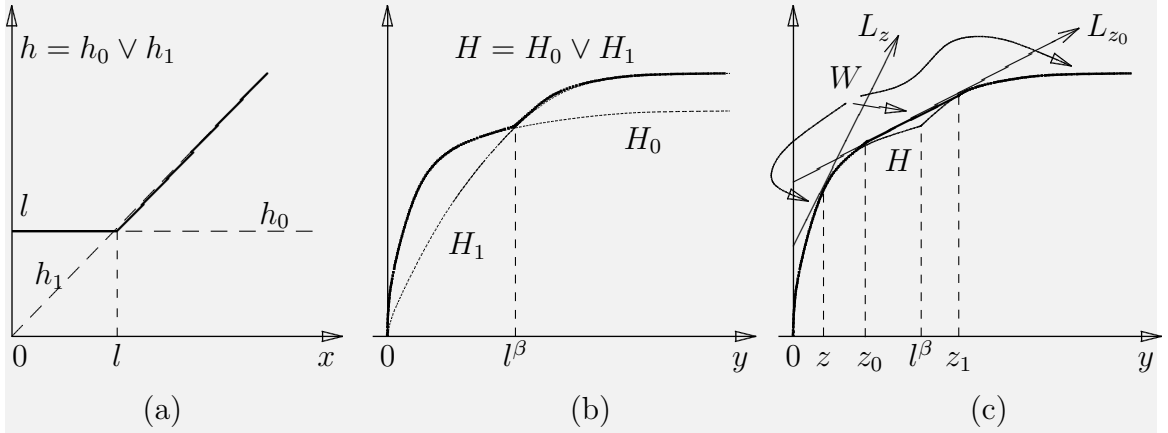
Figure 6.4: (Options for risk–averse investors) (a) h, (b) H, (c) H and W. h is maximum of $h_0 \equiv l$ and $h_1 \equiv x$. The transformation $h \to H$ of Proposition 5.20 preserves the monotonicity. If we similarly let $h_0 \to H_0$ and $h_1 \to H_1$, then H becomes maximum of H_0 and H_1 as in (b).

Both H_0 and H_1 turn out to be concave. Since two curves intersect (at l^β), H will however not be concave. Therefore, the smallest nonnegative concave majorant W of H will not be trivial.

To find W, we observe in (c) how the straight lines L_z, tangent to H at z, change direction as z moves to right. Since $H'(0+) = +\infty$ and H is concave near 0, every tangent L_z near 0 has so large slope that it never meets H other than at z. The slope of L_z however decreases steadily as z shifts right thanks to strict concavity of H_0. For the first time tangent L_{z_0} at some z_0, before H_0 meets H_1 at l^β, touches H_1 at some $z_1 > l^\beta$. We obtain W from H by replacing it on $[z_0, z_1]$ with L_{z_0}: We stick to $H \vee 0 \equiv H$ as much as we can as we maintain a non–increasing slope for our new curve. This will guarantee minimality and concavity as well as nonnegativity of the curve which we just described.

(l^β, ∞) (i.e. H is in fact the maximum of two concave functions, H_0 and H_1, on $(0, \infty)$). It is therefore clear that $H_0'(l^\beta) < H_1'(l^\beta)$ (See Figure 6.4(b)).

It is clear from Figure 6.4(c) that there exist unique $z_0 \in (0, l^\beta)$ and unique

$z_1 \in (l^\beta, \infty)$ such that

$$H'(z_0) = \frac{H(z_1) - H(z_0)}{z_1 - z_0} = H'(z_1). \tag{6.16}$$

This can be rigorously proven by studying the family of straight lines $L_z : [0, \infty) \to \mathbb{R}$,

$$L_z(y) \triangleq H(z) + H'(z)(y - z), \quad y \in (0, \infty),$$

indexed by $z \in (0, l^\beta]$ (see Figure 6.4(c)). These are the lines tangent to H at $z \in (0, l^\beta]$. The proof will use $H'(0+) = +\infty$, and the strict concavity of H on $[0, l^\beta]$ and $[l^\beta, \infty)$. In fact L_{z_0} is the straight line that both majorizes $\max\{H, 0\}$ on $[0, \infty)$, and cuts off the (only) convexity of H in the vicinity of l^β. Since L_{z_0} both majorizes H, and touches H at z_0 and z_1 where H is differentiable, there is a smooth–fit between L_{z_0} and H at both z_0 and z_1 (see Proposition 7.1 for a precise argument), i.e. the slope of the line L_{z_0} is equal to $H'(z_1)$. Therefore $H'(z_0) = H'(z_1)$. On the other hand since L_{z_0} connects $(z_0, H(z_0))$ and $(z_1, H(z_1))$, the slope of L_{z_0} is also same as the quotient in (6.16).

It is evident from Figure 6.4(c) that the smallest nonnegative concave majorant W of H on $[0, \infty)$ is given by

$$W(y) = \begin{cases} H(y), & y \in [0, z_0] \cup [z_1, \infty), \\ L_{z_0}(y), & y \in (z_0, z_1). \end{cases}$$

Strict concavity of H on $(0, l^\beta)$ and (l^β, ∞) also implies that

$$\widetilde{\mathbf{C}} \triangleq \{y \in (0, \infty) : W(y) > H(y)\} = (z_0, z_1).$$

Before we write V, we shall calculate z_0 and z_1 explicitly by solving the two equations in the defining relation (6.16) of z_0 and z_1. Using (6.14) we can write (6.16) as

$$-\frac{\gamma_1}{\beta} l z_0^{-\frac{\gamma_0}{\beta}} = \frac{z_1^{\frac{1-\gamma_1}{\beta}} - l z_0^{-\frac{\gamma_1}{\beta}}}{z_1 - z_0} = \frac{1 - \gamma_1}{\beta} z_1^{\frac{1-\gamma_0}{\beta}}. \tag{6.17}$$

The first equality in (6.17) gives

$$\left(-\frac{\gamma_1}{\beta}\, l z_0^{-\frac{\gamma_0}{\beta}}\right) z_1 = z_1^{\frac{1-\gamma_1}{\beta}} - \frac{\gamma_0}{\beta}\, l z_0^{-\frac{\gamma_1}{\beta}}.$$

We can replace the term in the parenthesis on the left with the third term in (6.17). After rearranging terms and simplifying the expression, we obtain $(\gamma_0 - 1)z_1^{(1-\gamma_1)/\beta} = l\gamma_0 z_0^{-\gamma_1/\beta}$. We solve the last equation for z_0 in terms of z_1, and find

$$z_0 = \left[\frac{\gamma_0 - 1}{l\gamma_0} z_1^{\frac{1-\gamma_1}{\beta}}\right]^{-\beta/\gamma_1}. \tag{6.18}$$

We plug this into $-\frac{\gamma_1}{\beta}\, l z_0^{-\gamma_0/\beta} = \frac{1-\gamma_1}{\beta} z_1^{(1-\gamma_0)/\beta}$ which is the equality of first and third terms in (6.17). Straightforward algebra gives z_1, and by plugging this back into (6.18), we finally find

$$z_0 = l^\beta \left(\frac{\gamma_1}{\gamma_1 - 1}\right)^{1-\gamma_1} \left(\frac{\gamma_0 - 1}{\gamma_0}\right)^{1-\gamma_0} \quad \text{and} \quad z_1 = l^\beta \left(\frac{\gamma_1}{\gamma_1 - 1}\right)^{-\gamma_1} \left(\frac{\gamma_0 - 1}{\gamma_0}\right)^{-\gamma_0}. \tag{6.19}$$

We shall write V explicitly. Let $x_0 \triangleq F^{-1}(z_0) = z_0^{1/\beta}$ and $x_1 \triangleq F^{-1}(z_1) = z_1^{1/\beta}$. By Proposition 5.13, $V(x) = \varphi(x) W(F(x))$, $x \in (0, \infty)$. Since $W \equiv H$ outside (z_0, z_1), $V \equiv h$ outside (x_0, x_1). Since $z_0 < l^\beta < z_1$ and F is increasing, we have $x_0 < l < x_1$. Therefore $V \equiv l$ on $(0, x_0)$, and $V(x) = x$ on $x \in (x_1, \infty)$. If $x \in (x_0, x_1)$, then $F(x) \in (z_0, z_1)$ where $W \equiv L_{z_0}$, and

$$\begin{aligned} V(x) &= \varphi(x)W(F(x)) = x^{\gamma_1} L_{z_0}(x^\beta) = x^{\gamma_1}\left[H(z_0) + H'(z_0)(x^\beta - z_0)\right] \\ &= x^{\gamma_1}\left[l z_0^{-\frac{\gamma_1}{\beta}} - l\frac{\gamma_1}{\beta} z_0^{-\frac{\gamma_0}{\beta}}(x^\beta - z_0)\right] = \frac{l x^{\gamma_1}}{\beta}\left[\beta x_0^{-\gamma_1} - \gamma_1 x_0^{-\gamma_0}(x^\beta - x_0^\beta)\right] \\ &= \frac{l}{\beta}\left[\gamma_0\left(\frac{x}{x_0}\right)^{\gamma_1} - \gamma_1\left(\frac{x}{x_0}\right)^{\gamma_0}\right]. \end{aligned}$$

In summary, we have

$$V(x) = \begin{cases} l, & \text{if } 0 < x \le x_0, \\[2ex] \dfrac{l}{\beta}\left[\gamma_0\left(\dfrac{x}{x_0}\right)^{\gamma_1} - \gamma_1\left(\dfrac{x}{x_0}\right)^{\gamma_0}\right], & \text{if } x_0 < x < x_1, \\[2ex] x, & \text{if } x \ge x_1, \end{cases} \tag{6.20}$$

and $\mathbf{C} \triangleq \{x \in (0, \infty) : V(x) > h(x)\} = F^{-1}(\widetilde{\mathbf{C}}) = F^{-1}((z_0, z_1)) = (x_0, x_1)$. Hence $\tau^* \triangleq \inf\{t \geq 0 : X_t \notin (x_0, x_1)\}$ is an optimal stopping rule by Proposition 5.21. Compare (6.20) with (19) in Guo and Shepp [4] (*al* and *bl* of Guo and Shepp [4] correspond to x_0 and x_1 in our calculations).

6.5 Another Exotic Option of Guo and Shepp [4]

The following example is quite instructive since it gives the opportunity to illustrate new ways to find W of Proposition 5.20. It will also enrich the imagination of the reader about different forms of least nonnegative concave majorants, and how they rise.

Let X be a geometric Brownian motion with constant drift $r > 0$ and dispersion $\sigma > 0$. Guo and Shepp [4] study the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x\big[e^{-r\tau}\big(\max\{l, X_\tau\} - K\big)^+\big], \quad x \in (0, \infty),$$

where l and K are positive constants and $l > K$. The reward function $h(x) \triangleq \big(\max\{l, x\} - K\big)^+$ can be seen as the payoff of some exotic option of American type. r is the riskless interest rate and K is the strike price of the option. The buyer of the option will be guaranteed to be paid at least $l - K > 0$ at the time of exercise. V is the maximum expected discounted payoff that the buyer can earn. If exists, we want to determine the best time to exercise the option. See Guo and Shepp [4] for more discussion about the option's properties.

From our discussion in the first section, we know that the generator of X coincides with the second–order differential operator $\mathcal{A} = \frac{\sigma^2}{2}x^2\frac{d^2}{dx^2} + rx\frac{d}{dx}$. The positive increasing and decreasing solutions of $\mathcal{A}u = ru$ are given by

$$\psi(x) = x \quad \text{and} \quad \varphi(x) = x^{-\frac{2r}{\sigma^2}}, \quad x \in (0, \infty).$$

The process is free to diffuse in $(0, \infty)$. Both boundaries are natural. Indeed $\psi(0+) =$

$\varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(0+) = +\infty$. Note that h is continuous in $(0,\infty)$ and

$$\ell_0 \triangleq \limsup_{x\to 0} \frac{h^+(x)}{\varphi(x)} = \lim_{x\to 0} \frac{h(x)}{\varphi(x)} = 0 \quad \text{and} \quad \ell_\infty \triangleq \limsup_{x\to\infty} \frac{h^+(x)}{\psi(x)} = \lim_{x\to\infty} \frac{h(x)}{\psi(x)} = 1.$$

Since h is bounded on every compact subset of $(0,\infty)$, and both ℓ_0 and ℓ_∞ are finite, V is finite by Proposition 5.18. Moreover V is the smallest nonnegative majorant of h such that $\frac{V}{\varphi}$ is F–concave by Proposition 5.19.

We shall use Proposition 5.20 in order to find V. Note that

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = x^\beta, \ x \in (0,\infty), \quad \text{where} \quad \beta \triangleq 1 + \frac{2r}{\sigma^2} > 1.$$

Therefore $F(0+) = 0$ and $F(+\infty) = +\infty$. If $W : [0,\infty) \to \mathbb{R}$ is the smallest nonnegative concave majorant of $H : [0,\infty) \to \mathbb{R}$, given by

$$H(y) \triangleq \begin{cases} \dfrac{h}{\varphi} \circ F^{-1}(y), & y \in (0,\infty) \\ \ell_0, & y = 0. \end{cases} = \begin{cases} (l - K)y^{1-1/\beta}, & 0 \le y \le l^\beta \\ (y^{1/\beta} - K)y^{1-1/\beta}, & y > l^\beta \end{cases},$$

then the value function is given by $V(x) = \varphi(x)W(F(x)) = x^{1-\beta}W(x^\beta), \ x \in (0,\infty)$.

In order to find W explicitly, we shall identify the concavities of H. Note that H is piecewise twice differentiable. In fact, we have

$$H'(y) = \begin{cases} \left(1 - \dfrac{1}{\beta}\right)(l - K)y^{-1/\beta}, & 0 < y \le l^\beta \\ 1 - \left(1 - \dfrac{1}{\beta}\right)Ky^{-1/\beta}, & y > l^\beta \end{cases},$$

and,

$$H''(y) = \begin{cases} -\dfrac{1}{\beta}\left(1 - \dfrac{1}{\beta}\right)(l - K)y^{-(1+1/\beta)}, & 0 < y \le l^\beta \\ \dfrac{1}{\beta}\left(1 - \dfrac{1}{\beta}\right)Ky^{-(1+1/\beta)}, & y > l^\beta \end{cases}.$$

Note that $H' > 0$ and $H'' < 0$ on $(0, l^\beta)$, i.e. H is strictly increasing and strictly concave on $[0.l^\beta]$. Furthermore $H'(0+) = +\infty$. On the other hand, $H'' > 0$, i.e. H is strictly convex on $(l^\beta, +\infty)$. We also have

$$0 < H'(l^\beta-) = 1 - \left(1 - \frac{1}{\beta}\right)\frac{K}{l} - \frac{1}{\beta} = H'(l^\beta+) - \frac{1}{\beta} < H'(l^\beta+).$$

Therefore H is also increasing on $(l^\beta, +\infty)$. One important observation which is key to our investigation of W is that H' is bounded, and asymptotically grows to one:

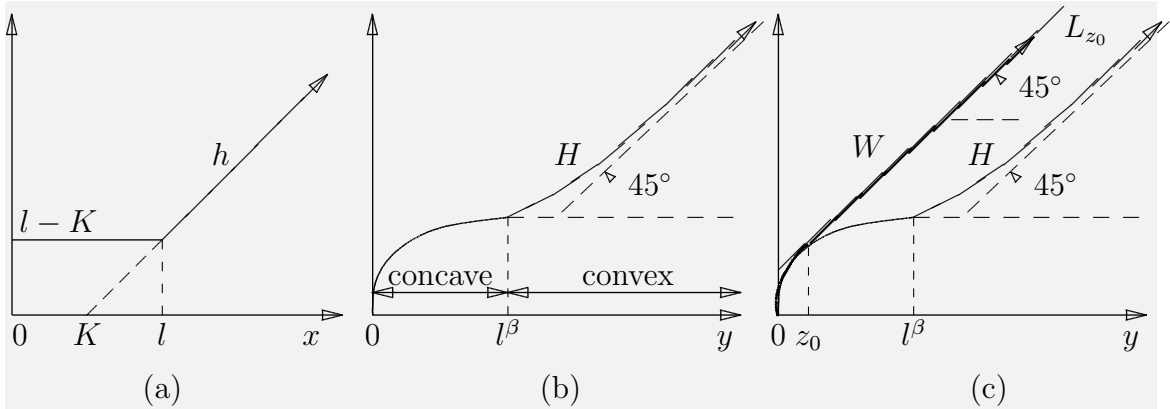$$0 < H'(l^\beta -) < H'(y) < 1,\ y > l^\beta; \quad \text{and} \quad \lim_{y \to +\infty} H'(y) = 1.$$

Figure 6.5: (Another Exotic Option) (a) h, (b) H, (c) H and W. Since h is piecewise, so is H. H is first concave increasing, and then convex increasing. $H'(0+) = +\infty$ whereas $H'(y)$ increases to one as y goes to infinity.

To find W, imagine $H \vee 0 \equiv H$ as a landscape. Suppose we tie one end of an infinite–length rope at $H(0) \equiv 0$ and release the rope by maintaining its tightness as we climb the hill. As we go farther, the tight rope increases to W in (c).

Because $H'(0+) = +\infty$, and H' decreases to $H'(l^\beta) < 1$, thanks to strict concavity, the tangent L_{z_0} of H at some z_0 has slope one for the first time before H becomes convex at l^β. Thus W agrees with H before z_0, and switches on L_{z_0} thereafter.

Figure 6.5(b) illustrates a sketch of H. Since $H'(0+) = +\infty$ and $H'(l^\beta -) < 1$, continuity of H' and strict concavity of H in $(0, l^\beta)$ imply that there exists unique $z_0 \in (0, l^\beta)$ such that $H'(z_0) = 1$. Let

$$L_{z_0}(y) \triangleq H(z_0) + H'(z_0)(y - z_0) = H(z_0) + y - z_0, \qquad y \in [0, \infty),$$

be the straight line tangent to H at z_0 (See Figure 6.5(c)). We claim that

$$W(y) = \widehat{W}(y) \triangleq \begin{cases} H(y), & 0 \leq y \leq z_0, \\ L_{z_0}(y), & y > z_0. \end{cases}, \qquad y \in [0, \infty).$$

We first show that $\widehat{W}$ is a nonnegative concave majorant of H on $[0, \infty)$. Since h is positive, H is also positive. Therefore $\widehat{W} \equiv H$ is positive on $[0, z_0]$. Since L_{z_0} is increasing, we also have $\widehat{W}(y) = L_{z_0}(y) \geq L_{z_0}(z_0) = H(z_0) > 0$ for every $y > z_0$. Therefore $\widehat{W}$ is nonnegative.

It is also obvious that $\widehat{W}$ is differentiable. Since H is concave on $[0, l^\beta] \supset [0, z_0]$, and $\widehat{W}'(y) = H'(z_0)$ for all $y \geq z_0$, $\widehat{W}'$ is non–increasing on $(0, \infty)$. Because we also have $\widehat{W}(0) < \widehat{W}(y)$ for all y, $\widehat{W}$ is concave on $[0, \infty)$.

We readily have $\widehat{W} \geq H$ on $[0, z_0]$. Since L_{z_0} is tangent to H at $z_0 \in [0, l^\beta]$ where H is concave, Proposition A.6(iii) implies that $\widehat{W} \equiv L_{z_0} \geq H$ on $[z_0, l^\beta]$. Finally, because $H'(y) < 1 = \widehat{W}'(y)$, $y > l^\beta$, we have

$$H(y) = H(l^\beta) + \int_{l^\beta}^{y} H'(z)dz \leq \widehat{W}(l^\beta) + \int_{l^\beta}^{y} \widehat{W}'(z)dz = \widehat{W}(y), \qquad y > l^\beta.$$

Since $\widehat{W}$ is a nonnegative concave majorant of H on $[0, \infty)$, we have $\widehat{W} \geq W$. Since $H = \widehat{W} \geq W \geq H$ on $[0, z_0]$, we immediately have $\widehat{W} = W$ on $[0, z_0]$. It therefore remains to show that $W \geq \widehat{W}$ on (z_0, ∞).

Fix any $y > z_0$. Since W is concave, and majorizes H, for any $z > \max\{y, l^\beta\}$ we have

$$W(y) \geq W(z_0)\frac{z - y}{z - z_0} + W(z)\frac{y - z_0}{z - z_0} \geq H(z_0)\frac{z - y}{z - z_0} + H(z)\frac{y - z_0}{z - z_0}$$

Since $\lim_{z \to +\infty} H(z)/(z - z_0) = \lim_{z \to +\infty} H'(z) = 1$ by L'Hospital rule, the limit of both sides as $z \to +\infty$ gives

$$W(y) \geq H(z_0) + y - z_0 = L_{z_0}(y) = \widehat{W}(y), \qquad y \in (z_0, \infty).$$

This completes the proof of $W \equiv \widehat{W}$.

We are now ready to write the original value function V. Let $x_0 \triangleq F^{-1}(z_0) = z_0^{1/\beta}$. Since F is strictly increasing, $0 < x_0 < l$. If $0 < x < x_0$, then $0 < F(x) < z_0$ and $V(x) = \varphi(x)W(F(x)) = \varphi(x)H(F(x)) = h(x) = l - K$ by Proposition 5.20.

Remember that $z_0 \in (0, l^\beta)$ has been uniquely determined by its defining relation

$$1 = H'(z_0) = \left(1 - \frac{1}{\beta}\right)(l - K)z_0^{-1/\beta} \tag{6.21}$$

We also have $H(z_0) = (l - K)z_0^{1-1/\beta}$. If $x > x_0$, then $F(x) > z_0$ and

$$
\begin{aligned}
V(x) &= \varphi(x)W(F(x)) = x^{1-\beta}L_{z_0}(x^\beta) = x^{1-\beta}\left[H(z_0) + H'(z_0)(x^\beta - z_0)\right]\\
&= x^{1-\beta}\left[(l-K)z_0^{1-1/\beta} + \left(1 - \frac{1}{\beta}\right)(l-K)z_0^{-1/\beta}(x^\beta - z_0)\right]\\
&= (l-K)\left[\left(1 - \frac{1}{\beta}\right)z_0^{-1/\beta}x + \frac{1}{\beta}x^{1-\beta}z_0^{1-1/\beta}\right]\\
&= (l-K)\left[\left(1 - \frac{1}{\beta}\right)\frac{x}{x_0} + \frac{1}{\beta}\left(\frac{x}{x_0}\right)^{1-\beta}\right].
\end{aligned}
$$

In summary, we have

$$V(x) = \begin{cases} l - K, & 0 < x < x_0,\\[2mm] (l-K)\left[\left(1 - \dfrac{1}{\beta}\right)\dfrac{x}{x_0} + \dfrac{1}{\beta}\left(\dfrac{x}{x_0}\right)^{1-\beta}\right], & x > x_0. \end{cases} \tag{6.22}$$

Furthermore, it follows from (6.21) that

$$x_0 = z_0^{1/\beta} = \left(1 - \frac{1}{\beta}\right)(l - K).$$

Compare (6.22) with Corollary 3 in Guo and Shepp [4] (In their notation $\gamma_0 = 1$, $\gamma_0 - \gamma_1 = \beta$, $l^* = x_0$). Finally

$$\mathbf{C} \triangleq \{x \in (0, \infty) : V(x) > h(x)\}$$
$$= F^{-1}(\{x \in (0, \infty) : W(x) > H(x)\}) = F^{-1}((z_0, \infty)) = (x_0, \infty).$$

Since $\ell_\infty = 1 > 0$, Lemma 5.6 implies that $\tau^* \triangleq \inf\{t \geq 0 : X_t \notin (x_0, \infty)\}$ is not an optimal stopping time. Hence there is no optimal stopping time (finite a.s. or not) for this problem.

6.6 An Example of H. Taylor [13]

Let X be one–dimensional Brownian motion with constant drift $\mu \leq 0$ and variance coefficient $\sigma^2 = 1$ in $\mathbb{R}$. Taylor [13, Example 1] studies the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta\tau}\max\{0, X_\tau\}], \qquad x \in \mathbb{R},$$

where the discounting rate $\beta > 0$ is constant. He guesses the value function and verifies that his guess is indeed the nonnegative β–excessive majorant of the reward function $h(x) \triangleq \max\{0, x\}$, $x \in \mathbb{R}$.

The infinitesimal generator of X is given by the operator $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2} + \mu\frac{d}{dx}$ acting on smooth functions. The positive increasing and decreasing solutions of the second–order ODE $\mathcal{A}u = \beta u$ exist, and are given by

$$\psi(x) = e^{\kappa x} \quad \text{and} \quad \varphi(x) = e^{\omega x}, \qquad x \in \mathbb{R},$$

respectively, where $\kappa = -\mu + \sqrt{\mu^2 + 2\beta} > 0 > \omega \triangleq -\mu - \sqrt{\mu^2 + 2\beta}$ are the roots of $(1/2)m^2 + \mu m - \beta = 0$. The boundaries $\pm\infty$ are natural. Observe that $\psi(-\infty) = \varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(-\infty) = +\infty$. The reward function h is continuous and

$$\ell_{-\infty} \triangleq \limsup_{x\to-\infty} \frac{h^+(x)}{\varphi(x)} = \lim_{x\to-\infty} \frac{h(x)}{\varphi(x)} = 0 \text{ and } \ell_{+\infty} \triangleq \limsup_{x\to+\infty} \frac{h^+(x)}{\psi(x)} = \lim_{x\to+\infty} \frac{h(x)}{\psi(x)} = 0.$$

By Proposition 5.18, the value function V is finite. It is the smallest nonnegative majorant of the reward function h such that V/ψ is G–concave in $\mathbb{R}$ by Proposition 5.19. As usual, we shall use Proposition 5.20 in order to calculate V explicitly. Note that

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -e^{(\omega-\kappa)x}, \quad x \in \mathbb{R},$$

and $G(-\infty) = -\infty$ and $G(+\infty) = 0$.

Let $W : (-\infty, 0] \to \mathbb{R}$ be the smallest nonnegative concave majorant of $H : (-\infty, 0] \to \mathbb{R}$ defined as in Proposition 5.20 by

$$H(y) = \begin{cases} \dfrac{h}{\psi} \circ G^{-1}(y), & y \in (-\infty, 0) \\ \ell_{+\infty}, & y = 0 \end{cases} \Bigg\} = \begin{cases} 0, & y \in (-\infty, -1] \cup \{0\} \\ \dfrac{(-y)^\alpha}{\omega - \kappa}\log(-y), & y \in (-1, 0) \end{cases} \Bigg\},$$

where $\alpha \triangleq \frac{\kappa}{\kappa-\omega}$ $(0 < \alpha < 1)$. Note that H is piecewise twice differentiable. In fact,

$$H'(y) = \begin{cases} 0, & y \in (-\infty, -1) \\ \dfrac{(-y)^{\alpha-1}}{\kappa - \omega}[\alpha \log(-y) + 1], & y \in (-1, 0) \end{cases}$$

and

$$H''(y) = \begin{cases} 0, & y \in (-\infty, -1) \\ -\dfrac{(-y)^{\alpha-2}}{\kappa - \omega}[\alpha(\alpha-1)\log(-y) + \alpha + (\alpha-1)], & y \in (-1, 0) \end{cases}.$$

Observe that

$$H''(y) < 0 \quad \Longleftrightarrow \quad \alpha(\alpha-1)\log(-y) + \alpha + (\alpha-1) > 0 \quad \Longleftrightarrow \quad y > T \triangleq -e^{\frac{\alpha+(\alpha-1)}{\alpha(\alpha-1)}}.$$

Since we have

$$\frac{\alpha + (\alpha-1)}{\alpha(\alpha-1)} = \frac{\kappa^2 - \omega^2}{\kappa\omega} = \frac{2(\theta\kappa + \beta) - 2(\theta\omega + \beta)}{-2\beta} = -\frac{2\theta\sqrt{\theta^2 + 2\beta}}{\beta},$$

H is strictly concave on $[T, 0]$ with

$$T \triangleq -e^{-(2\theta/\beta)\sqrt{\theta^2 + 2\beta}} \in (-1, 0).$$

One can also check that $H'(M) = 0$ gives the unique maximum of H and

$$M = -e^{-1/\alpha} \in (T, 0).$$

Figure 6.6(b) illustrates a sketch of H. We claim that

$$W(y) = \widehat{W}(y) \triangleq \begin{cases} H(M), & y \in (-\infty, M) \\ H(y), & y \in [M, 0] \end{cases}.$$

We shall first prove that $\widehat{W}$ is a nonnegative concave majorant of H. This will imply $\widehat{W} \geq W$ by Proposition 5.20.

Since h is nonnegative, H, as well as $\widehat{W}$, is nonnegative. It is also clear that $\widehat{W}$ is differentiable and $\widehat{W}' = 0$ on $(-\infty, M)$. Since H is concave on $[T, 0] \supset [M, 0]$, H'
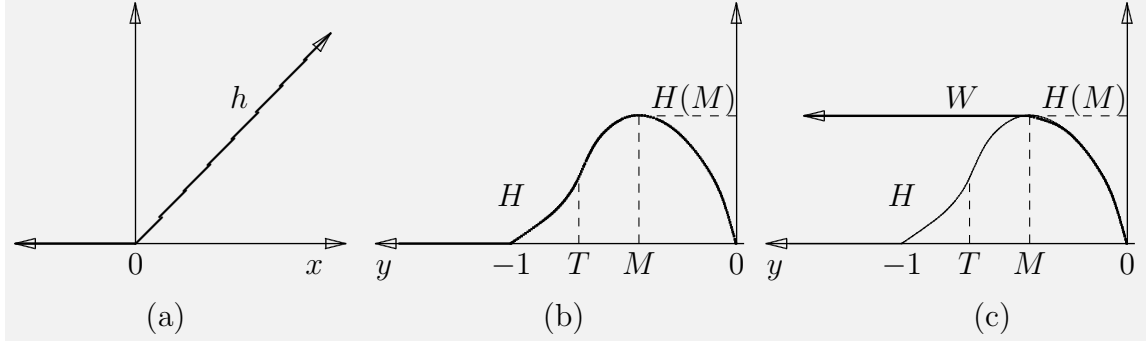
Figure 6.6: (Taylor's Example) (a) h, (b) H, (c) H and W. H is strictly concave on $[T, 0]$ and strictly convex on $(-\infty, T]$. It has unique maximum at M, and $-1 < T < M < 0$.

In order to visualize the smallest nonnegative concave majorant W of H, imagine we tie one end of an infinite–length rope at $H(0) \vee 0 \equiv 0$. We release the rope as we walk along $H \vee 0 \equiv H$ to the left by maintaining tightness of the rope. The rope increases to W as we continue our indefinitely long walk to $-\infty$.

Since $V(\cdot) = \psi W(G(\cdot))$ and $h = \psi(\cdot)H(G(\cdot))$ by Proposition 5.20, $V(x) > h(x)$ if and only if $W(G(x)) > H(G(x))$. Therefore the optimal continuation region becomes $\{x \in \mathbb{R} : V(x) > h(x)\} = G^{-1}(-\infty, M) = \big(-\infty, G^{-1}(M)\big)$.

is non–increasing on $[M, 0)$. Therefore $\widehat{W}'$ is non–increasing in $(-\infty, 0)$, i.e. $\widehat{W}$ is concave in $(-\infty, 0)$. Because $\widehat{W}(0) = 0 < \widehat{W}(y)$ for every $y \in (-\infty, 0)$, $\widehat{W}$ is also concave on $(-\infty, 0]$. It is obvious that $\widehat{W}$ majorizes H.

Observe that $H = \widehat{W} \geq W \geq H$ on $[M, 0]$. Hence $\widehat{W} = W$ on $[M, 0]$. It remains to show that $W \geq \widehat{W}$ in $(-\infty, M)$. Fix any $y < M$. Because W is concave, and majorizes H, for every $z < \min\{-1, x\}$ we have

$$W(y) \geq W(z)\frac{y - M}{M - z} + W(M)\frac{y - z}{M - z} \geq H(z)\frac{y - M}{M - z} + H(M)\frac{y - z}{M - z} = H(M)\frac{y - z}{M - z}.$$

By taking limit of both sides as $z \to -\infty$, we finally obtain $W(y) \geq H(M) = \widehat{W}(y)$ for every $y \in (-\infty, M)$.

We are know ready to write the value function V. Let

$$x_0 \triangleq G^{-1}(M) = \frac{1}{\alpha(\kappa - \omega)} = \frac{1}{\kappa} > 0.$$

Since G is strictly increasing, for every $x < x_0$ we have $G(x) < M$, and $V(x) = \psi(x)W(G(x)) = e^{\kappa x}H(M) = (1/\kappa)e^{\kappa x - 1}$ by Proposition 5.20. On the other hand, if $x > x_0 > 0$, then $G(x_0) \in [M, 0)$ and $V(x) = \psi(x)W(G(x)) = \psi(x)H(G(x)) = h(x) = x$. Hence

$$V(x) = \begin{cases} \dfrac{1}{\kappa} e^{\kappa x - 1}, & x < \dfrac{1}{\kappa}, \\ x, & x \geq \dfrac{1}{\kappa}. \end{cases}$$

Compare this with $f(\cdot)$ of Taylor [13, page 1337, Example 1] (In his notation, $a = \frac{1}{\kappa}$). Finally, note that

$$\mathbf{C} \triangleq \{x \in \mathbb{R} : V(x) > h(x)\}$$
$$= G^{-1}\big(\{y \in (-\infty, 0) : W(y) > H(y)\}\big) = G^{-1}\big((-\infty, M)\big) = (-\infty, 1/\kappa).$$

Because $\ell_{-\infty} = \ell_{+\infty} = 0$, Proposition 5.21 implies

$$\tau^* \triangleq \inf \{t \geq 0 : X_t \notin \mathbf{C}\} = \inf \{t \geq 0 : X_t \geq 1/\kappa\}$$

is an optimal stopping time (although $\mathbb{P}_x\{\tau^* = +\infty\} > 0$ for $x < 1/\kappa$ if $\mu < 0$).

6.7   An Example of P. Salminen [12]

Let X be a one–dimensional Brownian motions with drift $\mu \in \mathbb{R}$. Salminen [12, page 98, Example (iii)] studies the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta \tau} h(X_\tau)], \qquad x \in \mathbb{R}$$

with the reward function

$$h(x) \triangleq \begin{cases} 1, & \text{if } x \leq 0 \\ 2, & \text{if } x > 0 \end{cases} \equiv \begin{cases} h_1(x), & \text{if } x \leq 0 \\ h_2(x), & \text{if } x > 0 \end{cases}, \quad h_1 \equiv 1, \; h_2 \equiv 2, \text{ on } \mathbb{R},$$

and positive discounting rate β. He solves the problem by assuming $\mu = 0$ and by us-ing Corollary 7.3. Since h is not differentiable at 0 (which turns out to be a boundary of the optimal stopping region), he needed to calculate the mass of the "representing measure" for h. Salminen shows that every β–excessive function, including V, can uniquely be represented as an integral of a functional of minimal β–excessive functions with respect to a "representing measure". Therefore finding V is equivalent to calculating the representing measure of the value function.

Even though h is not differentiable at 0, we can use our results of Chapter 5 to calculate V. Note that $X_t = \mu t + B_t$, $t \geq 0$, and $X_0 = x \in \mathbb{R}$, where B is standard one–dimensional Brownian motion. Its generator coincides with $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2} + \mu\frac{d}{du}$, and the positive increasing and decreasing solutions of $\mathcal{A}u = \beta u$ are given by

$$\psi(x) = e^{\kappa x} \quad \text{and} \quad \varphi(x) = e^{\omega x}, \qquad x \in \mathbb{R},$$

respectively, where $\kappa \triangleq -\mu + \sqrt{\mu^2 + 2\beta} > 0 > \omega \triangleq -\mu - \sqrt{\mu^2 + 2\beta}$ are the roots of $\frac{1}{2}m^2 + \mu m - \beta$.

The boundaries $\pm\infty$ are natural. We have $\psi(-\infty) = \varphi(+\infty) = 0$ and $\psi(+\infty) = \varphi(-\infty) = +\infty$. Note that

$$\ell_{-\infty} \triangleq \limsup_{x\to-\infty} \frac{h^+(x)}{\varphi(x)} = 0 \quad \text{and} \quad \ell_{+\infty} \triangleq \limsup_{x\to+\infty} \frac{h^+(x)}{\psi(x)} = 0.$$

Since furthermore h is bounded (on every compact subset of $\mathbb{R}$), Proposition 5.18 implies that V is finite. Furthermore, V is the smallest nonnegative majorant of h such that $\frac{V}{\varphi}$ is F-concave in $\mathbb{R}$ by Proposition 5.19, where

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = e^{(\kappa-\omega)x}, \qquad x \in \mathbb{R}.$$

We shall use Proposition 5.20 in order to calculate V explicitly. Let $W : [0, \infty)$ be the smallest nonnegative concave majorant of

$$H(y) \triangleq \begin{cases} \dfrac{h}{\varphi} \circ F^{-1}(y), & y \in (0, +\infty) \\ \ell_{-\infty}, & y = 0 \end{cases} = \begin{cases} y^\gamma, & 0 \leq y < 1 \\ 2y^\gamma, & y \geq 1. \end{cases} \equiv \begin{cases} H_1(y), & 0 \leq y < 1 \\ H_2(y), & y \geq 1. \end{cases}$$

where

$$H_1(y) \triangleq y^\gamma, \quad H_2(y) \triangleq 2y^\gamma, \quad y \in [0, +\infty), \quad \text{and} \quad 0 < \gamma \triangleq \frac{-\omega}{\kappa - \omega} < 1,$$

as in Proposition 5.20. Then $V(x) = \varphi(x)W(F(x))$, $x \in \mathbb{R}$. Note that H is piecewise continuous and twice differentiable. Observe that H is the mixture of two nonnegative strictly concave and increasing functions H_1 and H_2. After $y = 1$, we switch from curve H_1 onto H_2. Thus H is strictly concave on $[0, 1]$ and $(1, +\infty)$. It is nonnegative and increasing. Also note that $H'(0+) = +\infty$ (See Figure 6.7(a) and Figure 6.7(b) for sketches of h and H).

Strict concavity of H on $[0, 1]$ and $H'(0+) = +\infty$ imply that there exists unique $z_0 \in (0, 1)$ such that

$$H'(z_0) = \frac{H(1+) - H(z_0)}{1 - z_0} = \frac{H_2(1) - H_1(z_0)}{1 - z_0}. \tag{6.23}$$

This is equivalent to saying that there exists unique $z_0 \in (0, 1)$ such that the straight line L_{z_0} tangent to H at z_0 also passes through the point $(1, H(1+))$ (See Figure 6.7(c)).

One can prove this by formalizing geometry depicted in Figure 6.7 with the family of tangent lines L_z of H at every $z \in (0, 1)$. We shall however verify directly that (6.23) has unique solution in $z_0 \in (0, 1)$. We can rewrite (6.23) as

$$\gamma z_0^{\gamma-1} = \frac{2 - z_0^\gamma}{1 - z_0} \quad \Longleftrightarrow \quad \gamma z_0^{\gamma-1} + (1 - \gamma)z_0^\gamma = 2.$$

Let $f(y) \triangleq \gamma y^{\gamma-1} + (1 - \gamma)y^\gamma$, $y \in (0, 1)$. Note that $+\infty = f(0+) > 2 > f(1-) = 1$, and

$$f'(y) = \gamma(\gamma - 1)y^{\gamma-1}\big[1 - y\big] < 0, \quad y \in (0, 1),$$

i.e. $f(\cdot)$ is strictly decreasing in $(0, 1)$. Therefore there is indeed unique $z_0 \in (0, 1)$ such that $f(z_0) = 2$.

Let L_{z_0} be as above, i.e.

$$L_{z_0}(y) \triangleq H(z_0)\frac{1 - y}{1 - z_0} + H(1+)\frac{y - z_0}{1 - z_0} = z_0^\gamma \frac{1 - y}{1 - z_0} + 2\frac{y - z_0}{1 - z_0}, \quad x \in [0, +\infty),$$
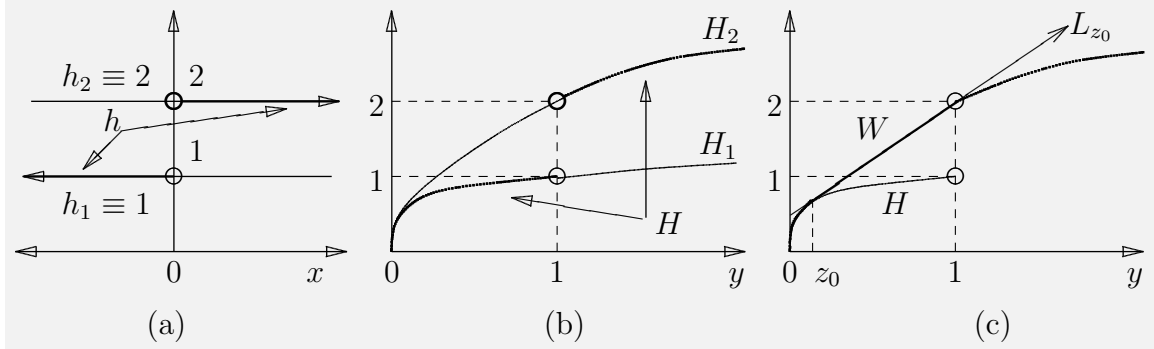
Figure 6.7: (Salminen's Example) (a) h, (b) H, (c) H and W. H is mixture of two positive strictly concave functions H_1 and H_2. We start with H at 0, and jump on H_2 after $y = 1$.

To find the smallest nonnegative concave majorant W of H, we shall look at the change of the direction of tangent lines L_z of H at $z \in (0, 1)$. Since $H = H_1$ is concave on $[0, 1]$ and $H'(0+) = +\infty$, for every z near 0, $L'_z \equiv H'(z)$ is so large that L_z never meets H_2 on $[1, +\infty)$. As z moves to the right, the slope of L_z decreases steadily thanks to the concavity of H_1. Therefore we reach some $z_0 < 1$ such that L_{z_0} meets H_2 at some $z_1 \geq 1$ for the first time. Using the concavity of H_2 and $H_2 > H_1$, we shall prove that $z_1 = 1$.

We obtain W by piecing L_{z_0} restricted to $[z_0, 1]$ with H elsewhere together. Hence we stick to H as much as possible as we maintain non–increasing right–derivative of our trajectory. Minimality and concavity of the resulting nonnegative majorant are thus obtained at the same time.

see Figure 6.7(c). We claim that

$$W(y) = \widehat{W}(y) \triangleq \begin{cases} H(y), & y \in [0, z_0] \cup (1, +\infty), \\ L_{z_0}(y), & y \in (z_0, 1]. \end{cases}$$

First we shall show that $\widehat{W}$ is a nonnegative concave majorant of H on $[0, +\infty)$. Since W is the smallest function with the same properties, this will prove $\widehat{W} \geq W$.

Since h is nonnegative, H is also nonnegative. Therefore $\widehat{W}$ is nonnegative on

$[0, z_0] \cup (1, +\infty)$. On the other hand, $\widehat{W}$ coincides with the straight line L_{z_0} on $[z_0, 1]$. Since $L_{z_0}(z_0) = H(z_0) > 0$ and $L_{z_0}(1) = H(1+) > 0$, $\widehat{W} \equiv L_{z_0}$ is nonnegative on $[z_0, 1]$. Hence $\widehat{W}$ is nonnegative on $[0, +\infty)$.

Observe that $\widehat{W}$ has right–derivative everywhere, and

$$D^+\widehat{W}(y) = \begin{cases} H_1'(y), & y \in (0, z_0] \\ H_1'(z_0), & y \in (z_0, 1) \\ H_2'(y), & y \in [1, +\infty) \end{cases}.$$

We claim that $D^+\widehat{W}$ is non–increasing. Since H_1 and H_2 are concave, H_1' and H_2' are non–increasing. Therefore $D^+\widehat{W}$ is non–increasing on $(0, z_0]$ and $(1, +\infty)$. However, for every $0 < y_1 < z_0 \leq y_2 < 1 \leq y_3$, we have

$$D^+\widehat{W}(y_1) = H_1'(y_1) \geq H_1'(z_0) = D^+\widehat{W}(y_2)$$
$$= \frac{H_2(1) - H_1(z_0)}{1 - z_0} \geq \frac{H_2(1) - H_2(z_0)}{1 - z_0} \geq H_2'(1) \geq H_2'(y_3) = \widehat{W}(y_3).$$

The third equality follows from the defining relation (6.23) of z_0. The second inequality is due $H_1 \leq H_2$. Third inequality is because $z_0 < 1$ and H_2 is concave. Last inequality also follows from concavity of H_2. This proves that $D^+\widehat{W}$ is non–increasing. Therefore $\widehat{W}$ is concave in $(0, +\infty)$. Since $\widehat{W}(0) = 0 < \widehat{W}(y)$, $y > 0$, it is concave also on $[0, +\infty)$.

Finally, we need to prove that $\widehat{W}$ majorizes H. Clearly $\widehat{W} \geq H$ on $[0, z_0] \cup (1, +\infty)$. On the other hand, since $\widehat{W}$ coincides on $[z_0, 1]$ with the tangent line L_{z_0} of H at $z_0 \in [0, 1]$ where H is concave, Proposition A.6(iii) implies that $\widehat{W} \equiv L_{z_0} \geq H$ on $[z_0, 1]$.

Hence $\widehat{W}$ is a nonnegative concave majorant of H on $[0, +\infty)$. Therefore $\widehat{W} \geq W$. On $[0, z_0] \cup (1, +\infty)$, $H = \widehat{W} \geq W \geq H$, i.e. $\widehat{W} = W$. Since W is concave, it is continuous in $(0, +\infty) \ni 1$ by Proposition A.1. Therefore $W(1) = \lim_{y \downarrow 1} W(y) =$

$\lim_{y\downarrow 1} \widehat{W}(y) = \widehat{W}(1)$. Finally, because $\widehat{W}$ coincides with linear L_{z_0} on $[z_0, 1]$, we have

$$\widehat{W}(y) = L_{z_0}(y) = L_{z_0}(z_0)\frac{1-y}{1-z_0} + L_{z_0}(1)\frac{y-z_0}{1-z_0} = \widehat{W}(z_0)\frac{1-y}{1-z_0} + \widehat{W}(1)\frac{y-z_0}{1-z_0}$$

$$= W(z_0)\frac{1-y}{1-z_0} + W(1)\frac{y-z_0}{1-z_0} \le W(y), \quad y \in [z_0, 1].$$

This proves that $\widehat{W} = W$ everywhere.

We are now ready to write V explicitly. Let $x_0 \triangleq F^{-1}(z_0)$. Since F is strictly increasing and $z_0 \in (0, 1)$, we have $x_0 \in (-\infty, 0)$. If $x \notin (x_0, 0]$, then $F(x) \notin (z_0, 1]$ and $V(x) = \varphi(x)W(F(x)) = \varphi(x)H(F(x)) = h(x)$ by Proposition 5.20. Thus if $x < x_0 < 1$, then $V(x) = 1$, and if $x > 1$, then $V(x) = 2$.

Suppose now $x \in (x_0, 0]$. Then $F(x) \in (z_0, 1]$ and

$$V(x) = \varphi(x)W(F(x)) = \varphi(x)L_{z_0}(F(x))$$

$$= \varphi(x)H(z_0)\frac{1-F(x)}{1-z_0} + \varphi(x)H(1+)\frac{F(x)-z_0}{1-z_0} = \frac{\left(1-2e^{\kappa x_0}\right)e^{\omega x} - \left(1 - 2e^{\omega x_0}\right)e^{\kappa x}}{e^{\omega x_0} - e^{\kappa x_0}}$$

In summary, we have

$$V(x) = \begin{cases} 1, & \text{if } x \le x_0 \\ \dfrac{\left(1-2e^{\kappa x_0}\right)e^{\omega x} - \left(1-2e^{\omega x_0}\right)e^{\kappa x}}{e^{\omega x_0} - e^{\kappa x_0}}, & \text{if } x_0 < x \le 0 \\ 2, & \text{if } x > 0 \end{cases}.$$

Since h is not continuous, we cannot use Proposition 5.21 to check if there is an optimal stopping time. However, if there were any optimal stopping time, then τ^* of (5.17) must also be optimal. Note that $\mathbf{C} \triangleq \{x \in \mathbb{R} : V(x) > h(x)\} = (x_0, 0]$. Remember that every Brownian motion with drift such as X is a standard Brownian motion under an equivalent change of measure. Since standard Brownian motion hits both half lines infinitely often in every arbitrarily small time interval with probability one, we have $\mathbb{P}_0\{\tau^* = 0\} = 1$. However

$$\mathbb{E}_0[e^{-\beta\tau^*}h(X_{\tau^*})] = h(0) = 1 < 2 = V(0),$$

i.e. τ^* is not optimal. Therefore there is no optimal stopping time, either.

Salminen [12] calculates the critical value x_0 explicitly for $\mu = 0$. When we set $\mu = 0$, we get $\kappa = -\omega = \sqrt{2\beta}$, $\gamma = 1/2$, and the defining relation (6.23) of z_0 becomes

$$\frac{1}{2}z_0^{-1/2} + \frac{1}{2}z_0^{1/2} = 2 \quad \Longleftrightarrow \quad z_0 - 4z_0^{1/2} + 1 = 0,$$

after simplifications. If we let $y_0 \triangleq z_0^{1/2}$, then y_0 is the only root in $(0,1)$ of $y^2 - 4y + 1 = 0$, i.e. $y_0 = 2 - \sqrt{4-1} = 2 - \sqrt{3}$. Therefore $z_0 = (2 - \sqrt{3})^2$. Finally,

$$x_0 = F^{-1}(z_0) = \frac{1}{\kappa - \omega} \log z_0 = \frac{1}{2\sqrt{2\beta}} \log (2 - \sqrt{3})^2 = \frac{1}{\sqrt{2\beta}} \log (2 - \sqrt{3}) \quad \text{if} \ \ \mu = 0,$$

which agrees with the calculations of Salminen [12, page 99].

6.8   A New Optimal Stopping Problem

Let B be one-dimensional standard Brownian motion in $[0, \infty)$ with absorption at 0. Consider

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\beta\tau}(B_\tau)^p], \quad x \in [0, \infty).$$

for some $\beta > 0$ and $p > 0$. Hence our reward function $h : [0, \infty) \to \mathbb{R}$ is given as $h(x) \triangleq x^p$, which is locally bounded on $[0, +\infty)$ for any choice of $p > 0$. The infinitesimal generator of Brownian motion is

$$\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$$

acting on the twice–continuously differentiable functions which vanish at $\pm\infty$. The usual solutions of the second–order ordinary differential equation $\mathcal{A}u = \beta u$ are

$$\psi(x) = e^{x\sqrt{2\beta}}, \quad \text{and} \quad \varphi(x) = e^{-x\sqrt{2\beta}}, \quad x \in \mathcal{I} = \mathbb{R} \supset [0, \infty).$$

The left boundary $c = 0$ is attainable in finite time with probability one, whereas the right boundary $b = \infty$ is a natural boundary for the (stopped) process. Note that h

is continuous on $[0, \infty)$, and

$$\ell_{+\infty} \triangleq \limsup_{x \to +\infty} \frac{h^+(x)}{\psi(x)} = \lim_{x \to +\infty} \frac{h(x)}{\psi(x)} = \lim_{x \to +\infty} x^p e^{-x\sqrt{2\beta}} = 0.$$

Since h is bounded on every compact subset of $[0, +\infty)$, and $\ell_{+\infty} < +\infty$, the value function V is finite. V is the smallest nonnegative majorant of h such that $\frac{V}{\psi}$ is $G-$concave on $[0, \infty)$ by Proposition 5.3. Furthermore, since h is continuous on $[0, +\infty)$ and $\ell_{+\infty} = 0$, Proposition 5.8 also implies $\tau^* \triangleq \inf\{t \geq 0 : B_t \in \mathbf{\Gamma}\}$ is an optimal stopping time where $\mathbf{\Gamma} \triangleq \{x \in [0, \infty) : V(x) = h(x)\}$.

Thanks to Proposition 5.6, $V(x) = \psi(x)W(G(x))$, $x \in [0, \infty)$, where

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -e^{-2x\sqrt{2\beta}}, \quad \forall\, x \in [0, \infty),$$

and $W : [-1, 0] \to \mathbb{R}$ is the smallest nonnegative concave majorant of

$$H(y) \triangleq \frac{h}{\psi} \circ G^{-1}(y) = \left(\frac{1}{2\sqrt{2\beta}}\right)^p \left[-\log(-y)\right]^p \cdot \sqrt{-y}, \qquad y \in [G(c), G(b-)] \equiv [-1, 0),$$

and $H(0) \triangleq \ell_{+\infty} = 0$.

$W(\cdot)$ can be obtained analytically by cutting off the convexities of $H(\cdot)$ with straight lines (geometrically speaking, the holes on $H(\cdot)$, due to the convexity, have to be bridged across the concave hills of $H(\cdot)$, see for example Figure 6.8).

Note that H is twice continuously differentiable in $(-1, 0)$, and we have

$$H'(y) = \frac{(-y)^{-1/2}\left[-\log(-y)\right]^{p-1}}{2\left(2\sqrt{2\beta}\right)^p} \cdot \left[2p + \log(-y)\right] \qquad y \in (-1, 0),$$

$$H''(y) = \frac{(-y)^{-3/2}\left[-\log(-y)\right]^{p-2}}{4\left(2\sqrt{2\beta}\right)^p} \left[4p(p-1) - \left(-\log(-y)\right)^2\right], \quad y \in (-1, 0).$$

If $0 < p \leq 1$, then $H''(\cdot) \leq 0$, so $H(\cdot)$ is concave on $[-1, 0]$, and $W = H$. Therefore Proposition 5.6 implies that $V = h$, and $\tau^* \equiv 0$ (i.e., stopping immediately) is optimal.

In the rest of this Section, we shall assume that p is strictly greater than 1. With $T \triangleq -e^{-2\sqrt{p(p-1)}}$, H is concave on $[-1, T]$, and convex on $[T, 0]$. It has unique maximum at $M \triangleq -e^{-2p} > T$, and nonnegative everywhere on $[-1, 0]$ (cf. Figure 6.8(a)).

We claim that $W = H$ on $[M, 0]$. Fix any $z \in [M, 0)$ and let

$$L_z(y) \triangleq H(z) + H'(z)(y - z), \quad y \in [-1, 0],$$

be the line tangent to H at z (See Figure 6.8(b)). Using Proposition A.6(iii), one can easily prove that L_z is a nonnegative concave majorant of H on $[-1, 0]$. Therefore, we have $H(z) = L_z(z) \geq W(z) \geq H(z)$. This proves $W(z) = H(z)$, for $z \in [M, 0)$. However, Proposition 5.6 also guarantees $W(0) = 0 = H(0)$.
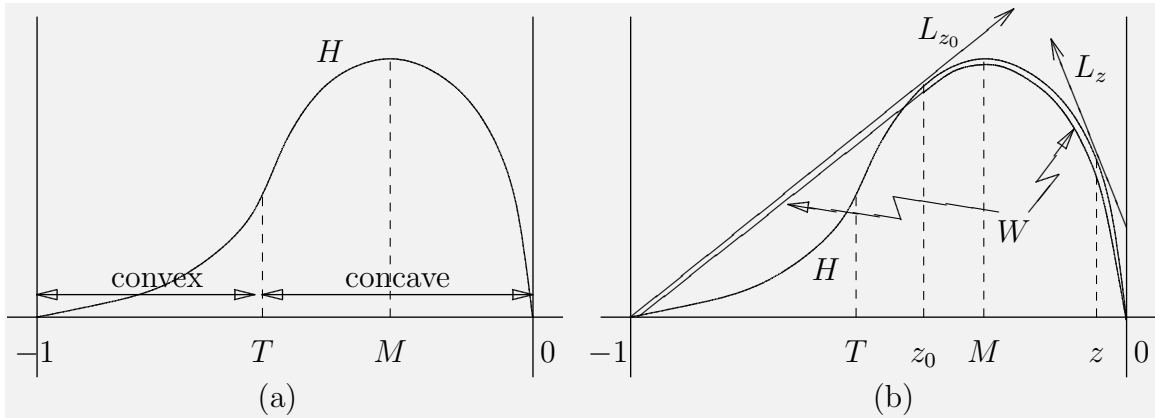
Figure 6.8: Sketches of (a) H, and (b) H and W (Note that the graphs are not scaled). In order to find the smallest nonnegative concave majorant W of H, we study the tangent lines L_z of H at z. Since H is concave on $[T, 0] \ni M$, no line L_z meets H other than at z, for every $z \in [M, 0]$. Since H is concave on $[T, 0]$, the slope of L_z, i.e. $H'(z)$, steadily increases as z moves to the left. Finally because H is convex on $[-1, T]$, $L_T < H$ on $[-1, T]$. Therefore as z moves to the left, we shall reach some z_0 such that L_{z_0} meets H (when it meets, it meets at $y = -1$) for the first time.

We obtain W by piecing L_{z_0} restricted to $[-1, z_0]$ with H elsewhere. In other words, we obtain W by stick to H as much as possible as we maintain a non–increasing slope for our trajectory. Minimality and concavity of our nonnegative majorant follows as a result of this.

It is clear from Figure 6.8 that there exists unique $z_0 \in [T, M)$ such that $L_{z_0}(-1) =$

$H(-1)$ (this is so-called *Smooth–Fit* in the context of variational inequalities).

We claim that

$$W(y) = \widehat{W}(y) \triangleq \begin{cases} L_{z_0}(y), & \text{if } y \in [-1, z_0), \\ \\ H(y), & \text{if } y \in [z_0, 0]. \end{cases} \tag{6.24}$$

(See Figure 6.8(b)). In order to show $\widehat{W} \geq W$, it is enough to prove that $\widehat{W}$ is a nonnegative concave majorant of H on $[-1, 0]$. As it is however clear from Figure 6.8, $\widehat{W}$ is minimum of two nonnegative concave functions majorizing H, and the result follows from Proposition A.5.

We shall next show the reverse inequality. We already have $\widehat{W} = H \leq W$ on $[z_0, 0]$, by the definitions of $\widehat{W}$ and W. Note also that $\widehat{W}(-1) = L_{z_0}(-1) = H(-1)$. Hence $\widehat{W}$ on $[-1, z_0]$ coincides with the line that spans between the points $(-1, H(-1))$ and $(z_0, H(z_0))$ on the graph of H. Since W majorizes H on $[-1, 0]$, especially at -1 and z_0, we have

$$\begin{aligned} \widehat{W}(y) = L_{z_0}(y) &= L_{z_0}(-1)\frac{z_0 - y}{z_0 - (-1)} + L_{z_0}(z_0)\frac{y - (-1)}{z_0 - (-1)} \\ &= H(-1)\frac{z_0 - y}{z_0 - (-1)} + H(z_0)\frac{y - (-1)}{z_0 - (-1)} \\ &\leq W(-1)\frac{z_0 - y}{z_0 - (-1)} + W(z_0)\frac{y - (-1)}{z_0 - (-1)} \leq W(y), \quad \forall\, y \in [-1, z_0], \end{aligned}$$

since W is also concave on $[-1, 0]$. This proves $\widehat{W}$ is dominated by W. Therefore $\widehat{W} = W$ on $[-1, 0]$.

We shall next identify z_0. By definition, it satisfies $H(-1) = L_{z_0}(-1)$. Since H is strictly concave on $[T, M] \ni z_0$, the uniqueness is clear from Figure 6.8(b)). $H(-1) = L_{z_0}(-1)$ implies

$$\frac{H(z_0) - H(-1)}{z_0 - (-1)} = H'(z_0), \tag{6.25}$$

which finally leads to

$$\log(-z_0) = 2p\frac{z_0 + 1}{z_0 - 1} \quad \Longleftrightarrow \quad -z_0 = e^{2p\frac{z_0+1}{z_0-1}}. \tag{6.26}$$

It can easily be shown that (6.26) uniquely determines $z_0 \in (-1, 0)$.

Finally, by Proposition 5.6, we have $V(x) = \psi(x)W(G(x))$, for every $x \in [0, \infty)$, that is

$$V(x) = \begin{cases} \dfrac{H(z_0)}{1 + z_0} \left[e^{x\sqrt{2\beta}} - e^{-x\sqrt{2\beta}} \right], & \text{if } 0 \le x \le -\dfrac{1}{2\sqrt{2\beta}} \log\left(-z_0\right), \\ x^2, & \text{if } x > -\dfrac{1}{2\sqrt{2\beta}} \log\left(-z_0\right). \end{cases} \tag{6.27}$$

The strict concavity of H on $[-1, z_0]$ leads to $\widetilde{\mathbf{C}} \triangleq \{y \in [-1.0) : W(y) > H(y)\} = (-1, z_0)$. Following the remarks at page 44 after Lemma 5.1, the optimal continuation region and optimal stopping time for our original optimal stopping problem become

$$\mathbf{C} \triangleq \left(0, -\frac{1}{2\sqrt{2\beta}} \log\left(-z_0\right) \right) \quad \text{and} \quad \tau^* = \inf\left\{ t \ge 0 : B_t \ge -\frac{1}{2\sqrt{2\beta}} \log\left(-z_0\right) \right\},$$

respectively.

6.9 Optimal Stopping Problem of Karatzas and Ocone [6]

Karatzas and Ocone [6] study a special optimal stopping problem in order to solve a stochastic control problem. In this section, we shall take another look at the same optimal stopping problem.

Suppose that the process X is governed by the dynamics

$$dX_t = -\theta dt + dB_t$$

for some positive constant θ. Its infinitesimal generator coincides with the second–order differential operator

$$\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2} - \theta\frac{d}{dx}$$

acting on the smooth functions in the domain of the generator. Since $\pm\infty$ are natural boundaries for X on $\mathbb{R}$, the usual solutions of $\mathcal{A}u = \beta u$, $\beta > 0$, subject to the

boundary conditions $\psi(-\infty) = \varphi(\infty) = 0$, become

$$\psi(x) = e^{\kappa x}, \quad \text{and,} \quad \varphi(x) = e^{\omega x}, x \in \mathbb{R},$$

where $\kappa \triangleq \theta + \sqrt{\theta^2 + 2\beta}$ and $\omega \triangleq \theta - \sqrt{\theta^2 + 2\beta}$.

Now consider the stopped process, which we shall denote by the same notation X, which is started in $[0, \infty)$, and is absorbed at 0 when it reaches. Since the original process reaches 0 with positive probability, we have $0 < \psi(0), \varphi(0) < \infty$ (In fact, we have $\psi(0) = \varphi(0) = 1$ as we explicitly calculated above). Consider the optimal stopping problem

$$\inf_{\tau \geq 0} \mathbb{E}_x \left[\int_0^\tau e^{-\beta t} \pi(X_t) dt + e^{-\beta \tau} g(X_\tau) \right], \quad x \in [0, \infty),$$

with $\pi(x) \triangleq x^2$ and $g(x) \triangleq \delta x^2$. If we introduce $R_\beta \pi : [0, \infty) \to \mathbb{R}$, defined as

$$R_\beta \pi(x) \triangleq \mathbb{E}_x \left[\int_0^\infty e^{-\beta t} \pi(X_t) dt \right] = \frac{1}{\beta} x^2 - \frac{2\theta}{\beta^2} x + \frac{2\theta^2 + \beta}{\beta^3} - \frac{2\theta^2 + \beta}{\beta^3} e^{\omega x}, \quad \forall \, x \in [0, \infty),$$

$$(6.28)$$

then, by using strong Markov property of X, one can show that

$$\mathbb{E}_x \left[\int_0^\tau e^{-\beta t} \pi(X_t) dt + e^{-\beta \tau} g(X_\tau) \right] = R_\beta \pi(x) - \mathbb{E}_x [e^{-\beta \tau} (R_\beta \pi(x) - g(x))], \, \forall \, x \in [0, \infty).$$

Therefore, our task is to solve the auxiliary optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x [e^{-\beta \tau} h(X_\tau)], \quad x \in [0, \infty), \tag{6.29}$$

with

$$h(x) \triangleq R_\beta \pi(x) - g(x) = \frac{1 - \delta\beta}{\beta} x^2 - \frac{2\theta}{\beta^2} x + \frac{2\theta^2 + \beta}{\beta^3} - \frac{2\theta^2 + \beta}{\beta^3} e^{\omega x}, \quad x \in [0, \infty).$$

Note that h is continuous and bounded on every compact subinterval of $[0, \infty)$ and

$$\ell_\infty \triangleq \limsup_{x \to \infty} \frac{h^+(x)}{\psi(x)} = \lim_{x \to \infty} \frac{h(x)}{\psi(x)} = 0.$$

Proposition 5.2 implies that V is finite. By Proposition 5.3, V is the smallest non-negative majorant of h such that $\frac{V}{\psi}$ is G–concave. Since $\ell_\infty = 0$, Proposition 5.8 also shows that an optimal stopping time exists.

To solve for V of (6.29), we shall use Proposition 5.6. Note that

$$G(x) \triangleq -\frac{\varphi(x)}{\psi(x)} = -e^{(\omega - \kappa)x}, \quad x \in [0, \infty),$$

and $G(0) = -1$. As in Proposition 5.6, let $W : [-1, 0] \to \mathbb{R}$ be the smallest nonnegative concave majorant of $H : [-1, 0] \to \mathbb{R}$, given by

$$H(y) \triangleq \frac{h}{\psi} \circ G^{-1}(y) = (-y)^\alpha \left[a \left(\log (-y) \right)^2 + b \log (-y) + c \right] + c\,y, \quad y \in [-1, 0),$$

and $H(0) \triangleq \ell_\infty = 0$, where α, a, b and c are constants defined by

$$\alpha \triangleq \frac{\kappa}{\kappa - \omega}, \quad a \triangleq \frac{1 - \delta\beta}{\beta} \frac{1}{(\omega - \kappa)^2}, \quad b \triangleq -\frac{2\theta}{\beta^2} \frac{1}{(\omega - \kappa)}, \quad c \triangleq \frac{2\theta^2 + \beta}{\beta^3}. \tag{6.30}$$

Observe that

$$0 < \alpha < 1, \quad a \in \mathbb{R}, \quad b \geq 0, \quad c > 0.$$

We hope to find W analytically by cutting off the convexities of H. Therefore, we need to find out where H is convex and concave. Note that H is twice–continuously differentiable in $(-1, 0)$. Therefore, we can determine convexities by looking at the sign of H''. One can easily calculate

$$H'(y) = -(-y)^{\alpha-1} \left[\alpha a \left(\log (-y) \right)^2 + (\alpha b + 2a) \log (-y) + \alpha c + b \right] + c, \tag{6.31}$$

$$H''(y) = (-y)^{\alpha-2} Q_1 \left(\log (-y) \right), \quad y \in (-1, 0), \tag{6.32}$$

where

$$Q_1(x) \triangleq \alpha(\alpha - 1)a\,x^2 + \left[\alpha(\alpha - 1)b + 2a(2\alpha - 1) \right]x + 2a + (2\alpha - 1)b + \alpha(\alpha - 1)c$$

for every $x \in \mathbb{R}$, is a second–order polynomial. Since $(-y)^{\alpha-2} > 0$, $y \in (-1, 0)$, the sign of H'' is determined by the sign of $Q_1 \left(\log (-y) \right)$. Since $\log (-y) \in (-\infty, 0)$ as $y \in (-1, 0)$, we are only interested in the behavior of $Q_1(x)$ when $x \in (-\infty, 0)$. Let

$$\Delta_1 \triangleq \left[\alpha(\alpha - 1)b + 2a(2\alpha - 1) \right]^2 - 4 \cdot \left[\alpha(\alpha - 1)a \right] \cdot \left[2a + (2\alpha - 1)b + \alpha(\alpha - 1)c \right]$$

be the discriminant of Q_1. After some algebra, we find

$$\Delta_1 = \frac{\theta^2 + \beta}{4(\theta^2 + 2\beta)^3 \beta^2} \widetilde{Q}_1(1 - \delta\beta), \tag{6.33}$$

where

$$\widetilde{Q}_1(x) \triangleq x^2 - 2x + 1 - \frac{\delta\beta^2}{\theta^2 + \beta} = (x - 1)^2 - \frac{\delta\beta^2}{\theta^2 + \beta}, \quad x \in \mathbb{R},$$

is also a second–order polynomial. Note that $\widetilde{Q}_1$ always has two real roots,

$$\widetilde{q}_1 = 1 - \sqrt{\frac{\delta\beta^2}{\theta^2 + \beta}} \quad \text{and} \quad \widetilde{q}_2 = 1 + \sqrt{\frac{\delta\beta^2}{\theta^2 + \beta}}.$$

One can show that

$$\Delta_1 < 0 \iff \delta(\theta^2 + \beta) < 1.$$

Therefore, Q_1 has no real roots if $\delta(\theta^2+\beta) < 1$, has a repeated real root if $\delta(\theta^2+\beta) = 1$, and two distinct real roots if $\delta(\theta^2 + \beta) > 1$. The sign of H'', and therefore the regions where H is convex and concave, depend on the choice of the parameters δ, θ and β.

Case I. Suppose $\delta(\theta^2 + \beta) < 1$. Then Q_1 has no real roots, and $Q_1 < 0$ everywhere. Therefore (6.32) implies that $H'' < 0$ everywhere in $(-1, 0)$. Since furthermore $H(-1) = H(0) = 0$, H is strictly concave and nonnegative on $[-1, 0]$. Thus, H is a nonnegative concave majorant of itself, and $W \equiv H$. By Proposition 5.6, we have $V = h$, and $\tau^* \equiv 0$ is an optimal stopping time thanks to Proposition 5.8.

Suppose now $\delta(\theta^2 + \beta) \geq 1$. Then Q_1 has two real roots (repeated if equality holds). The sign of Q_1 (hence the sign of H'' by (6.32)) will be determined by the sign of the coefficient of the leading term of Q_1, namely by $\alpha(\alpha - 1)a$. Note that $\alpha(\alpha - 1)$ is always negative, whereas a has the same sign as $1 - \delta\beta$ thanks to (6.30).

Case II. Suppose $\delta(\theta^2 + \beta) \geq 1$ and $1 - \delta\beta \leq 0$. Since $1 - \delta\beta \leq 0$, $\alpha(\alpha - 1)a \geq 0$. Therefore, Q_1 has two real roots (possibly repeated), and is upward directed. Denote the roots by $q_1 \leq q_2$. Since $Q_1(0) < 0$, we have $q_1 < 0 < q_2$, and $Q_1 > 0$ in $(-\infty, q_1)$ and $Q < 0$ in $(q_1, 0]$. Thus, (6.32) implies that

$$H'' < 0 \text{ in } (-1, -e^{q_1}), \quad \text{and} \quad H'' > 0 \text{ in } (-e^{q_1}, 0), \quad \text{and} \quad H''(-e^{q_1}) = 0. \tag{6.34}$$

Hence H is strictly concave on $[-1, -e^{q_1}]$, and strictly convex on $[-e^{q_1}, 0]$. Note also that $-1 < -e^{q_1} < 0$.

We claim that H has unique maximum at some $M \in (-1, -e^{q_1})$, and $H(M) > 0$. We shall first show that $\max_{y \in [-1,0]} H(y) > 0$ by proving that H is positive in some neighborhood of -1. Thanks to (6.31), we have

$$H'(-1^+) = -(\alpha - 1)c - b = -\frac{\omega}{\kappa - \omega}\frac{2\theta^2 + \beta}{\beta^3} - \frac{2\theta}{\beta^2(\kappa - \omega)} = -\frac{\omega(2\theta^2 + \beta) + 2\theta\beta}{\beta^3(\kappa - \omega)}.$$

Since $H'(-1^+) > 0 \Leftrightarrow \beta^2 > 0$, and β is strictly greater than 0, we have $H'(-1^+) > 0$. Therefore, there exists some sufficiently small $\varepsilon > 0$ such that $H'(y) > 0$ for all $y \in (-1, -1 + \varepsilon)$. Hence H is strictly increasing in $(-1, -1 + \varepsilon]$. Therefore, $H(y) > H(-1) = 0$ for all $y \in (-1, -1 + \varepsilon)$. This proves that $\max_{y \in [-1,0]} H(y) > 0$.

Let $x \in [-1, 0]$ be such that $H(x) = \max_{y \in [-1,0]} H(y) > 0$. Since $H(x) > 0 = H(-1) = H(0)$, we must have $x \in (-1, 0)$. Therefore $H'(x) = 0$ and $H''(x) \leq 0$, and $x \in (-1, -e^{q_1})$ thanks to (6.34). Because H is strictly concave on $[-1, -e^{q_1}]$, there can be at most one interior maximizer of H on $[-1, -e^{q_1}]$. Therefore H has unique maximum at some $M \in (-1, -e^{q_1})$ and $H(M) > 0$ (See Figure 6.9(a)).
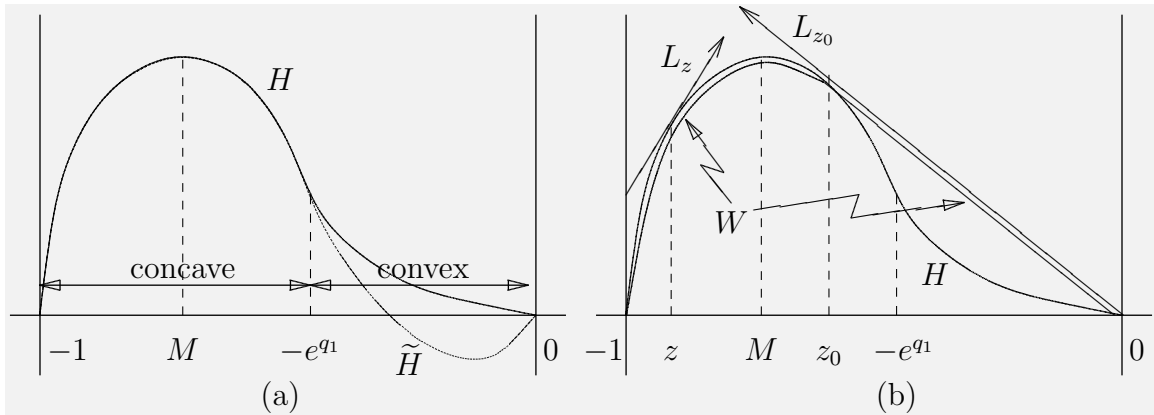

Figure 6.9: Sketches of (a) H (may become negative in the neighborhood of zero as $\widetilde{H}$ looks like), (b) H and W, in **Case II**. The idea behind how we find W is similar to that of Chapter 6.8. Especially, read the caption of Figure 6.8.

We shall first prove that $W \equiv H$ on $[-1, M]$. Since $H(-1^+) = 0$, and H is

increasing on $[-1, M]$, H is nonnegative on $[-1, M]$. Next fix any $z \in (-1, M]$ and

let

$$L_z(y) \triangleq H(z) + H'(z)(y - z), \quad y \in [-1, 0],$$

be the line tangent to H at z (See Figure 6.9(b)). It is evident that L_z is a nonnegative

concave majorant of H on $[-1, 0]$. Since W is the *smallest* nonnegative concave

majorant of H on $[-1, 0]$, we have $L_z \geq W$ on $[-1, 0]$. However $H(z) = L_z(z) \geq$

$W(z) \geq H(z)$, that is, $W(z) = H(z)$, $z \in [-1, M]$.

We shall continue our investigation of W on $[M, 0]$ by considering the family of

linear functions

$$L_z(y) \triangleq H(z) + H'(z)(y - z), \quad y \in [-1, 0],$$

indexed by $z \in [M, -e^{q_1}]$. In fact, every L_z is the line segment that is tangent to H at

$z \in [M, -e^{q_1}]$. It is clear from Figure 6.9(b) that there exists unique $z_0 \in (M, -e^{q_1}]$

such that $L_{z_0}(0) = H(0)$. Similar to the previous examples, one can finally show (cf.

Figure 6.9(b)) that

$$W(x) = \begin{cases} H(y), & \text{if } y \in [-1, z_0] \\ L_{z_0}(y), & \text{if } y \in (z_0, 0] \end{cases}. \tag{6.35}$$

Moreover, trivial calcultions show that $\log (-z_0)$ is the unique solution of

$$(1 - \alpha)\big[a\, x^2 + b\, x + c\big] = 2ax + b, \quad x \in \big[\log (-M), q_1\big],$$

and $\widetilde{\mathbf{C}} \triangleq \{y \in [-1, 0] : W(y) > H(y)\} = (z_0, 0)$ (cf. Figure 6.9(b)). Proposition 5.6

implies

$$V(x) = \begin{cases} h(x), & \text{if } 0 \leq x \leq x_0 \\ \dfrac{\varphi(x)}{\varphi(x_0)} h(x_0), & \text{if } x_0 < x < \infty \end{cases}, \tag{6.36}$$

with $x_0 \triangleq G^{-1}(z_0)$, and the optimal continuation region becomes $\mathbf{C} = G^{-1}(\widetilde{\mathbf{C}}) =$

$G^{-1}((z_0, 0)) = (x_0, \infty)$. We shall next look at the final case, namely

Case III. Suppose $\delta(\theta^2 + \beta) \geq 1$ and $1 - \delta\beta > 0$. Therefore $\alpha(\alpha - 1)a < 0$, and Q_1 is downward directed with two real roots. As in Case II, we shall denote the roots of Q_1 by q_1 and q_2, $q_1 \leq q_2$. By investigating the signs of the roots of Q_1, one can show that

$$\left\{ \begin{array}{c} H'' > 0 \quad \text{in } (-e^{q_2}, -e^{q_1}), \quad H'' < 0 \quad \text{in } (-1, -e^{q_2}) \cup (-e^{q_1}, 0) \\ H''(-e^{q_2}) = H''(-e^{q_1}) = 0 \end{array} \right\}. \qquad (6.37)$$

Furthermore $H'(-1^+) > 0$, and H is positive in some neighborhood of 0 (See Figure 6.10(a)). Since in addition $H'(0^-) = -\infty$, we have $W \equiv H$ in some neighborhood of 0 unlike Case II. To see this, we shall introduce linear functions

$$L_z \triangleq H(z) + H'(z)(y - z), \quad y \in [-1, 0],$$

indexed by $z \in [-e^{q_1}, 0)$. They are line segments, tangent to H at z. Since H is continuous on $[-1, 0]$, it is bounded. Denote $+\infty > K \triangleq \max_{y \in [-1,0]} H(y) > 0$. Since $e^{q_1} > 0$ and $H'(0^-) = -\infty$, there exists some $-e^{q_1} \leq -\varepsilon < 0$ such that $H'(z) \leq -K/e^{q_1}$ for every $z \in [-\varepsilon, 0)$. We now claim that L_z, $z \in [-\varepsilon, 0)$, is a nonnegative concave majorant of H on $[-1, 0]$ (See Figure 6.10(b) for the geometric interpretation).

Let's fix some $z \in [-\varepsilon, 0)$. Since L_z is linear, it is concave. Because L_z is tangent to H at $z \in [-e^{q_1}, 0]$ where H is concave, Proposition A.6(iii) implies $L_z \geq H$ on $[-e^{q_1}, 0]$. To prove that $L_z \geq H$ on $[-1, -e^{q_1}]$, note that L_z is decreasing, and $L_z(-e^{q_1}) \geq K$. For the first one we have $L'_z \equiv H'(z) \leq -K/e^{q_1} < 0$. Since $L_z(0) \geq H(0) = 0$ as proved above, we also have

$$L_z(-e^{q_1}) = H(z) + H'(z)(-e^{q_1} - z) = \left[H(z) - zH'(z)\right] + H'(z)(-e^{q_1})$$

$$= L_z(0) + H'(z)(-e^{q_1}) \geq H(0) + H'(z)(-e^{q_1}) = H'(z)(-e^{q_1}) \geq -\frac{K}{e^{q_1}}(-e^{q_1}) = K.$$

Therefore, for every $y \in [-1, -e^{q_1}]$, we have $H(y) \leq K \leq L_z(-e^{q_1}) \leq L_z(y)$, i.e. $L_z \geq H$ also on $[-1, -e^{q_1}]$. Finally, since L_z is decreasing and L_z majorizes H on $[-1, 0]$, we have $L_z(y) \geq L_z(0) \geq H(0) = 0$, $y \in [-1, 0]$. Therefore L_z is nonnegative.

We proved that L_z is a nonnegative concave majorant of H on $[-1, 0]$. Since W is the smallest of such functions, we have $L_z \geq W$ on $[-1, 0]$. However $L_z(z) = H(z)$. Since W also majorizes H, this implies $W(z) \geq H(z) = L_z(z) \geq W(z)$, i.e. $W(z) = H(z)$. This proves that $W \equiv H$ in $[-\varepsilon, 0]$ (remember that $W(0) = 0 = H(0)$ always by Proposition 5.6). It is clear from Figure 6.10(b) that

$$W(y) = \begin{cases} L_{z_1}(y), & \text{if } y \in [z_2, z_1], \\ H(y), & \text{if } y \in [-1, z_2) \cup (z_1, 0], \end{cases} \tag{6.38}$$

for some unique $-1 < z_2 < z_1 < 0$, where z_2 is the second tangent point of L_{z_1} to H. In fact, the pair $(z, \widetilde{z}) = (z_2, z_1)$ is the unique solution of *exactly* one of

$$H'(z) = \frac{H(z) - H(\widetilde{z})}{z - \widetilde{z}} = H'(\widetilde{z}), \ \widetilde{z} > -1, \quad \text{or} \quad H'(z) = \frac{H(z) - H(-1)}{z - (-1)}, \ \widetilde{z} = -1,$$

for some $\widetilde{z} \in [-1, -e^{q_2}], z \in [-e^{q_1}, 0)$. Furthermore

$$\widetilde{\mathbf{C}} = \{y \in [-1, 0] : W(y) > H(y)\} = (z_2, z_1).$$

The value function V of (6.29) can be calculated using Proposition 5.6, and the optimal continuation region becomes

$$\mathbf{C} \triangleq \{x \in [0, \infty) : V(x) > h(x)\} = G^{-1}(\widetilde{\mathbf{C}}) = (G^{-1}(z_2), G^{-1}(z_1)).$$
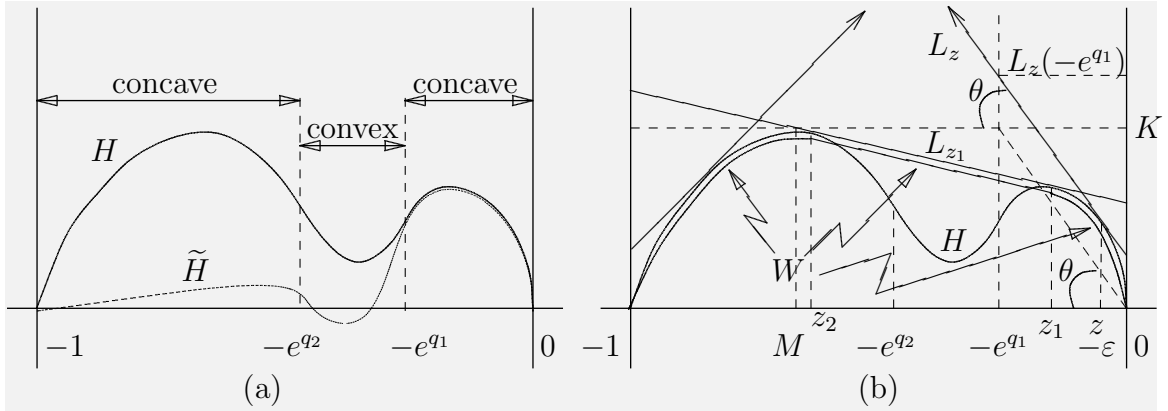
Figure 6.10:   Sketches of (a) H, (b) H and W, in **Case III**. In (a), $\widetilde{H}$ depicts another possibility where $\widetilde{H}$ takes negative values, and its global maximum is contained in $[-e^{q_1}, 0]$. In (b), $\tan\theta = -K/e^{q_1}$, and $\varepsilon > 0$ is chosen as described in **12**. For illustration, we have also chosen $z = \varepsilon$.

In order to find the smallest nonnegative concave majorant W of H, we shall study the tangent lines L_z of H at $z \in [-e^{q_1}, 0)$. Since $H'(0-) = -\infty$ and H is concave on $[-e^{q_1}, 0]$, the slope of L_z near 0 ($z \in (-\varepsilon, 0)$) is so large that it never meets H except at z. On the other hand, as z moves to the left, the slope of L_z steadily decreases thanks to the concavity of H on $[-e^{q_1}, 0]$. Finally $L_{-e^{q_1}} < H$ in $(-e^{q_2}, -e^{q_1})$ because of the convexity of H on the same interval. Therefore as z moves to the left, we reach some $z_1 \in (-e^{q_1}, 0)$ such that L_{z_1} meets H (if it meets, it meets H at some $z_2 \in [-1, -e^{q_2})$) for the first time.

We obtain W by piecing L_{z_1} restricted to $[z_2, z_1]$ together with H elsewhere. Hence we stick H as much as possible as we maintain a non–increasing slope for our new curve. Therefore both minimality and concavity of our nonnegative majorant are guaranteed.

Chapter 7

Smooth–Fit Principle and Variational Inequalities

In this chapter, we shall continue to investigate the properties of the value function V. For concreteness, we shall focus on the discounted optimal stopping problem introduced in Chapter 4, although all results can be carried over for the optimal stopping problems of Chapter 3 and Chapter 5.

7.1 Smooth–Fit Principle and Necessary Conditions for Optimal Stopping Boundaries

In Chapter 4, we started by assuming that h is bounded, and showed that $\frac{V}{\varphi}$ is the smallest nonnegative F–concave majorant of $\frac{h}{\varphi}$ on $[c,d]$ (cf. Proposition 4.2). The continuity of V in (c,d) immediately followed from concavity, as pointed out before Lemma 4.2. The F–concavity property of $\frac{V}{\varphi}$ has further implications. From Proposition A.6(i), we know that $\frac{d^\pm}{dF}\left(\frac{V}{\varphi}\right)$ exist, and are nondecreasing in (c,d). Furthermore,[1]

---

[1]The fact that the left–derivative of the value function V is always greater than or equal to the right–derivative of V was pointed by Salminen [12, page 86].

$$\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x), \quad x \in (c,d). \tag{7.1}$$

Lemma A.6(ii) implies that equality holds in (7.1) everywhere in (c,d), except possibly on a subset N which is at most countable, i.e.

$$\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \equiv \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x), \quad x \in (c,d)\backslash N.$$

Hence $\frac{V}{\varphi}$ is essentially F–differentiable in (c,d). Let

$$\mathbf{\Gamma} \triangleq \{x \in [c,d] : V(x) = h(x)\} \quad \text{and} \quad \mathbf{C} \triangleq [c,d]\backslash\mathbf{\Gamma} = \{x \in [c,d] : V(x) > h(x)\}.$$

When the F–concavity of $\frac{V}{\varphi}$ is combined with the fact that V majorizes h on $[c,d]$, we obtain the key result of Proposition 7.1, which leads, in turn, to the celebrated Smooth–Fit principle.

Proposition 7.1. *At every $x \in \mathbf{\Gamma} \cap (c,d)$, where $\frac{d^\pm}{dF}\left(\frac{h}{\varphi}\right)(x)$ exist, we have*

$$\frac{d^-}{dF}\left(\frac{h}{\varphi}\right)(x) \geq \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{h}{\varphi}\right)(x).$$

Proof. The second inequality is the same as (7.1). For the rest, first remember that $V = h$ on $\mathbf{\Gamma}$. Since V majorizes h on $[c,d]$, and F is strictly increasing, this leads to

$$\frac{\frac{h(y)}{\varphi(y)} - \frac{h(x)}{\varphi(x)}}{F(y) - F(x)} \geq \frac{\frac{V(y)}{\varphi(y)} - \frac{V(x)}{\varphi(x)}}{F(y) - F(x)} \quad \text{and} \quad \frac{\frac{V(z)}{\varphi(z)} - \frac{V(x)}{\varphi(x)}}{F(z) - F(x)} \geq \frac{\frac{h(z)}{\varphi(z)} - \frac{h(x)}{\varphi(x)}}{F(z) - F(x)}, \tag{7.2}$$

for every $x \in \mathbf{\Gamma}$, $y < x < z$. Suppose $x \in \mathbf{\Gamma} \cap (c,d)$, and $\frac{d^\pm}{dF}\left(\frac{h}{\varphi}\right)(x)$ exist. As we summarized before stating Proposition 7.1, we know that $\frac{d^\pm}{dF}\left(\frac{V}{\varphi}\right)(x)$ always exist in (c,d). Therefore, the limits of both sides of the inequalities in (7.2), as $y \uparrow x$ and $z \downarrow x$ respectively, exist, and give

$$\frac{d^-}{dF}\left(\frac{h}{\varphi}\right)(x) \geq \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) \quad \text{and} \quad \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) \geq \frac{d^+}{dF}\left(\frac{h}{\varphi}\right)(x),$$

respectively. $\qquad\square$

Corollary 7.1 (Smooth–Fit Principle). *At every $x \in \Gamma \cap (c, d)$ where $\frac{h}{\varphi}$ is F–differentiable, $\frac{V}{\varphi}$ is also F–differentiable, and touches $\frac{h}{\varphi}$ at x smoothly, in the sense that the F–derivatives of both functions also agree at x:*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x).$$

Corollary 7.1 raises the question when we should expect $\frac{V}{\varphi}$ to be F–differentiable in (c, d). If $\frac{h}{\varphi}$ is F–differentiable in (c, d), then it is immediate from Corollary 7.1 that $\frac{V}{\varphi}$ is F–differentiable in $\Gamma \cap (c, d)$. However, we know little about the behavior of $\frac{V}{\varphi}$ on $\mathbf{C} = [c, d] \backslash \Gamma$ if h is only bounded. On the contrary, if h is continuous on $[c, d]$, then Proposition 4.4 shows that an optimal stopping time exists. More precisely, it implies that $\tau^* \triangleq \inf\{t \geq 0 : X_t \notin \mathbf{C}\}$ is an optimal stopping time. This finding will help us better characterize V on $\mathbf{C}$ in our attempt to answer standing question of F–differentiability of $\frac{V}{\varphi}$.

Suppose $h : [c, d] \to \mathbb{R}$ is continuous on $[c, d]$. Since $[c, d]$ is closed and bounded, h is bounded and previous results still hold. Lemma 4.2 implies that V is also continuous at the boundaries c and d, i.e. V is a continuous function on $[c, d]$.

$\mathbf{C}$ is now an open subset of $[c, d]$. Therefore, it is the union of a countable family $(J_\alpha)_{\alpha \in \Lambda}$ of disjoint open (relative to $[c, d]$) subintervals of $[c, d]$. Using Lemma 4.3, one can show that

$$\frac{V(x)}{\varphi(x)} = \frac{\mathbb{E}_x[e^{-\beta\tau^*}h(X_{\tau^*})]}{\varphi(x)} = \frac{V(l_\alpha)}{\varphi(l_\alpha)} \cdot \frac{F(r_\alpha) - F(x)}{F(r_\alpha) - F(l_\alpha)} + \frac{V(r_\alpha)}{\varphi(r_\alpha)} \cdot \frac{F(x) - F(l_\alpha)}{F(r_\alpha) - F(l_\alpha)}, \quad x \in J_\alpha, \tag{7.3}$$

where l_α and r_α are the left– and right–boundary of J_α, $\alpha \in \Lambda$, respectively. Observe that $\frac{V}{\varphi}$ coincides with an F–linear function on every J_α, i.e. it is F–differentiable in $J_\alpha \cap (c, d)$ for every $\alpha \in \Lambda$. By taking the F–derivative of (7.3), we find that

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{1}{F(r_\alpha) - F(l_\alpha)}\left[\frac{V(r_\alpha)}{\varphi(r_\alpha)} - \frac{V(l_\alpha)}{\varphi(l_\alpha)}\right], \quad x \in J_\alpha \cap (c, d) \tag{7.4}$$

is constant, i.e. is itself F–differentiable in $J_\alpha \cap (c, d)$. Since $\mathbf{C}$ is the union of disjoint

J_α, $\alpha \in \Lambda$, this implies that $\frac{V}{\varphi}$ is twice continuously F–differentiable in $\mathbf{C} \cap (c,d)$. We are ready to prove the following result.

Proposition 7.2. *Suppose h is continuous on $[c,d]$. Then V is continuous on $[c,d]$, and $\frac{V}{\varphi}$ is twice continuously F–differentiable in $\mathbf{C} \cap (c,d)$. Furthermore,*

*(i) if $\frac{h}{\varphi}$ is F–differentiable in (c,d), then $\frac{V}{\varphi}$ is **continuously**[2] F–differentiable in (c,d), and*

(ii) if $\frac{h}{\varphi}$ is twice (continuously) F–differentiable in (c,d), then $\frac{V}{\varphi}$ is twice (continuously) F–differentiable in $(c,d) \backslash \partial \mathbf{C}$,

where $\partial \mathbf{C}$ is the boundary of $\mathbf{C}$ relative to $\mathbb{R}$ or $[c,d]$.

Proof. Since h and F are continuous, V is continuous by Lemma 4.2. We also proved above that $\frac{V}{\varphi}$ is F–differentiable in $\mathbf{C} \cap (c,d)$ (this is always true even if $\frac{h}{\varphi}$ were not F–differentiable).

(i) If $\frac{h}{\varphi}$ is F–differentiable in (c,d), then the F–differentiability of $\frac{V}{\varphi}$ in $(c,d) \backslash \mathbf{C} = (c,d) \cap \Gamma$ follows from Corollary 7.1. Therefore $\frac{V}{\varphi}$ is F–differentiable in $(c,d) = [(c,d) \backslash \mathbf{C}] \cup \mathbf{C}$ by the discussion above. However, $\frac{V}{\varphi}$ is also F–concave on $[c,d]$, and F is continuous on $[c,d]$. Therefore Proposition A.7 implies that $\frac{d}{dF} \left(\frac{V}{\varphi} \right)$ is continuous in (c,d).

(ii) We only need prove that $\frac{V}{\varphi}$ is twice (continuously) F–differentiable in $(c,d) \backslash \overline{\mathbf{C}}$ where $\overline{\mathbf{C}}$ is the closure of $\mathbf{C}$ relative to $[c,d]$. However $(c,d) \backslash \overline{\mathbf{C}}$ is an open set (relative to $\mathbb{R}$) contained in Γ where V and h coincide. Because we assume $\frac{h}{\varphi}$ is twice (continuously) F–differentiable, the conclusion follows immediately. $\square$

Even if $\frac{h}{\varphi}$ is not smooth everywhere in (c,d), it is still possible to draw conclusions based on its local properties. In (7.3), suppose l_α and r_α are contained in Γ. Therefore

---

[2]Note that this is always true no matter whether $\frac{d}{dF} \left(\frac{h}{\varphi} \right)$ is continuous or not. As the proof indicates, this is as a result of F–concavity of $\frac{V}{\varphi}$ and continuity of F on $[c,d]$.

$V(l_\alpha) = h(l_\alpha)$ and $V(r_\alpha) = h(r_\alpha)$, and $\frac{V}{\varphi}$ coincides with the F–linear function

$$L_\alpha(x) \triangleq \frac{h(l_\alpha)}{\varphi(l_\alpha)} \cdot \frac{F(r_\alpha) - F(x)}{F(r_\alpha) - F(l_\alpha)} + \frac{h(r_\alpha)}{\varphi(r_\alpha)} \cdot \frac{F(x) - F(l_\alpha)}{F(r_\alpha) - F(l_\alpha)}, \quad x \in [c, d],$$

on J_α. If we envisage $\frac{h}{\varphi}$ as the outermost boundary of a pile of rocks on the ground, then L_α may be thought of a tree which has fallen and come to a rest on the pile of rocks. l_α and r_α can be seen the points where rocks support the tree. We expect the surfaces of rock and the fallen tree to match at those supporting points where the rock has a smooth surface (See Figure 7.1).
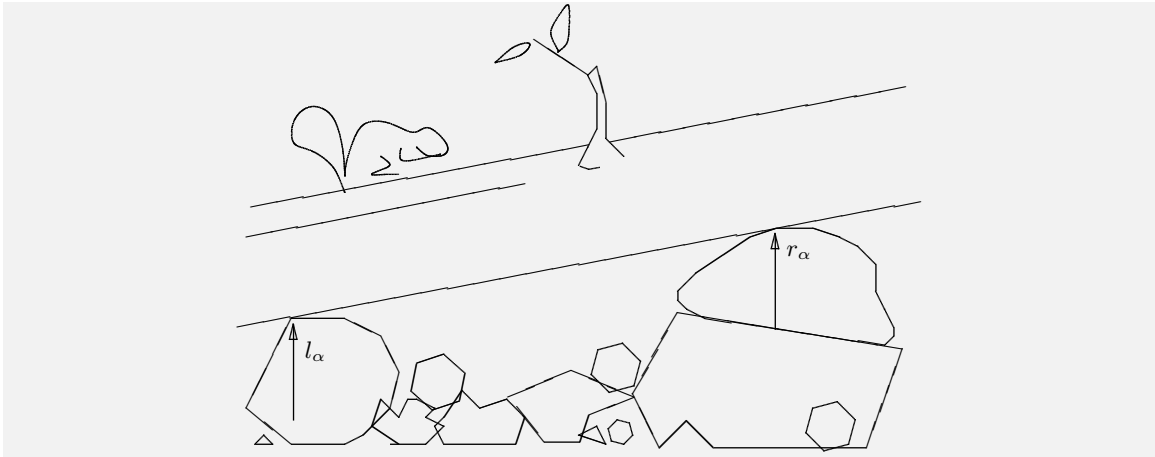
Figure 7.1: The smooth–fit principle may hold even if the reward function is not smooth everywhere.

Proposition 7.3 (Necessary conditions for the boundaries of the optimal continuation region). *Suppose h is continuous on $[c, d]$. Suppose $l, r \in \mathbf{\Gamma} \cap (c, d)$, and $\frac{h}{\varphi}$ has F–derivatives at l and r. Then $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$ exists at l and r. Moreover*

(i) *If $(l, r) \subseteq \mathbf{C}$, then*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(l) = \frac{\frac{h(r)}{\varphi(r)} - \frac{h(l)}{\varphi(l)}}{F(r) - F(l)} = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(r) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r),$$

and,

$$\frac{V(x)}{\varphi(x)} = \frac{h(\theta)}{\varphi(\theta)} + [F(x) - F(\theta)]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(\theta), \quad x \in [l, r], \ \theta = l, r.$$

(ii) If $[c, r) \subseteq \mathbf{C}$, then

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(r) = \frac{1}{F(r) - F(c)} \cdot \frac{h(r)}{\varphi(r)},$$

and,

$$\frac{V(x)}{\varphi(x)} = \frac{h(r)}{\varphi(r)} + \left[F(x) - F(r)\right]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) = \left[F(x) - F(c)\right]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(r), \quad x \in [c, r).$$

(iii) If $(l, d] \subseteq \mathbf{C}$, then

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(l) = -\frac{1}{F(d) - F(l)} \cdot \frac{h(l)}{\varphi(l)},$$

and,

$$\frac{V(x)}{\varphi(x)} = \frac{h(l)}{\varphi(l)} + \left[F(x) - F(l)\right]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \left[F(x) - F(d)\right]\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l), \quad x \in (l, d].$$

Proof. Existence of $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$, and equality of $\frac{d}{dF}\left(\frac{h}{\varphi}\right)$ and $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$ at l and r, respectively, follow from Corollary 7.1. Therefore, the first and last equality in (i), and the first equalities in (ii) and (iii) are clear.

Note that the intervals (l, r), $[c, r)$ and $(l, b]$ are all three possible forms that J_α, $\alpha \in \Lambda$ can take. Let l_α and r_α denote the left– and right–boundaries of intervals, respectively. Then (7.4) is true for all three cases.

In (i), both $l_\alpha = l$ and $r_\alpha = r$ are in $\mathbf{\Gamma}$. Therefore, $V(l) = h(l)$ and $V(r) = h(r)$, and (7.4) implies

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{1}{F(r) - F(l)}\left[\frac{h(r)}{\varphi(r)} - \frac{h(l)}{\varphi(l)}\right], \quad x \in (l, r). \tag{7.5}$$

Since $\frac{V}{\varphi}$ is F–concave on $[c, d] \supset [l, r]$, and F is continuous on $[c, d]$, Proposition A.6(iii) implies that $\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)$ and $\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)$ are right– and left–continuous in (c, d). Because $\frac{V}{\varphi}$ is F–differentiable on $[l, r]$, $\frac{d^\pm}{dF}\left(\frac{V}{\varphi}\right)$ and $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$ coincide on $[l, r]$. Therefore $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$ is continuous on $[l, r]$, and second and third equalities in (i) immediately follow from

(7.5). In a more direct way,

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(l) = \frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(l) = \lim_{x\downarrow l}\frac{d^+}{dF}\left(\frac{V}{\varphi}\right)(x) = \lim_{x\downarrow l}\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{\frac{h(r)}{\varphi(r)} - \frac{h(l)}{\varphi(l)}}{F(r) - F(l)}.$$

$$\frac{d}{dF}\left(\frac{V}{\varphi}\right)(r) = \frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(r) = \lim_{x\downarrow l}\frac{d^-}{dF}\left(\frac{V}{\varphi}\right)(x) = \lim_{x\downarrow l}\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) = \frac{\frac{h(r)}{\varphi(r)} - \frac{h(l)}{\varphi(l)}}{F(r) - F(l)}.$$

Same equalities could have also been proved by direct calculation using (7.3).

Proofs of second equalities in (ii) and (iii) are similar once we note that $V(c) = 0$ if $c \in \mathbf{C}$, and $V(d) = 0$ if $d \in \mathbf{C}$.

Finally, the expressions for $\frac{V}{\varphi}$ follow from (7.3) by direct calculation. Simply note that $\frac{V}{\varphi}$ is an F–linear function passing through $\left(l_\alpha, \frac{V}{\varphi}(l_\alpha)\right)$ and $\left(r_\alpha, \frac{V}{\varphi}(r_\alpha)\right)$. $\qquad\square$

Proposition 5.3 is more useful in the applications than Proposition 5.6, although the first one is merely a restatement of the second. We therefore would like to see what Proposition 7.3 may imply for the (H,W) pair of Proposition 5.6. Suppose h, ψ and φ are differentiable at $x \triangleq F^{-1}(y)$ for some $y \in (F(c), F(d))$. Then both F and h/φ are differentiable at x. Therefore $\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x)$ exists. Moreover H is differentiable at y and

$$\frac{dH}{dy}(y) = \frac{d}{dy}\left(\frac{h}{\varphi} \circ F^{-1}\right)(y) = \frac{1}{F'(F^{-1}(y))} \cdot \left[\frac{d}{dy}\left(\frac{h}{\varphi}\right)\right] \circ F^{-1}(y)$$
$$= \left[\frac{dy}{dF} \cdot \frac{d}{dy}\left(\frac{h}{\varphi}\right)\right] \circ F^{-1}(y) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(x).$$

If x is also in $\mathbf{\Gamma}$, equivalently $y = F(x) \in \{z \in [F(c), F(d)] : W(z) = H(z)\}$, then $\frac{d}{dF}\left(\frac{V}{\varphi}\right)(x)$ also exist by Corollary 7.1. Therefore $W = \frac{V}{\varphi} \circ F^{-1}$ is differentiable at y, and

$$\frac{dW}{dy}(y) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x).$$

Finally define $\widetilde{\mathbf{\Gamma}} \triangleq \{y \in [F(c), F(d)] : W(y) = H(y)\}$ and $\widetilde{\mathbf{C}} \triangleq [F(c), F(d)]\backslash\widetilde{\mathbf{\Gamma}}$. Then Proposition 7.3 immediately implies

Corollary 7.2. *Suppose h is continuous on $[c,d]$. Let H and W be as in Propo-sition 5.6. Suppose h, ψ and φ are differentiable at some $l, r \in \Gamma \cap (c,d)$. Then $\widetilde{l} \triangleq F(l) \in \widetilde{\Gamma}$ and $\widetilde{r} \triangleq F(r) \in \widetilde{\Gamma}$, and H' and W' exist at $\widetilde{l}$ and $\widetilde{r}$. Moreover*

(i) If $(l, r) \subseteq \mathbf{C}$, then $(\widetilde{l}, \widetilde{r}) \subset \widetilde{\mathbf{C}}$, and

$$H'(\widetilde{l}) = W'(\widetilde{l}) = \frac{H(\widetilde{r}) - H(\widetilde{l})}{\widetilde{r} - \widetilde{l}} = W'(\widetilde{r}) = H'(\widetilde{r}),$$

and

$$W(y) = H(\widetilde{l}) + H'(\widetilde{l})(y - \widetilde{l}) = H(y) = H(\widetilde{r}) + H'(\widetilde{r})(y - \widetilde{r}), \qquad y \in [\widetilde{l}, \widetilde{r}].$$

(ii) If $[c, r) \subset \mathbf{C}$, then $[F(c), \widetilde{r}) \subset \widetilde{\mathbf{C}}$ and

$$H'(\widetilde{r}) = W'(\widetilde{r}) = \frac{H(\widetilde{r})}{\widetilde{r} - \widetilde{l}},$$

and

$$W(y) = H(\widetilde{r}) + H'(\widetilde{r})(y - \widetilde{r}) = (y - F(c))H'(\widetilde{r}), \qquad y \in [F(c), \widetilde{r}].$$

(iii) If $(l, d] \subseteq \mathbf{C}$, then $(\widetilde{l}, F(d)] \subseteq \widetilde{\mathbf{C}}$ and

$$H'(\widetilde{l}) = W'(\widetilde{l}) = -\frac{H(\widetilde{l})}{F(d) - \widetilde{l}},$$

and

$$W(y) = H(\widetilde{l}) + (y - \widetilde{l})H'(\widetilde{l}) = (y - F(d))H'(\widetilde{l}), \qquad y \in (\widetilde{l}, F(d)].$$

Remark 7.1. *All results of this chapter also hold when we replace (φ, F) with (ψ, G) (equivalently, replace (φ, ψ) with $(\psi, -\varphi)$ since $\psi = \varphi \cdot F \leftrightarrow \psi \cdot G = -\varphi$). Alternatively one can use relations*

$$F \cdot G = -1 \quad and \quad \frac{h}{\psi} = -\frac{h}{\varphi} \cdot G$$

to derive counterparts. For example,

$$\frac{d}{dG}\left(\frac{h}{\psi}\right) = \frac{d}{dG}\left(-\frac{h}{\varphi} \cdot G\right) = -G \cdot \frac{d}{dG}\left(\frac{h}{\varphi}\right) - \frac{h}{\varphi} = -G \cdot \frac{dF}{dG} \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right) - \frac{h}{\varphi}$$

and

$$\frac{dF}{dG} = -\frac{d}{dG}\left(\frac{1}{G}\right) = \frac{1}{G^2} = F^2$$

will together imply

$$\frac{d}{dG}\left(\frac{h}{\psi}\right) = -G \cdot F^2 \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right) - \frac{h}{\varphi} = F \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right) - \frac{h}{\varphi}. \tag{7.6}$$

We can obtain $\frac{d}{dG}\left(\frac{V}{\psi}\right)$ by replacing h with V. Now at every $x \in \Gamma$ where $\frac{d}{dF}\left(\frac{h}{\varphi}\right)$ exists, $\frac{d}{dF}\left(\frac{V}{\varphi}\right)$, $\frac{d}{dG}\left(\frac{h}{\psi}\right)$ and $\frac{d}{dG}\left(\frac{V}{\psi}\right)$ also exist, and

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{d}{dF}\left(\frac{V}{\varphi}\right)(x) \quad \Longleftrightarrow \quad \frac{d}{dG}\left(\frac{h}{\psi}\right)(x) = \frac{d}{dG}\left(\frac{V}{\psi}\right)(x)$$

by (7.6) (Observe that $h(x) = V(x)$ since $x \in \Gamma$). We can therefore rewrite Proposition 7.3 and its Corollary in terms of G and h/ψ instead of F and h/φ easily.

We shall verify that our necessary conditions agree with those of Salminen [12, Theorem 4.7]. To do this, we first remember his

Definition 7.1 (Salminen [12], page 95)**.** *A point $x^* \in \Gamma$ is called a **left boundary** of Γ if for $\varepsilon > 0$ small enough $(x^*, x^* + \varepsilon) \subseteq \mathbf{C}$ and $(x^* - \varepsilon, x^*] \subseteq \Gamma$. A point $y^* \in \Gamma$ is called a **right boundary** of Γ if for $\varepsilon > 0$ small enough $(y^* - \varepsilon, y^*) \subseteq \mathbf{C}$ and $[y^*, y^* + \varepsilon) \subseteq \Gamma$ (cf. Figure 7.2 for illustration).*

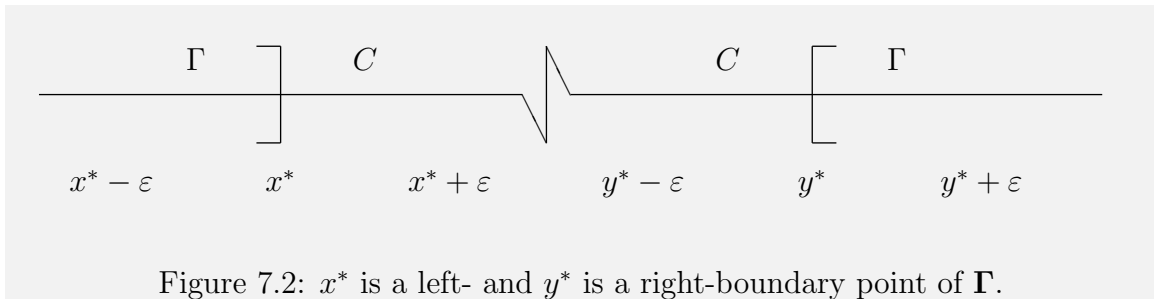Figure 7.2: x^* is a left- and y^* is a right-boundary point of Γ.

We shall also remind the definitions of the key functions G_b and G_a of Salminen's conclusion. At every $x \in (c,d)$ where h is S–differentiable, let

$$G_b(x) \triangleq \varphi(x)\frac{dh}{dS}(x) - h(x)\frac{d\varphi}{dS}(x) \quad \text{and} \quad G_a(x) \triangleq h(x)\frac{d\psi}{dS} - \psi(x)\frac{dh}{dS}(x). \tag{7.7}$$

Proposition 7.4. *Suppose h is continuous on $[c, d]$. If h, ψ and φ are S–differentiable at some $x \in (c, d)$, then $\frac{h}{\varphi}$ and $\frac{h}{\psi}$ are F– and G–differentiable at x, respectively. Moreover,*

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{G_b(x)}{W(\psi, \varphi)} \quad and \quad \frac{d}{dG}\left(\frac{h}{\psi}\right)(x) = -\frac{G_a(x)}{W(\psi, \varphi)}, \tag{7.8}$$

where $G_b(x)$ and $G_a(x)$ are defined as in (7.7), and $W(\psi, \varphi) \triangleq \varphi\frac{d\psi}{dS} - \psi\frac{d\varphi}{dS}$ (Wronskian of ψ and φ) is constant and positive (cf. Chapter 2).

Proof. Since h, ψ and φ are S–differentiable at x, $\frac{h}{\varphi}$ and F are S–differentiable at x. Therefore, $\frac{d}{dF}\left(\frac{h}{\varphi}\right)$ exist at x, and equals

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(x) = \frac{\frac{d}{dS}\left(\frac{h}{\varphi}\right)}{\frac{dF}{dS}}(x) = \frac{D_S h \cdot \varphi - h \cdot D_S \varphi}{D_S \psi \cdot \varphi - \psi \cdot D_S \varphi}(x)$$

$$= \frac{1}{W(\psi, \varphi)}\left[\varphi(x)\frac{dh}{dS}(x) - h(x)\frac{d\varphi}{dS}(x)\right] = \frac{G_b(x)}{W(\psi, \varphi)}, \tag{7.9}$$

where $D_S \equiv \frac{d}{dS}$. Following Remark 7.1 and noting the symmetry in (φ, F) versus (ψ, G), we can repeat all arguments by replacing (φ, ψ) with $(\psi, -\varphi)$. Therefore it can be similarly shown that $\frac{d}{dG}\left(\frac{h}{\psi}\right)(x)$ exists and $\frac{d}{dG}\left(\frac{h}{\psi}\right)(x) = -\frac{G_a(x)}{W(\psi, \varphi)}$ (note that $W(-\varphi, \psi) = W(\psi, \varphi)$). $\square$

Corollary 7.3 (Salminen [12], Theorem 4.7). *Let h be continuous on $[c, d]$. Suppose l and r are left– and right–boundary points of $\mathbf{\Gamma}$, respectively, such that $(l, r) \subseteq \mathbf{C}$. Assume that h, ψ and φ are S(scale function)–differentiable on the set $A \triangleq (l - \varepsilon, l] \cup [r, r + \varepsilon)$ for some $\varepsilon > 0$ such that $A \subseteq \mathbf{\Gamma}$. Then on A, the functions G_b and G_a of (7.7) are non–increasing and non–decreasing, respectively, and*

$$G_b(l) = G_b(r), \quad G_a(l) = G_a(r).$$

Proof. Proposition 7.4 implies that $\frac{d}{dF}\left(\frac{h}{\varphi}\right)$ and $\frac{d}{dG}\left(\frac{h}{\psi}\right)$ exist on A. Since $l, r \in \mathbf{\Gamma}$ and $(l, r) \subseteq \mathbf{C}$, Proposition 7.3(i) and (7.8) imply

$$\frac{G_b(l)}{W(\psi, \varphi)} = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) = \frac{G_b(r)}{W(\psi, \varphi)},$$

i.e. $G_b(l) = G_b(r)$ (Remember also that the Wronskian $W(\psi, \varphi) \triangleq \frac{d\psi}{dS} \varphi - \psi \frac{d\varphi}{dS}$ of ψ and φ is constant and positive. See Chapter 2). From Remark 7.1, it also follows $G_a(l) = G_a(r)$.

On the other hand, observe that $\frac{d}{dF} \left(\frac{V}{\varphi} \right)$ and $\frac{d}{dG} \left(\frac{V}{\psi} \right)$ also exist and, are equal to $\frac{d}{dF} \left(\frac{h}{\varphi} \right)$ and $\frac{d}{dG} \left(\frac{h}{\psi} \right)$ on A, respectively, by Corollary 7.1. Therefore

$$\frac{d}{dF} \left(\frac{V}{\varphi} \right)(x) = \frac{G_b(x)}{W(\psi, \varphi)} \quad \text{and} \quad \frac{d}{dG} \left(\frac{V}{\psi} \right)(x) = -\frac{G_a(x)}{W(\psi, \varphi)}, \quad x \in A, \qquad (7.10)$$

by Proposition 7.8. Because $\frac{V}{\varphi}$ is F–concave, and $\frac{V}{\psi}$ is G–concave, Proposition A.6(i) implies that both $\frac{d}{dF} \left(\frac{V}{\varphi} \right)$ and $\frac{d}{dG} \left(\frac{V}{\psi} \right)$ are non–increasing on A. Therefore (7.10) implies that G_b is non–increasing, and G_a is non–decreasing on A. □

Remark 7.2. All results of this chapter can exactly be translated for the optimal stopping problem of Chapter 3 with no discounting by replacing φ and G with 1(constant) and S(scale function), (and, S and $\overline{S}$,) respectively. They can also be adapted to the optimal stopping problem of Chapter 5 with little more work. Remember that if one or both of the boundaries are natural, then V may be infinite, or an optimal stopping time may not exist even if V is finite. In the latter case, it is still possible to derive results similar to those above. However, we no longer can refer to formulas for $U(x) \triangleq \mathbb{E}_x[e^{-\beta \tau^*} h(X_{\tau^*})]$, derived in the proof of Proposition 5.8 for example, as we explicitly write V on $\mathbf{C}$ since τ^* of (5.17) is no longer an optimal stopping time (Simply because U is not equal to V any more). Instead, we should use Proposition B.2 in Appendix B when we need to write V explicitly on $\mathbf{C}$.

7.2 Variational Inequalities

In this Section, we shall discuss the shortcomings of the Smooth–Fit principle when used in conjunction with the variational inequalities in solving optimal stopping problems. To this end, we shall construct an example and try to understand why sometimes this principle fails. First, we shall give a brief

7.2.1 Overview of the Method of Variational Inequalities

We stick to the same setup of Chapter 4. Namely, we have a stopped diffusion process X in $[c, d]$ where both c and d can be reached with probability one. The reward function h is continuous on $[c, d]$. We shall further assume that h has as many continuous derivatives in (c, d) as we need in the forthcoming discussion.

Proposition 4.2 and Proposition 4.4 guarantee the existence of the (finite) value function V of (4.1) and an optimal stopping time.

Intuitively, there are two possible actions that a decision maker can take at any point of state space. He either stops immediately, or continues for a while and acts optimally thereafter. The first alternative leads to $V \leq h$, whereas a formal argument by using Taylor series suggests that

$$\frac{1}{2}\sigma^2 V'' + \mu V' - \beta V \geq 0.$$

Since at least one of those alternatives will be optimal, one expects that V should formally solve the variational inequality

$$\min\left\{\frac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) - \beta V(x), h(x) - V(x)\right\} = 0, \quad \text{for } x \in \mathcal{I}, \quad (7.11)$$

and $(\mathcal{A} - \beta)V(x) \equiv \frac{1}{2}\sigma^2(x)V''(x) + \mu(x)V'(x) - \beta V(x) = 0$ holds in $\mathbf{C}$, and $V = h$ holds in $\boldsymbol{\Gamma}$. One common practice in the literature is to admit as *an ansatz* that (1) V is sufficiently smooth, and (2) it should solve the variational inequality (7.11). Later, a *verification lemma* guarantees, under further conditions, that the variational inequality has a unique solution, which is by the first part equal to the value function. Hence, if (l, r) or $(r, l) \subseteq \mathbf{C}$ and $l \in \boldsymbol{\Gamma}$, we expect that $v \triangleq V$ solves the boundary–value problem

$$\left\{\begin{array}{r}\frac{1}{2}\sigma^2(x)v''(x) + \mu(x)v'(x) - \beta v(x) = 0, \quad x \in (l, r) \text{ or } (r, l) \\ v(l) = h(l) \\ v'(l) = h'(l)\end{array}\right\}, \quad (7.12)$$

where the boundary conditions follow from Corollary 7.1. Denote the boundary value problem in (7.12) by $\underline{\text{BVP}}(l, r)$ if $l < r$, and $\overline{\text{BVP}}(r, l)$ otherwise.

Lemma 7.1. *Suppose that h is S–differentiable at l. Then the boundary value problem in (7.12) has unique solution*

$$v(x) = \frac{G_b(l)}{W(\psi, \varphi)}\psi(x) + \frac{G_a(l)}{W(\psi, \varphi)}\varphi(x), \quad x \in (l, r) \ or \ (r, l), \tag{7.13}$$

where G_a and G_b are given by (7.7).

Proof. Let us recall that ψ and φ are linearly independent solutions of $(\mathcal{A} - \beta)u \equiv \frac{1}{2}\sigma^2 u'' + \mu u' - \beta u = 0$. Therefore $v(x) = \alpha_1 \psi(x) + \alpha_2 \varphi(x)$ for some constants α_1 and α_2, which can be found by using the boundary conditions. $\qquad\square$

Proposition 7.5. *Let $c < l < r < d$. Suppose that h is S–differentiable at l and/or r.*

(a) *$v(x) \triangleq \mathbb{E}_x\left[e^{-\beta\tau_r}h(X_{\tau_r})\right]$, $x \in [c, r]$, is the unique solution of $\overline{\text{BVP}}(c, r)$ if and only if $G_a(r) + F(c) \cdot G_b(r) = 0$.*

(b) *$v(x) \triangleq \mathbb{E}_x\left[e^{-\beta\tau_l}h(X_{\tau_l})\right]$, $x \in [l, d]$, is the unique solution of $\underline{\text{BVP}}(l, d)$ if and only if $G(d) \cdot G_a(l) - G_b(l) = 0$.*

(c) *$v(x) \triangleq \mathbb{E}_x\left[e^{-\beta\tau_l \wedge \tau_r}h(X_{\tau_l \wedge \tau_r})\right]$, $x \in [l, r]$, is the unique solution of $\overline{\text{BVP}}(l, r)$ and $\underline{\text{BVP}}(l, r)$ simultaneously if and only if $G_a(l) = G_a(r)$ and $G_b(l) = G_b(r)$.*

Proof. It is clear from Lemma 4.3 that all three expectations satisfy the differential equation $\mathcal{A}v = \beta v$, and the first boundary condition of (7.12). Therefore, we only need to verify the second boundary conditions.

Suppose v is defined as in (a). By Lemma 4.3,

$$\frac{dv}{dS}(x) = \frac{d}{dS}\left(\varphi(x)\frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(c)}{F(r) - F(c)}\right) = \frac{h}{S}(r)\frac{1}{F(r) - F(c)}\left(\frac{d\psi}{dS}(x) - F(c)\frac{d\varphi}{dS}(x)\right),$$

for every $a < x < r$. One can show easily that $\frac{dv}{dS}(r-) = \frac{dh}{dS}(r)$ if and only if $G_a(r) + F(c) \cdot G_b(r) = 0$. (b) can be proved similarly.

For (c), after the derivatives of $v(x) \triangleq \mathbb{E}_x\left[e^{-\beta \tau_l \wedge \tau_r} h(X_{\tau_l \wedge \tau_r})\right]$, $l < x < r$ given by Lemma 4.3, are set to $\frac{dh}{dS}(l)$ and $\frac{dh}{dS}(r)$, respectively at $x = l$ and $x = r$, we obtain two equations

$$\frac{G_b(l)}{W(\psi, \varphi)} \cdot \psi(r) + \frac{G_a(l)}{W(\psi, \varphi)} \cdot \varphi(r) = h(r) \text{ and } \frac{G_b(r)}{W(\psi, \varphi)} \cdot \psi(l) + \frac{G_a(r)}{W(\psi, \varphi)} \cdot \varphi(l) = h(l),$$

respectively. Lemma 7.1 however implies that

$$\frac{G_b(r)}{W(\psi, \varphi)} \cdot \psi(r) + \frac{G_a(r)}{W(\psi, \varphi)} \cdot \varphi(r) = h(r) \text{ and } \frac{G_b(l)}{W(\psi, \varphi)} \cdot \psi(l) + \frac{G_a(l)}{W(\psi, \varphi)} \cdot \varphi(l) = h(l).$$

By subtracting the latter equations from the former, we obtain

$$\left[\frac{G_b(r)}{W(\psi, \varphi)} - \frac{G_b(l)}{W(\psi, \varphi)}\right] \cdot \psi(r) + \left[\frac{G_a(r)}{W(\psi, \varphi)} - \frac{G_a(l)}{W(\psi, \varphi)}\right] \cdot \varphi(r) = 0,$$

and

$$\left[\frac{G_b(r)}{W(\psi, \varphi)} - \frac{G_b(l)}{W(\psi, \varphi)}\right] \cdot \psi(l) + \left[\frac{G_a(r)}{W(\psi, \varphi)} - \frac{G_a(l)}{W(\psi, \varphi)}\right] \cdot \varphi(l) = 0.$$

After we multiply the first equation with $\varphi(l)$ and the second equation with $-\varphi(r)$, we sum the resulting equations. By rearranging the terms, we obtain

$$\left[\frac{G_b(r)}{W(\psi, \varphi)} - \frac{G_b(l)}{W(\psi, \varphi)}\right] \cdot \left[\frac{\psi(r)}{\varphi(r)} - \frac{\psi(l)}{\varphi(l)}\right] = 0.$$

Since $\frac{\psi(\cdot)}{\varphi(\cdot)}$ is strictly increasing, this implies $G_b(l) = G_b(r)$. Similarly, $G_a(l) = G_a(r)$. Conversely, if those equalities hold, then both equations above hold, thanks to Lemma 7.1. Therefore the second boundary condition is satisfied. $\square$

Since we know that the value function must coincide with one of the expectations in the form of $\mathbb{E}_x\left[e^{-\beta \tau_l} h(X_{\tau_l})\right]$, $\mathbb{E}_x\left[e^{-\beta \tau_r} h(X_{\tau_r})\right]$ and $\mathbb{E}_x\left[e^{-\beta \tau_l \wedge \tau_r} h(X_{\tau_l \wedge \tau_r})\right]$ for some $c \leq l < r \leq d$, we can eliminate the free boundary problem in (7.12), and directly work with $\overline{\text{BVP}}(c, r)$ for $G_a(r) + F(c)G_b(r) = 0$, $\underline{\text{BVP}}(l, d)$ for $G(d)G_a(l) - G_b(l) = 0$,

and, $\overline{\mathrm{BVP}}(l,r)$ and $\underline{\mathrm{BVP}}(l,r)$ for $G_a(l) - G_a(r) = G_b(l) - G_b(r) = 0$. Salminen[12, Theorem 4.7 and its remarks] also arrives at the same conclusion. In order to find the value function and optimal stopping rule, it is enough to compare solutions of boundary value problems against h and each other.

7.2.2 Why does and does not the Method of Variational Inequalities work?

Suppose that, after a brief study of the reward function $h(\cdot)$, we thought that it is is quite likely that $(l,r) \subset \mathbf{C}$ with some $c < l < r < d$ being the boundaries of optimal stopping region. Namely, once the process X starts in (l,r), we have reasons to believe that, it is optimal to let the process diffuse until it exits the interval. We expect to calculate three unknowns, l and r and $V(\cdot)$ on $[l,r]$, by solving the free boundary problem

$$\mathcal{A}v = \beta v \ \text{ in } (l,r) \quad \text{with} \quad v(x) = h(x) \ \text{ and } \ v'(x) = h'(x), \quad x = l, r, \qquad (7.14)$$

for some piecewise twice continuously differentiable[3] v on (c,d). Thanks to the Smooth–Fit principle $v' = h'$ at the boundaries of (l,r), (7.14) has unique solution $v(\cdot)$ for fixed l and r if and only if

$$G_b(l) = G_b(r) \quad \text{and} \quad G_a(l) = G_a(r)$$

by Proposition 7.5(c), where $G_a(\cdot)$ and $G_b(\cdot)$ are defined in (7.7). However the same system of equations is equivalent to

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) \quad \text{and} \quad \frac{d}{dG}\left(\frac{h}{\psi}\right)(l) = \frac{d}{dG}\left(\frac{h}{\psi}\right)(r), \qquad (7.15)$$

---

[3]i.e. continuously differentiable everywhere, and twice continuous differentiable everywhere except on at most a countable subset N of (c,d). Still left– and right–limits of second derivatives exist and are finite at every point in N.

by Proposition 7.4. We prefer however to work with only one of the pairs $(F, h/\varphi)$ and $(G, h/\psi)$ since we shall soon turn to the graphs and we do not want to bother with two different graphs for each pair. By using (7.6) however we can rewrite the second equation in (7.15) as

$$F(l) \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) - \frac{h}{\varphi}(l) = F(r) \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) - \frac{h}{\varphi}(r)$$

$$F(r) \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) - F(l) \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{h}{\varphi}(r) - \frac{h}{\varphi}(l)$$

$$\left(F(r) - F(l)\right) \cdot \frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{h}{\varphi}(r) - \frac{h}{\varphi}(l)$$

by using the first equality in (7.15). Therefore the system of equations in (7.15) is equivalent to

$$\frac{d}{dF}\left(\frac{h}{\varphi}\right)(l) = \frac{d}{dF}\left(\frac{h}{\varphi}\right)(r) = \frac{\frac{h}{\varphi}(r) - \frac{h}{\varphi}(l)}{F(r) - F(l)} \qquad (7.16)$$

which are in the mean time the same equations in Proposition 7.3(i) about the necessary conditions on the boundaries of optimal stopping region. Finally, if we define $H : [F(c), F(d)] \to \mathbb{R}$ by

$$H(y) \triangleq \frac{h}{\varphi} \circ F^{-1}(y), \quad y \in [F(c), F(d)],$$

as in Proposition 4.3, then Corollary 7.2(i) implies that l and r satisfy (7.16) if and only if

$$H'(F(l)) = \frac{H(F(r)) - H(F(l))}{F(r) - F(l)} = H'(F(r)). \qquad (7.17)$$

In summary, the boundary value problem for fixed l and r of (7.14) has unique solution if and only if l and r solve (7.17). Therefore success of method of variational inequalities often relies on the expectation that (7.17) has unique solution.

In the case that (7.17) has more than one solution, we still have to deal with selection of the "best" pair(s) of (l, r): *Not every solution, (l, r), of (7.17) necessarily gives a connected component of the optimal continuation region.* We shall investigate, in the rest of this section, how exactly the solutions of the free boundary value problem are related to the optimal stopping problem.

We can rewrite (7.17) as

$$H(F(r)) = H(F(l)) + (F(r) - F(l))H'(F(l)) \quad \text{and} \quad H(F(l)) = (F(l) - F(r))H'(F(r)).$$

These two equations tell us nothing more than that the straight line

$$L_{l,r}(y) \triangleq H(F(l)) \cdot \frac{F(r) - y}{F(r) - F(l)} + H(F(r)) \cdot \frac{y - F(l)}{F(r) - F(l)}, \quad y \in [F(c), F(d)], \quad (7.18)$$

is tangent to H at $F(l)$ and $F(r)$.

Hence solving the free boundary problem of (7.14) is exactly the attempt to find points l and r in (c, d) such that the straight line $L_{l,r}$ of (7.18) on $[F(l), F(r)]$ is tangent (i.e. smoothly fits to) H at $F(l)$ and $F(r)$.

We have not yet made a word about $v(\cdot)$, which is also to be determined as part of the the same free boundary problem. It is however not unreasonable to expect that the line $L_{l,r}$ itself gives rise to $v(\cdot)$. Remembering how the smallest nonnegative concave majorant W of H is related to the value function V, by the identity $V(x) = \varphi(x)W(F(x))$, $x \in [c, d]$ (cf. Proposition 4.3), suggests that

$$v(x) \triangleq \varphi(x)L_{l,r}(F(x)), \qquad x \in [l, r], \qquad (7.19)$$

should solve the ODE in (7.14), as well as, satisfy the boundary conditions at l and r. Observe that v of (7.19) is smooth enough, and

$$v(x) = \varphi(x) \cdot \left[H(F(l)) \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + H(F(r)) \cdot \frac{F(x) - F(l)}{F(r) - F(l)} \right] = \mathbb{A}\psi(x) + \mathbb{B}\varphi(x),$$

$x \in [l, r]$, for some constants $\mathbb{A}$ and $\mathbb{B}$, is a linear combination of β–harmonic functions ψ and φ (remember they are independent solutions of $\mathcal{A}u = \beta u$). Hence $v(\cdot)$ of (7.19) is itself β–harmonic, i.e. solves $\mathcal{A}v = \beta v$. On the other hand

$$v(x) = \varphi(x)L_{l,r}(F(x)) = \varphi(x)H(F(x)) = h(x) \quad \text{for both } x = l \text{ and } x = r.$$

We can now summarize our findings as in

Proposition 7.6. *The free–boundary problem of (7.14) admits the triplet $(l, r, v(\cdot))$ as a solution if and only if the straight line $L_{l,r}$ of (7.18) on $[F(c), F(d)]$ is tangent to H at $F(l)$ and $F(r)$, and $v(\cdot)$ is given by (7.19).*

This connection in conjunction with Proposition 4.3 will now reveal why *any* solution $(l, r, v(\cdot))$ of the free–boundary problem may not correctly match with optimal stopping boundaries and the value function V.

Let W be the smallest nonnegative concave majorant of H on $[F(c), F(d)]$. Proposition 4.3 shows that $V(x) = \varphi(x)W(F(x))$, $x \in [c, d]$. Therefore a solution $(l, r, v(\cdot))$ of (7.14) is a pair of optimal stopping region boundaries such that $(l, r) \subset \mathbf{C}$, and the value function on $[l, r]$, if and only if

$$L_{l,r} \equiv W > H \quad \text{on } (F(l), F(r)), \quad \text{and} \quad W(y) = H(y) \text{ for } y = F(l) \text{ and } F(r).$$
$$(7.20)$$

Is it possible to find sufficient conditions that will identify the solution for the optimal stopping problem among the solutions $(l, r, v(\cdot))$ of (7.14)?

Suppose (7.20) holds. Since W is concave, and $L_{l,r}$ is a straight line intersecting with W at $(l, W(F(l))) \equiv (l, H(F(l)))$ and $(r, W(F(r))) \equiv (r, H(F(r)))$, Proposition A.3 implies that $L_{l,r} \geq W$ outside $[F(l), F(r)]$. However W is a nonnegative majorant of H. Therefore (7.20) implies that

$$L_{l,r} \geq \max\{H, 0\} \quad \text{on } [F(c), F(d)], \quad \text{with strict inequality in } (F(l), F(r)). \quad (7.21)$$

Note now that (7.21) also implies (7.20). Since $L_{l,r}$ is a nonnegative concave majorant of H on $[F(c), F(d)]$ under (7.21), we have $L_{l,r} \geq W$ on $[F(c), F(d)]$. On the other hand

$$
\begin{aligned}
L_{l,r}(y) &= H(F(l)) \cdot \frac{F(r) - y}{F(r) - F(l)} + H(F(r)) \cdot \frac{y - F(l)}{F(r) - F(l)} \\
&\leq W(F(l)) \cdot \frac{F(r) - y}{F(r) - F(l)} + W(F(r)) \cdot \frac{y - F(l)}{F(r) - F(l)} \leq W(y), \qquad y \in [F(l), F(r)],
\end{aligned}
$$

since W is concave, and majorizes H. Therefore $L_{l,r} \equiv W$ on $[F(l), F(r)]$. This implies $H(y) = L_{l,r}(y) = W(y) \geq H(y)$ at $y = F(l)$ and $y = F(r)$. Therefore $W(y) = H(y)$ at $y = F(l)$ and $y = F(r)$. Since $L_{l,r} > H$ on $(F(l), F(r))$ by (7.21), we also have $L_{l,r} \equiv W > H$ in $(F(l), F(r))$. This proves (7.20). Hence (7.20) and (7.21) are one and the same conditions. Therefore

Proposition 7.7. *A solution $(l, r, v(\cdot))$ of (7.14) is a pair of optimal stopping region boundaries such that $(l, r) \subset \mathbf{C}$ and the value function on $[l, r]$, if and only if, $L_{l,r}$ of (7.18) on $[F(c), F(d)]$ is tangent to H at $F(l)$ and $F(r)$, and (7.21) holds.*

This conclusion will soon lead us to the necessary and sufficient conditions for a solution of the free boundary problem in (7.14) to be the solution for the optimal stopping problem at hand.

Let $(l, r, v(\cdot))$ be a solution of (7.14). Although v is found as a solution of $\mathcal{A}v = \beta u$ in (l, r), it is merely the restriction to $[l, r]$ of some continuous solution $\widetilde{v}(\cdot)$ of the same ODE in (c, d). To show that there exists indeed such $\widetilde{v}(\cdot)$ on $[c, d]$, remember that $v(\cdot)$ is necessarily related to $L_{l,r}$ of (7.18) as in (7.19), and $L_{l,r}$ is defined on the whole $[c, d]$. Hence if we define $\widetilde{v} : [c, d] \to \mathbb{R}$ by

$$\widetilde{v}(x) \triangleq \varphi(x) L_{l,r}(F(x)), \qquad x \in [c, d],$$

then $\widetilde{v} = \mathbb{A}\psi + \mathbb{B}\varphi$, for some constants $\mathbb{A}$ and $\mathbb{B}$, i.e. it is a linear combination of β–harmonic functions on $[c, d]$, and therefore is itself β–harmonic on $[c, d]$. Furthermore, $\widetilde{v} \equiv v$ on $[l, r]$.

As a matter of fact, $\widetilde{v}(\cdot)$ is unique. If $\widehat{v}(\cdot)$ is another continuous solution of $\mathcal{A}u = \beta u$ on $[c, d]$ that coincides with $v(\cdot)$ on $[l, r]$, then $\widehat{v}(\cdot) - \widetilde{v}(\cdot) \equiv 0$ on $[l, r]$. Since $\widehat{v} - \widetilde{v}$ also solves $\mathcal{A}u = \beta u$ in (c, d), $\widehat{v} - \widetilde{v} = \mathbb{C}\psi + \mathbb{D}\varphi$ for some constants $\mathbb{C}$ and $\mathbb{D}$ on $[c, d]$. Therefore, we have $\mathbb{C}\psi + \mathbb{D}\varphi \equiv 0$ on $[l, r]$, $l < r$. This however implies $\mathbb{C} = \mathbb{D} = 0$ since ψ and φ are *independent* solutions of $\mathcal{A}u = \beta u$.

Since $\widetilde{v}(\cdot)$ is unique, we shall omit the "tilde", and denote it also by $v(\cdot)$. Because v is related to $L_{l,r}$ by (7.19), and we also have $h(x) = \varphi(x)H(F(x))$, $x \in [c,d]$, we can rewrite (7.21) in terms of v and h as

$$v \geq \max\{h, 0\} \quad \text{on} \quad [c,d], \quad \text{with strict inequality in } (l,r), \tag{7.22}$$

and Proposition 7.7 implies

Proposition 7.8. *Let $(l, r, v(\cdot))$ be a solution of (7.14). Then $v(\cdot)$ can continuously and uniquely be extended onto $[c,d]$ such that $\mathcal{A}v = \beta v$ on (c,d). Moreover $(l, r, v(\cdot))$ is a pair of optimal stopping region boundaries such that $(l,r) \subset \mathbf{C}$, and the value function on $[l,r]$, if and only if, (7.22) holds.*

Finally, observe that we no longer need a verification lemma in order to show that the solution, described as in Proposition 7.8, coincides with the value function of the stopping problem in $[l,r]$.

An illustration of the connection between the solutions of the free boundary problem of (7.14) and the optimal stopping problem.

We shall now illustrate what we have done so far in an example which is deliberately constructed with the features that cause trouble when we use the method of variational inequalities.

Let B the standard one–dimensional Brownian motion in $[c,d]$ stopped, whenever it reaches $c \triangleq (1/2)\log 2\pi$ or $d \triangleq (1/2)\log 8\pi$. Consider the optimal stopping problem

$$V(x) \triangleq \sup_{\tau \geq 0} \mathbb{E}_x[e^{-\tau/2}h(X_\tau)], \qquad x \in [c,d],$$

with the reward function

$$h(x) \triangleq e^{-x}\left(5\pi - e^{2x}\right)\sin e^{2x}, \qquad x \in [c,d] \tag{7.23}$$

(we choose the discount rate $\beta = 1/2$). The infinitesimal generator is $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$, and the positive increasing and decreasing solutions of $\mathcal{A}u = u$ ($\beta = 1/2$) are given by

$$\psi(x) = e^x \quad \text{and} \quad \varphi(x) = e^{-x}, \qquad x \in \mathbb{R},$$

respectively. Therefore

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} = e^{2x}, \quad x \in R, \quad \text{and} \quad F^{-1}(y) = \frac{1}{2}\log y, \quad y \in [2\pi, 8\pi]. \tag{7.24}$$

Figure 7.3 is a (scaled) graph of h. The x–axis is labeled in terms of F^{-1} of (7.24) in order to relate it later with the graph of H. h has zeros at $z_k \triangleq F^{-1}(k\pi)$, for every $k = 1, \ldots, 8$. There are three minima at $m_1 \triangleq F^{-1}(3.5\pi)$, $m_2 \triangleq F^{-1}(5\pi)$ and $m_3 \triangleq F^{-1}(6.5\pi)$.

When the process starts at any of m_1, m_2 or m_3, there is nothing to lose by letting the process diffuse for some time. Therefore, m_1, m_2 and m_3 must be contained in optimal continuation region. It is however not clear if there are one– or two– or three– disconnected intervals of continuation region about those minima. Consider for example m_2. Should the connected component of $\mathbf{C}$ containing m_2 necessarily be larger than $[z_3, z_7]$? Or is it optimal to stop if the process can make almost at the top of any immediate small hills rising on $[z_4, z_6]$ (cf. Figure 7.3)?

First one is considered possible since the process will eventually reach either of two highest tops rising on $[z_2, z_3] \cup [z_7, z_8]$ in finite expected time. However, it may still take quite some time to cross over the valleys at either side of m_2 before it starts climbing higher tops. Therefore the possible gain earned by waiting, can be eroded by the discount factor. This is the rationale behind the second possibility.

In the first case, all three m_i, $i = 1, 2, 3$ would have been contained in a single connected component of the optimal continuation region. There might however be more than one connected components, if the second possibility turned out to be the fact. Therefore, when we attempt to solve the free–boundary problem of (7.14),
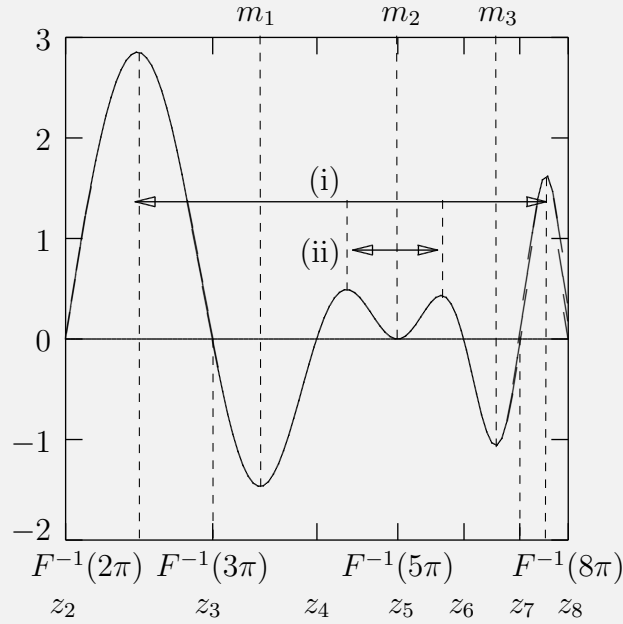
Figure 7.3: Reward function h of (7.23). $z_1, \ldots z_8$ are the zeros, and m_1, m_2, m_3 are the minima of h, respectively. When the process starts at m_2, there are two sensible optimal stopping strategies. We may choose either (i) to wait until the process reaches near the top of the first two highest hills, or (ii) to stop as soon as the process reaches near the top of either of the closest hills. The cost of waiting due to depreciation of the money value may wipe the additional expected gain of action (i) over (ii), and therefore we may favor (ii).

we should expect to find one or more solutions $(l, r, v(\cdot))$ depending on the actual topological properties of the optimal continuation region.

We shall not attack (7.14) itself. In order to learn about its solution(s), and to identify both the value function and the connected components of the optimal continuation region, we shall use Proposition 7.6 and Proposition 7.7. Note that

$$H(y) \triangleq \frac{h}{\varphi} \circ F^{-1}(y) = (5\pi - y)\sin y, \qquad y \in [F(c), F(d)] \equiv [2\pi, 8\pi].$$

Figure 7.4(a) is a scaled graph of H. The diagonal lines are $\mathcal{L}_1(y) \triangleq 5\pi - y$ and $\mathcal{L}_2(y) \triangleq -5\pi + y$. H oscillates between them. Note that H touches the lines at the
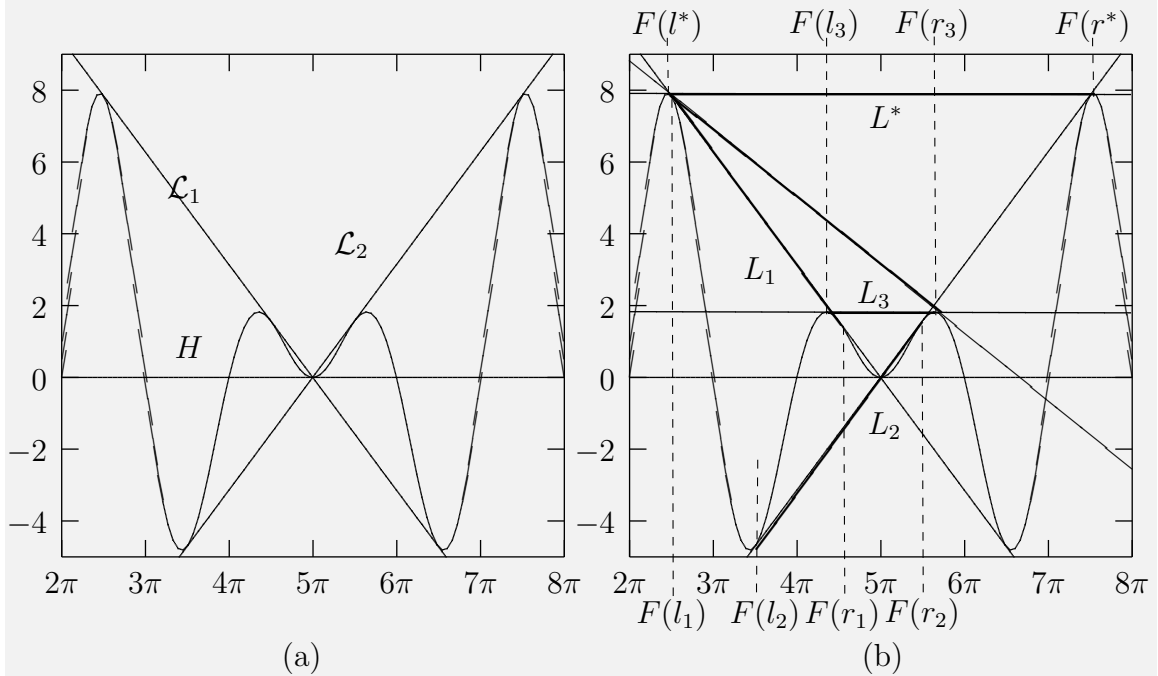
Figure 7.4: (a) H, (b) Some of the (many) tangent lines of H. In (b), we draw and label only four (out of ten) tangent lines of H. For readability, we replace L_{l_k, r_k} with L_k, and L_{l^*, r^*} with L^*. Every L_k corresponds to a unique solution $(l, r, v(\cdot))$ of the free–boundary problem (7.14), and vice versa, according to Proposition 7.6 (more precisely, L_k gives rise to the solution $(l_k, r_k, v_k(\cdot))$, where $v_k(\cdot)$ coincides with (7.19)). It is however easy to see that exactly one $L_{l,r}$, namely L^*, majorizes $\max\{H, 0\}$ on $[F(c), F(d)]$. Furthermore, Proposition 7.7 implies that the value function V coincides with v^* on $[F(l^*), F(r^*)]$ and $(l^*, r^*) \subseteq \mathbf{C}$, $\{l^*, r^*\} \subset \mathbf{\Gamma}$.

Note also that smallest nonnegative concave majorant W of H on $[F(c), F(d)]$ is obtained by piecing the restriction of L^* to $[F(l^*), F(r^*)]$ with H elsewhere. Therefore Proposition 4.3 and Corollary 4.1 also give the same result.

multiples of $\pi/2$ alternatingly. In particular, $\mathcal{L}_1$ is tangent to H at 2.5π, 4.5π and 6.5π. $\mathcal{L}_2$ is tangent to H at 3.5π, 5.5π, and 7.5π.

According to Proposition 7.6, solutions (l, r, v) of the free–boundary problem

(7.14) are in one–to–one correspondence with the lines $L_{l,r}$ on $[F(c), F(d)]$ that are tangent to H at $F(l)$ and $F(r)$. When Figure 7.4(b) is inspected, it is seen that H has a lot of tangent lines (once labeled pairwise with respect to the points where they are tangent to H, there are in total ten $L_{l,r}$, i.e. the free–boundary problem (7.14) has exactly ten solutions). For convenience, we identified only four of them in Figure 7.4(b). A brief inspection of Figure 7.4(b) reveals that none of the lines $L_{l,r}$, except $L_{l^*,r^*} \equiv L^*$, can majorize $\max\{H, 0\}$ on $[F(c), F(d)]$. Note also that $L^* > H$ on $(F(l^*), F(r^*))$. If we define

$$v^*(x) \triangleq \varphi(x)L^*(F(x)), \qquad x \in [l^*, r^*],$$

as in (7.19), then Proposition 7.7 guarantees that $(l, r, v(\cdot)) = (l^*, r^*, v^*(\cdot))$ is the only solution of (7.14) such that $(l, r) \subseteq \mathbf{C}$, $l, r \in \mathbf{\Gamma}$ and $V \equiv v^*$ on $[l, r]$.

Independently, it is also clear from Figure 7.4(b) that the smallest nonnegative concave majorant of H on $[F(c), F(d)]$ becomes

$$W(y) = \begin{cases} H(y), & y \notin (F(l^*), F(r^*)), \\ L^*(y), & y \in (F(l^*), F(r^*)). \end{cases}$$

Proposition 4.3 implies that $V(x) = \varphi(x)W(F(x))$, $x \in [c, d]$. In particular, $V(x) = \varphi(x)W(F(x)) = \varphi(x)L^*(F(x)) = v^*(x)$ for $x \in [l^*, r^*]$. Since on the other hand $W > H$ on $(F(l^*), F(r^*))$, Corollary 4.1 implies $(l^*, r^*) \subseteq \mathbf{C}$.

One may still wonder if the remaining $L_{l,r}$'s, equivalently solutions $(l, r, v(\cdot))$ of (7.14) have any probabilistic meaning. The answer of this question was already given by Proposition 7.5(c) in Subsubsection 7.2.1: If we let $\tau_{l,r} \triangleq \inf\{t \geq 0 : X_t \notin (l, r)\}$, then

$$v(x) = \mathbb{E}_x[e^{-\beta \tau_{l,r}} h(X_{\tau_{l,r}})], \qquad x \in [l, r].$$

This observation suggests a criterion in order to eliminate the uninteresting solutions of the free–boundary problem. Suppose $(l_1, r_1, v_1(\cdot))$ and $(l_2, r_2, v_2(\cdot))$ are

two solutions of (7.14) such that (l_1, r_1) and (l_2, r_2) have non–empty intersection. If $v_1(x) > v_2(x)$ for some $x \in (l_1, r_1) \cap (l_2, r_2)$, then we can eliminate $(l_2, r_2, v_2(\cdot))$ since, at least for one state, the stopping time τ_{l_1, r_1} returns better payoff than τ_{l_2, r_2}. Equivalently $V(x) \geq v_1(x) > v_2(x)$ for some $x \in (l_2, r_2)$ which implies $V \neq v_2$ on $[l_2, r_2]$.

In this Section, we learned that solving the free–boundary problem of (7.14) is equivalent to identifying tangent lines of H as summarized in Proposition 7.6. This was also a part of our task (see Examples in Chapter 6) in finding the smallest non-negative concave majorant for H of Proposition 4.3. The approach however shortfalls since a post–analysis is required in order to eliminate the unfit solutions.

Another difficulty with the method of variational inequalities is that the reward function h may not be differentiable at one of the optimal stopping boundaries. See Section 6.7 for an example and how our approach can still solve the problem.

Appendix A

Concave Functions

Let $F : [c, d] \to \mathbb{R}$ be a strictly increasing function. A real–valued function u is called F–concave on $[c, d]$ if, for every $a \le l < r \le b$ and $x \in [l, r]$, we have

$$u(x) \ge u(l)\frac{F(r) - F(x)}{F(r) - F(l)} + u(r)\frac{F(x) - F(l)}{F(r) - F(l)}. \tag{A.1}$$

Here, we shall state many facts about the properties of F–concave functions without proofs. Missing proofs can be reproduced from those for $F(x) = x$, which are available in many basic textbooks.

Lemma A.1. *The following are equivalent:*

(i) *u is F–concave on $[c, d]$.*

(ii) *For every $c \le x < y < z \le d$, we have: $\frac{u(y)-u(x)}{F(y)-F(x)} \ge \frac{u(z)-u(y)}{F(z)-F(y)}$.*

(iii) *For every $c \le x < y < z \le d$: $\frac{u(y)-u(x)}{F(y)-F(x)} \ge \frac{u(z)-u(x)}{F(z)-F(x)}$.*

(iv) *For every $c \le x < y < z \le d$: $\frac{u(z)-u(x)}{F(z)-F(x)} \ge \frac{u(z)-u(y)}{F(z)-F(y)}$.*

Proposition A.1. *Suppose u is real-valued F–concave and F is continuous on $[c, d]$. Then u is continuous in (c, d) and*

$$u(c) \le \liminf_{x \downarrow c} u(x), \quad and \quad u(d) \le \liminf_{x \uparrow d} u(x).$$

Let $I \subseteq \mathbb{R}$ be an interval with endpoints $-\infty \leq a < b \leq +\infty$. These endpoints may or may not be contained in I. Suppose $U : I \rightarrow \mathbb{R}$ is F–concave on I (i.e., for every closed and bounded interval $[c, d] \subseteq I$, the restriction of U to $[c, d]$ is F–concave on $[c, d]$). Proposition A.1 implies that U must be continuous in the interior of I, provided that it is F–continuous in I. In other words, U cannot oscillate in the interior of I. Proposition A.2 shows that U cannot oscillate near the end–points of I either, by the following

Proposition A.2. *Let I be as described above, and $U : I \rightarrow \mathbb{R}$ be F–concave on I. Then both $\lim_{x \downarrow a} U(x)$ and $\lim_{x \uparrow b} U(x)$ exist (possibly equal to $\pm\infty$).*

Proof. Let $\alpha \triangleq \lim\sup_{x \downarrow a} U(x)$ and $\beta \triangleq \lim\inf_{x \downarrow a} U(x)$. Both may be infinite. Assume on the contrary $\beta < \alpha$. We can always find two decreasing sequences (x_n) and (y_n) in I with limits a such that $\lim_{n \rightarrow +\infty} U(x_n) = \alpha$ and $\lim_{n \rightarrow +\infty} U(y_n) = \beta$.

Fix any $\gamma \in \mathbb{R}$ such that $\beta < \gamma < \alpha$. There exists then some $N > 0$ such that

$$n \geq N \quad \Longrightarrow \quad U(y_n) < \gamma < U(x_n). \tag{$*$}$$

Let $n_1, n_2, n_3 \geq N$ such that $x_{n_3} < y_{n_2} < x_{n_1}$. Because U is F–concave, and F is strictly increasing on I, we have

$$\gamma > U(y_{n_2}) \geq U(x_{n_3}) \frac{F(x_{n_1}) - F(y_{n_2})}{F(x_{n_1}) - F(x_{n_3})} + U(x_{n_1}) \frac{F(y_{n_2}) - F(x_{n_3})}{F(x_{n_1}) - F(x_{n_3})} > \gamma.$$

by $(*)$ above. Contradiction. Therefore, we must have $\beta = \alpha$. We can similarly show that $\lim\sup_{x \uparrow b} U(x) = \lim\inf_{x \uparrow b} U(x)$. $\square$

Remark A.1. If $a \in I$, then Proposition A.2 does not imply $U(a) = \lim_{x \downarrow a} U(x)$ by no means. In fact, Proposition A.1 implies $\lim_{x \downarrow a} U(x) \geq U(a)$ in this case. Same precaution also extends to b when $b \in I$.

Proposition A.3. *Let u be F–concave on $[c, d]$. Fix $c \leq l < r \leq d$ and define $L : [c, d] \rightarrow \mathbb{R}$ by*

$$L(x) \triangleq u(l) \frac{F(r) - F(x)}{F(r) - F(l)} + u(r) \frac{F(x) - F(l)}{F(r) - F(l)}, \quad \forall x \in [c, d].$$

Then we have $L \leq u$ on (l, r), and $L \geq u$ outside (l, r).

Proposition A.4. *Let $U : [c, d] \to \mathbb{R}$ be F–concave. Given $c \leq l < r_1 < r_2 \leq d$, let $L^{(i)} : [c, d] \to \mathbb{R}$ be defined by*

$$L^{(i)}(x) \triangleq U(l) \cdot \frac{F(r_i) - F(x)}{F(r_i) - F(l)} + U(r_i) \cdot \frac{F(x) - F(l)}{F(r_i) - F(l)}, \quad i = 1, 2,$$

respectively. Then $L^{(1)}(x) \geq L^{(2)}(x)$ for every $x \in [l, d]$, and, $L^{(1)}(x) \leq L^{(2)}(x)$ for every $x \in [c, l]$

Proposition A.5. *Let $(u_\alpha)_{\alpha \in \Lambda}$ is a family of F–concave functions on $[c, d]$. Then $u \triangleq \wedge_{\alpha \in \Lambda} u_\alpha$ is also F–concave on $[c, d]$.*

Let $v : [c, d] \to \mathbb{R}$ be any function. Since F is strictly increasing, for every $c \leq x < y \leq d$, the quotients $\frac{v(x) - v(y)}{F(x) - F(y)}$ are well-defined. Let $x \in [c, d)$. If $\lim_{y \downarrow x} \frac{v(x) - v(y)}{F(x) - F(y)}$ exists, then we say that v has right-derivative with respect to F at x, and denote the limit by

$$\frac{d^+ v}{dF}(x) \triangleq \lim_{y \downarrow x} \frac{v(x) - v(y)}{F(x) - F(y)}.$$

We similarly define the left-derivative of v with respect to F at $x \in (c, d]$ by

$$\frac{d^- v}{dF}(x) \triangleq \lim_{y \uparrow x} \frac{v(x) - v(y)}{F(x) - F(y)}.$$

provided that the limit on the right-hand side exists. If both $\frac{d^+ v}{dF}(x)$ and $\frac{d^- v}{dF}(x)$ exist and are equal , then we say that v is F-differentiable at x and denote their common value by

$$\frac{dv}{dF}(x) \triangleq \lim_{y \to x} \frac{v(x) - v(y)}{F(x) - F(y)}.$$

Proposition A.6. *Suppose $u : [c, d] \to \mathbb{R}$ is F–concave. Then we have the following:*

(i) $\frac{d^+ u}{dF}$ and $\frac{d^- u}{dF}$ exist in (c, d). Both are non–increasing and

$$\frac{d^+ u}{dF}(l) \geq \frac{d^- u}{dF}(x) \geq \frac{d^+ u}{dF}(x) \geq \frac{d^- u}{dF}(r), \quad c < l < x < r < d. \qquad \text{(A.2)}$$

(ii) *Let $x_0 \in (c, d)$. For every $\frac{d^+ u}{dF}(x_0) \leq \theta \leq \frac{d^- u}{dF}(x_0)$, we have*

$$u(x_0) + \theta[F(x) - F(x_0)] \geq u(x), \quad \forall x \in [c, d].$$

(iii) *Suppose F is continuous on $[c, d]$. Then there exist sequences $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$ such that*

$$u(x) = \inf_{n \geq 1} \left[\alpha_n F(x) + \beta_n\right], \quad \forall x \in (c, d).$$

(iv) *If F is continuous on $[c, d]$, then $\frac{d^+ u}{dF}$ is right–continuous, and $\frac{d^- u}{dF}$ is left–continuous.*

Lemma A.2. *Suppose $F : [c, d] \to \mathbb{R}$ is strictly increasing and continuous. Let $u : [c, d] \to \mathbb{R}$ be F–differentiable at $x \in [c, d]$. Then u is continuous at x.*

Lemma A.3. *Suppose $F : [c, d] \to 0$ is strictly increasing. Suppose $u : [c, d] \to \mathbb{R}$ has a local maximum or minimum at a point $x \in (c, d)$, and $\frac{du}{dF}(x)$ exists. Then $\frac{du}{dF}(x) = 0$.*

Lemma A.4. *Let $F : [c, d] \to \mathbb{R}$ be strictly increasing and continuous. Suppose $u : [c, d] \to \mathbb{R}$ is a continuous function which is F–differentiable in (c, d). Then there exists some $x \in (c, d)$ such that*

$$u(d) - u(c) = \frac{du}{dF}(x)\big(F(d) - F(c)\big). \tag{A.3}$$

Lemma A.5. *Suppose $F : [c, d] \to \mathbb{R}$ is strictly increasing and continuous. Suppose $u : [c, d] \to \mathbb{R}$ is a continuous function. If $\frac{du}{dF}$ exists and identically equals zero in (c, d), then u is constant on $[c, d]$.*

Suppose $F : \mathbb{R} \to \mathbb{R}$ is strictly increasing. A function $u : \mathbb{R} \to \mathbb{R}$ is called F–concave if the restriction of u to $[-n, n]$ is F–concave for every $n \geq 1$.

Lemma A.6 (Adapted from Karatzas and Shreve [7], Page 213, Problem 6.20)**.** *Let*
$F : \mathbb{R} \to \mathbb{R}$ *be strictly increasing and continuous, and* $v : \mathbb{R} \to \mathbb{R}$ *be non–increasing.*
Define

$$v_{\pm}(x) \triangleq \lim_{y \to x\pm} v(y), \quad V(x) \triangleq \int_0^x v(y) F(dy),$$

where the integral is understood in Lebesgue–Stieltjes sense.

(i) *The functions* v_+ *and* v_- *are right– and left–continuous, respectively, and*

$$v_+(x) \le v(x) \le v_-(x), \quad \forall x \in \mathbb{R}. \tag{A.4}$$

(ii) *The functions* $v_{\pm}$ *have the same set of continuity points, and equality holds in*
(A.4) on this set. In particular, except for x *in a countable set* N, *we have*
$v_{\pm}(x) = v(x)$.

(iii) *The function* V *is* F*–concave, and*

$$\frac{d^+ V}{dF}(x) = v_+(x) \le v(x) \le v_-(x) = \frac{d^- V}{dF}(x), \quad \forall x \in \mathbb{R}.$$

(iv) *If* $u : \mathbb{R} \to \mathbb{R}$ *is any other* F*–concave function for which*

$$\frac{d^+ u}{dF}(x) \le v(x) \le \frac{d^- u}{dF}(x), \quad \forall x \in \mathbb{R}, \tag{A.5}$$

then we have $u(x) = u(0) + V(x)$, *for all* $x \in \mathbb{R}$.

Proof. Proofs are similar to those of Problem 6.21 and Problem 6.22, Karatzas and
Shreve [7, page 213–214]. □

Proposition A.7. *Suppose* F *is strictly increasing and continuous on* $[c, d]$. *Let*
$U : [c, d] \to \mathbb{R}$ *be* F*–concave on* $[c, d]$, *and* F*–differentiable in* (c, d). *Then* $\frac{dU}{dF}$ *is*
continuous in (c, d).

Proof. Since F is continuous on $[c, d]$, and U is F–concave on $[c, d]$, Proposition A.6(i)
and (iv) imply that $\frac{d^- U}{dF}$ and $\frac{d^+ U}{dF}$ exist, and are left– and right–continuous in (c, d),

respectively. Since U is F–differentiable in (c, d) by hypothesis, $\frac{d^{\pm}U}{dF}$ agree with $\frac{dU}{dF}$ everywhere in (c, d). Therefore,

$$
\frac{dU}{dF}(x) = \frac{d^{+}U}{dF}(x) = \lim_{y \downarrow x} \frac{d^{+}U}{dF}(y) = \lim_{y \downarrow x} \frac{dU}{dF}(y),
$$
$$
\frac{dU}{dF}(x) = \frac{d^{-}U}{dF}(x) = \lim_{y \uparrow x} \frac{d^{-}U}{dF}(y) = \lim_{y \uparrow x} \frac{dU}{dF}(y),
$$

for every $x \in (c, d)$, i.e. $\frac{dU}{dF}$ is both left– and right– continuous, therefore continuous in (c, d). $\qquad \square$

Appendix B

Properties of Nonnegative Concave Majorants

Let I denote an interval contained in $\mathbb{R}$ with end–points a and b, $-\infty \leq a < b \leq +\infty$. The end–points a and b may or may not be contained in I. Suppose $\psi : I \to (0, \infty)$ and $\varphi : I \to (0, \infty)$ are continuous functions that are strictly increasing and strictly decreasing, respectively. Then

$$F(x) \triangleq \frac{\psi(x)}{\varphi(x)} \quad \text{and} \quad G(x) \triangleq -\frac{\varphi(x)}{\psi(x)}, \qquad x \in I,$$

are well–defined, continuous and strictly increasing functions on I. Note that $F \cdot G = -1$.

Let $h : I \to \mathbb{R}$ be a continuous function. In this chapter, we shall study the smallest nonnegative F–concave (G–concave, respectively) majorant of $\frac{h}{\varphi}$ ($\frac{h}{\psi}$, respectively) on I. We shall adopt the following

Convention B.1. *Every concave function on I is real–valued. Hence $U \equiv +\infty$ or $U \equiv -\infty$ on I will not be regarded as concave functions.*

Therefore, we first need to address existence of smallest nonnegative F–concave and G–concave majorants of $\frac{h}{\varphi}$ and $\frac{h}{\psi}$, respectively. We will see that their existence

are tied to

$$\ell_a \triangleq \limsup_{x\downarrow a} \frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad \ell_b \triangleq \limsup_{x\uparrow b} \frac{h^+(x)}{\psi(x)} \tag{B.1}$$

where $h^+ \triangleq \max\{0, h\}$ on I. Note that h^+ is itself continuous on I. Before stating the main results, we need some preliminary work which we now present in a sequence of Lemmata.

Lemma B.1. *A function $U : I \to \mathbb{R}$ is F–concave on I if and only if $(-G) \cdot U$ is G–concave on I. Similarly, U is G–concave if and only if $F \cdot U$ is F–concave.*

Proof. Let $x \in [l, r] \subseteq I$, $l < r$. By using $F(\cdot) \cdot G(\cdot) = -1$, we obtain

$$U(l)\frac{F(r) - F(x)}{F(r) - F(l)} + U(r)\frac{F(x) - F(l)}{F(r) - F(l)}$$
$$= \frac{1}{(-G)(x)}\left[(-G \cdot U)(l)\frac{G(r) - G(x)}{G(r) - G(l)} + (-G \cdot U)(r)\frac{G(x) - G(l)}{G(r) - G(l)}\right]. \tag{B.2}$$

If U is F–concave, then left-hand side is smaller than or equal to $U(x)$. Since $-G$ is positive, by multiplying both sides of inequality with $(-G)(x)$ we obtain

$$(-G \cdot U)(x) \geq (-G \cdot U)(l)\frac{G(r) - G(x)}{G(r) - G(l)} + (-G \cdot U)(r)\frac{G(x) - G(l)}{G(r) - G(l)}, \tag{B.3}$$

i.e. $(-G) \cdot U$ is G–concave on I.

Now suppose that $(-G) \cdot U$ is G–concave on I. Then we have (B.3). By dividing both sides with positive $-G(x)$, we realize that the right–hand side of (B.2) is smaller than or equal to $U(x)$. Hence U is F–concave on I.

As of the proof of the second statement, note that $U \equiv (-G) \cdot (F \cdot U)$ is G–concave (since $F \cdot G = -1$) if and only if $F \cdot U$ is F–concave, according to the first part. $\square$

Lemma B.2. $\frac{h}{\varphi}$ *has a nonnegative F–concave majorant on I if and only if $\frac{h}{\psi}$ has a nonnegative G–concave majorant on I.*

In particular, if U is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I, then $(-G) \cdot U$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$ on I. Similarly, if $\widetilde{U}$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$ on I, then $F \cdot \widetilde{U}$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I.

Proof. Let U be a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Then $(-G) \cdot U$ is nonnegative and G–concave by Lemma B.1. Furthermore since $U \geq \frac{h}{\varphi}$ on I, and $-G$ is positive, we have

$$(-G) \cdot U \geq -G \cdot \frac{h}{\varphi} = \frac{\varphi}{\psi} \cdot \frac{h}{\varphi} = \frac{h}{\psi}, \qquad \text{on } I.$$

Hence $(-G) \cdot U$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$ on I.

Let $\widetilde{U}$ be a nonnegative G–concave majorant of $\frac{h}{\psi}$ on I. Then $F \cdot \widetilde{U}$ is nonnegative and F–concave by Lemma B.1. Since $\widetilde{U}$ majorizes $\frac{h}{\psi}$, we have

$$F \cdot \frac{h}{\psi} = \frac{\psi}{\varphi} \cdot \frac{h}{\psi} = \frac{h}{\varphi}.$$

Therefore, $F \cdot \widetilde{U}$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. $\qquad\square$

Lemma B.2 shows that exactly one of

$$\mathbb{A} \triangleq \left\{ \begin{array}{c} \text{both } \frac{h}{\varphi} \text{ and } \frac{h}{\psi} \\ \text{have (smallest)}^1 \text{nonnegative} \\ F- \text{ and } G\text{–concave} \\ \text{majorants} \\ \text{on I, respectively} \end{array} \right\} \text{ and } \mathbb{B} \triangleq \left\{ \begin{array}{c} \text{neither } \frac{h}{\varphi} \text{ nor } \frac{h}{\psi} \\ \text{have (smallest) nonnegative} \\ F- \text{ and } G\text{–concave} \\ \text{majorants} \\ \text{on I, respectively} \end{array} \right\} \quad (\text{B.4})$$

must be true. Observe also that both ℓ_a and ℓ_b of (B.1) are contained in $[0, +\infty]$. Our result on existence of nonnegative concave majorants is

Proposition B.1. *At least one of ℓ_a and ℓ_b of (B.1) is equal to $+\infty$ if and only if $\mathbb{B}$ of (B.4) holds. In other words, both ℓ_a and ℓ_b are finite if and only if $\mathbb{A}$ holds.*

---

[1] Define $\mathcal{M} \triangleq \{f | f : I \to \mathbb{R} \text{ is a nonnegative } F\text{–concave majorant of } \frac{h}{\varphi}\}$. Let $U(x) \triangleq \inf_{f \in \mathcal{M}} f(x)$, $x \in I$ ($\inf \varnothing \equiv +\infty$). By Proposition A.5, we know that U is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I, if the set on the right–hand side is not empty. Minimality of U is also evident. Hence U is the smallest nonnegative F–concave majorant of $\frac{h}{\varphi}$.

This proves that the smallest nonnegative concave majorant exist if and only if a nonnegative concave majorant exists. Hence addition of "(smallest)" in the definitions of $\mathbb{A}$ and $\mathbb{B}$ in (B.4) do not impose any burden.

Suppose $\mathbb{A}$ holds. Let U_a (U_b, respectively) be the smallest nonnegative F–concave (G–concave, respectively) majorant of $\frac{h}{\varphi}$ ($\frac{h}{\psi}$, respectively) on I. Then $\lim_{x\downarrow a} U_a(x)$ and $\lim_{x\uparrow b} U_b(x)$ exist, and

$$\ell_a \leq \lim_{x\downarrow a} U_a(x) < +\infty \quad and \quad \ell_b \leq \lim_{x\uparrow b} U_b(x) < +\infty. \tag{B.5}$$

Proof. Suppose $\ell_a = +\infty$. Assume on the contrary that $\frac{h}{\varphi}$ has a nonnegative F–concave majorant, denoted by U, on I. Fix any $x \in \mathrm{int}(I)$. For every $[l, r] \subset I$ that contains x, we have

$$U(x) \geq U(l) \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + U(r) \cdot \frac{F(x) - F(l)}{F(r) - F(l)} \geq \frac{h^+(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)}.$$

First inequality follows from F–concavity of U whereas second inequality is valid since $U \geq \max\{0, h/\varphi\}$ on I. Since F is strictly increasing, $F(a+)$ exist and $0 \leq F(a+) < F(r) < \infty$. By taking limit supremum of both sides as $l \downarrow a$, we find

$$U(x) \geq \left(\limsup_{l \downarrow a} \frac{h^+(l)}{\varphi(l)} \right) \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} = \ell_a \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} = +\infty.$$

Hence $U \equiv +\infty$ in the interior of I. By our Convention B.1, U is not a concave function. Contradiction. Therefore $\mathbb{A}$ cannot be true.

Suppose $\ell_b = +\infty$, assume on the contrary that $\frac{h}{\psi}$ has a nonnegative G–concave majorant, denoted by $\widetilde{U}$, on I. Fix any $x \in \mathrm{int}(I)$. For every $[l, r] \subset I$ containing x, we have

$$\widetilde{U}(x) \geq \widetilde{U}(l) \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \widetilde{U}(r) \cdot \frac{G(x) - G(l)}{G(r) - G(l)} \geq \frac{h^+(r)}{\psi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}.$$

Since G is strictly increasing and negative on I, $G(b-)$ exists and $-\infty < G(l) < G(b-) \leq 0$. Therefore

$$\widetilde{U}(x) \geq \left(\limsup_{r \uparrow b} \frac{h^+(r)}{\psi(r)} \right) \cdot \frac{G(x) - G(l)}{G(b-) - G(l)} = \ell_b \cdot \frac{G(x) - G(l)}{G(b-) - G(l)} = +\infty.$$

Therefore $\widetilde{U} \equiv +\infty$ in the interior of I, and is not a concave function by Convention B.1. Contradiction. Hence $\mathbb{A}$ cannot be true.

Now suppose that both ℓ_a and ℓ_b are finite. We shall prove that there exists a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. This will imply that $\mathbb{B}$ cannot be correct, i.e. $\mathbb{A}$ must hold because of the dilemma in (B.4).

Since $\frac{h}{\varphi}(a+) = \ell_a \in [0, +\infty)$, there exists some $l \in (a, b)$ such that $\frac{h}{\varphi}(x) \leq 1 + \ell_a$ for every $x \in I \cap (-\infty, l)$. On the other hand

$$\frac{h}{\varphi} = \frac{h}{\psi} \cdot \frac{\psi}{\varphi} = \frac{h}{\psi} \cdot F.$$

Since $\frac{h}{\psi}(b-) = \ell_b \in [0, +\infty)$, there exists some $r \in (l, b)$ such that $\frac{h}{\psi}(x) \leq 1 + \ell_b$ for all $x \in I \cap (r, \infty)$. Therefore $\frac{h(x)}{\varphi(x)} \leq (1 + \ell_b)F(x)$ for every $x \in I \cap (r, \infty)$. Finally, since $\frac{h}{\varphi}$ is continuous on I, it is bounded on $[l, r] \subset I$. Therefore there are real numbers $c_1 \geq 1 + \ell_a > 0$ and $c_2 \geq 1 + \ell_b > 0$ such that $\frac{h}{\varphi} \leq c_1 + c_2 F$ on I. Hence $c_1 + c_2 F$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I.

Now suppose $\mathbb{A}$ holds, and let U_a and U_b be as described in statement of Proposition. $\lim_{x \downarrow a} U_a(x)$ and $\lim_{x \uparrow b} U_b(x)$ exist by Proposition A.2 in Appendix A. Since U_a and U_b majorize $\max\{0, h/\varphi\}$ and $\max\{0, h/\psi\}$, respectively, the first inequalities in (B.5) are clear. It remains to show that the limits are finite. We proved in the previous paragraph that there is a nonnegative F–concave majorant $U \triangleq c_1 + c_2 F$ of $\frac{h}{\varphi}$ on I for some positive real numbers c_1 and c_2. Therefore $U_a \leq U$ on I, and

$$\lim_{x \downarrow a} U_a(x) \leq \lim_{x \downarrow a} U(a) = c_1 + c_2 F(a+) < \infty.$$

By Lemma B.2, $\widetilde{U} \triangleq (-G) \cdot U = -c_1 G + c_2$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$. Therefore $U_b \geq \widetilde{U}$, and

$$\lim_{x \uparrow b} U_b(x) \leq \lim_{x \uparrow b} \widetilde{U}(x) = -c_1 G(b-) + c_2 < +\infty.$$

This completes the proof. $\qquad\square$

In the remaining part of this Chapter, we shall assume that

$$\text{the quantities } \ell_a \text{ and } \ell_b \text{ of (B.1) are finite.} \qquad (B.6)$$

By Proposition B.1, $\mathbb{A}$ of (B.4) holds. Suppose

$$
\left\{
\begin{array}{c}
U_a : I \to \mathbb{R} \text{ is} \\[4pt]
\text{the smallest nonnegative} \\[4pt]
F\text{–concave majorant} \\[4pt]
\text{of } \frac{h}{\varphi} \text{ on } I.
\end{array}
\right\}
\quad \text{and} \quad
\left\{
\begin{array}{c}
U_b : I \to \mathbb{R} \text{ is} \\[4pt]
\text{the smallest nonnegative} \\[4pt]
G\text{–concave majorant} \\[4pt]
\text{of } \frac{h}{\psi} \text{ on } I.
\end{array}
\right\}.
$$

Lemma B.3. *We have*

$$
\varphi(x)U_a(x) = \psi(x)U_b(x), \qquad x \in I. \tag{B.7}
$$

Proof. Lemma B.2 implies that $(-G) \cdot U_a$ is a nonnegative G–concave majorant of $\frac{h}{\psi}$ on I. Since U_b is the smallest of the functions with the same properties, we have $(-G) \cdot U_a \geq U_b$, i.e. $\varphi\, U_a \geq \psi\, U_b$.

By the same Lemma B.2, $F \cdot U_b$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Since U_a is the *smallest* nonnegative F–concave majorant, we have $F \cdot U_b \geq U_a$, i.e. $\varphi\, U_a(x) \leq \psi\, U_b$. $\qquad\square$

We shall denote the common value of $\varphi \cdot U_a$ and $\psi \cdot U_b$ by $V \triangleq \varphi \cdot U_a \equiv \psi \cdot U_b$. Thus $V : I \to \mathbb{R}$ enjoys the following properties:

$$
\left\{
\begin{array}{c}
\frac{V}{\varphi}(\equiv U_a) \text{ is the smallest} \\[4pt]
\text{nonnegative} \\[4pt]
F\text{–concave majorant} \\[4pt]
\text{of } \frac{h}{\varphi} \text{ on } I.
\end{array}
\right\}
\quad \text{and} \quad
\left\{
\begin{array}{c}
\frac{V}{\psi}(\equiv U_b) \text{ is the smallest} \\[4pt]
\text{nonnegative} \\[4pt]
G\text{–concave majorant} \\[4pt]
\text{of } \frac{h}{\psi} \text{ on } I.
\end{array}
\right\}.
$$

Furthermore, Proposition B.1 shows that $\lim_{x \downarrow a} \frac{V(x)}{\varphi(x)}$ and $\lim_{x \uparrow b} \frac{V(x)}{\psi(x)}$ exist, and

$$
\ell_a \leq \lim_{x \downarrow a} \frac{V(x)}{\varphi(x)} < +\infty \quad \text{and} \quad \ell_b \leq \lim_{x \uparrow b} \frac{V(x)}{\psi(x)} < +\infty. \tag{B.8}
$$

Define

$$
\mathbf{\Gamma} \triangleq \left\{ x \in I \,\middle|\, V(x) = h(x) \right\} \quad \text{and} \quad \mathbf{C} \triangleq I \backslash \mathbf{\Gamma} = \left\{ x \in I \,\middle|\, V(x) > h(x) \right\}.
$$

Since $\frac{V}{\varphi}$ is F–concave on I, Proposition A.1 implies that $\frac{V}{\varphi}$ is lower semi–continuous on I (in fact, it is continuous in (a,b) and lower semi–continuous at the boundaries

of I which are contained in I). Since φ and F are continuous and positive on I, this implies that V is itself lower semi–continuous on I. Finally, since h is continuous on I, $V - h$ is lower semi–continuous on I. Therefore **C** is open relative to I^2.

Since **C** is open relative to I, it is union of a countable family of disjoint and open, relative to I, subintervals of I. We shall denote this family by $(J_\alpha)_{\alpha \in \Lambda}$. Observe that J_α can take at most three different forms: (i) $J_\alpha = (l, r) \subseteq \mathbf{C}$, for some $l, r \in \mathbf{\Gamma}$, or (ii) $J_\alpha = I \cap (-\infty, r)$ for some $r \in \mathbf{\Gamma}$ or (iii) $J_\alpha = I \cap (l, \infty)$ for some $l \in \mathbf{\Gamma}$, for all $\alpha \in \Lambda$. Observe that $I \cap (-\infty, r)$ becomes $[a, r)$ if $a \in I$, or (a, r) if $a \notin I$. Similarly $I \cap (l, \infty)$ becomes $(l, b]$ if $b \in I$, or (l, b) if $b \notin I$.

Proposition B.2. *Let V be defined as above. Then $V = h$ on $\mathbf{\Gamma}$. $\mathbf{C}$ can be partitioned into countable disjoint open (relative to I) subintervals of I. The followings summarize all three possible forms of subintervals, and the form of V in each case:*

(i) Suppose $(l, r) \subseteq \mathbf{C}$ for some $l, r \in \mathbf{\Gamma}$, $l < r$. Then

$$\frac{V(x)}{\varphi(x)} = \frac{h(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)}, \qquad x \in [l, r].$$

Equivalently,

$$\frac{V(x)}{\psi(x)} = \frac{h(l)}{\psi(l)} \cdot \frac{G(r) - G(x)}{G(r) - G(l)} + \frac{h(r)}{\psi(r)} \cdot \frac{G(x) - G(l)}{G(r) - G(l)}, \qquad x \in [l, r].$$

(ii) Suppose $I \cap (-\infty, r) \subseteq \mathbf{C}$ for some $r \in \mathbf{\Gamma}$. Then

$$\frac{V(x)}{\varphi(x)} = \ell_a \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} + \frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(a+)}{F(r) - F(a+)}, \qquad x \in I \cap (-\infty, r].$$

If $a \in I$, then we have $F(a+) = F(a)$ by continuity of F on I. If $a \notin I$, then $F(a+)$ still exists, and is finite since F is strictly increasing and positive on I.

---

[2] A function $f : I \to \mathbb{R}$ is called *lower semi–continuous* at $x \in I$ if $f(x) \leq \liminf_{y \to x} f(y)$. It is called a lower semi–continuous function if it is lower semi–continuous at every $x \in I$. f is lower semi–continuous if and only if for every $\lambda \in \mathbb{R}$, $\{x \in I : f(x) > \lambda\}$ is open relative to I.

(iii) Suppose $I \cap (l, \infty) \subseteq \mathbf{C}$ for some $l \in \mathbf{\Gamma}$. Then

$$\frac{V(x)}{\psi(x)} = \frac{h(l)}{\psi(l)} \cdot \frac{G(b-) - G(x)}{G(b-) - G(l)} + \ell_b \cdot \frac{G(x) - G(l)}{G(b-) - G(l)}, \qquad x \in I \cap [l, \infty).$$

If $b \in I$, then we have $G(b-) = G(b)$ by continuity of G on I. If $b \notin I$, then $G(b-)$ still exists, and is finite since G is strictly increasing and negative on I.

Proof. We shall start by proving the first identity in (i). Define the F–linear function

$$L(x) \triangleq \frac{V(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)}, \qquad x \in I.$$

Since $l, r \in \mathbf{\Gamma}$, we have $V(l) = h(l)$ and $V(r) = h(r)$. Thus L is same as the right–hand side of the the first expression in (i). To prove the identity, we therefore need to show $L = \frac{V}{\varphi}$ on $[l, r]$.

Since $\frac{V}{\varphi}$ is F–concave, Proposition A.3 immediately implies $L \leq \frac{V}{\varphi}$ on $[l, r]$. In order to show the reverse inequality, remember first that $\frac{V}{\varphi}$ is the *smallest* nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. It is therefore enough to show that L is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I.

Proposition A.3 implies that $L \geq \frac{V}{\varphi} \geq 0$ on $I \backslash [l, r]$ since V is nonnegative. However L is also nonnegative on $[l, r]$ since both $\frac{V(l)}{\varphi(l)}$ and $\frac{V(r)}{\varphi(r)}$ are nonnegative. Since it is F–linear, L is F–concave on I. It remains to show that L majorizes $\frac{h}{\varphi}$ on I.

Assume on the contrary that $L < \frac{h}{\varphi}$ somewhere in I. Therefore

$$\theta \triangleq \sup_{x \in I} \left[\frac{h(x)}{\varphi(x)} - L(x) \right] > 0. \tag{B.9}$$

Proposition A.3 in fact implies that $L \geq \frac{V}{\varphi} \geq \frac{h}{\varphi}$ on $I \backslash [l, r]$. On the other hand, $\frac{h}{\varphi} - L$ is continuous. Therefore "$\sup_{x \in I}$" in (B.9) can be replaced with "$\max_{x \in [l, r]}$". This implies that $\theta > 0$ is finite, and is attained in $[l, r]$.

Because θ is finite, $\theta + L$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Therefore $\theta + L \geq \frac{V}{\varphi}$ on I. Let $x_0 \in (l, r)$ be where θ is attained (Since $l, r \in \mathbf{\Gamma}$, $\frac{h}{\varphi}(l) - L(l) = \frac{h}{\varphi}(r) - L(r) = 0$. Because $\theta > 0$, we must therefore have $x_0 \notin \{l, r\}$). Thus

$$\frac{h(x_0)}{\varphi(x_0)} = \theta + L(x_0) \geq \frac{V(x_0)}{\varphi(x_0)} \geq \frac{h(x_0)}{\varphi(x_0)}.$$

Hence $h(x_0) = V(x_0)$ or $x_0 \in \Gamma$. However $x_0 \in (l, r) \cap \Gamma \subseteq \mathbf{C} \cap \Gamma = \varnothing$. Contradiction. Therefore L is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Therefore $L \geq \frac{V}{\varphi}$ on $[l, r]$. Together with the reserve inequality on $[l, r]$ proved above, this completes the proof of the first identity in (i). Second identity can be proven similarly. Alternatively, it also follows from the first identity after some simple algebra using definitions of F and G.

We shall next prove (ii). The proof is easier if $a \in I$. To deal with the case $a \notin I$, especially when $a = -\infty$, we need to specialize. However, same approach also solves the case $a \in I$. Therefore, we would like to prove all cases at once despite of the expense of some little nuisance. Define this time the F–linear function L as in

$$L(x) \triangleq \ell_a \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(a+)}{F(r) - F(a+)}, \qquad x \in I.$$

Since $r \in \Gamma$, we have $V(r) = h(r)$. Therefore, the right–hand side of the expression in (ii) coincides with L on $I \cap (-\infty, r]$. Thus, our aim is to prove $L = \frac{V}{\varphi}$ on $I \cap (-\infty, r]$.

It follows from (B.5) and F–concavity of $\frac{V}{\varphi}$ that

$$
\begin{aligned}
L(x) &= \ell_a \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(a+)}{F(r) - F(a+)} \\
&\leq \left(\lim_{l \downarrow a} \frac{V(l)}{\varphi(l)} \right) \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(a+)}{F(r) - F(a+)} \\
&= \lim_{l \downarrow a} \left[\frac{V(l)}{\varphi(l)} \cdot \frac{F(r) - F(x)}{F(r) - F(l)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(l)}{F(r) - F(l)} \right] \leq \frac{V(x)}{\varphi(x)}, \qquad x \in (a, r).
\end{aligned}
$$

If $a \in I$, then $L(a) = \ell_a = \frac{h^+(a)}{\varphi(a)} \leq \frac{V(a)}{\varphi(a)}$ by continuity of h on I. Hence $\frac{V}{\varphi} \geq L$ on $I \cap (-\infty, r]$.

We need to prove the reverse inequality. We shall do this by showing that L is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Let $(a_n)_{n \geq 1} \subset (a, r)$ be a strictly decreasing sequence with limit a. Introduce

$$L_n(x) \triangleq \frac{V(a_n)}{\varphi(a_n)} \cdot \frac{F(r) - F(x)}{F(r) - F(a_n)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(a_n)}{F(r) - F(a_n)}, \qquad x \in I, \ n \geq 1,$$

and define $L_0(x) \triangleq \lim_{n \to +\infty} L_n(x)$, $x \in I$ (L_0 is well-defined because of (B.8)).

Hence

$$L_0(x) = \left(\lim_{y \downarrow a} \frac{V(y)}{\varphi(y)}\right) \cdot \frac{F(r) - F(x)}{F(r) - F(a+)} + \frac{V(r)}{\varphi(r)} \cdot \frac{F(x) - F(a+)}{F(r) - F(a+)}, \qquad x \in I,$$

Proposition A.4 shows that $(L_n(x))_{n \geq 1}$ is decreasing at every $x \in I \cap (-\infty, r]$, and increasing at every $x \in I \cap [r, \infty)$.

Since Proposition A.3 implies that $L_n \geq \frac{V}{\varphi}$ on $I \cap [r, \infty)$ for every $n \geq 1$, we have $L_0 \geq \frac{V}{\varphi}$ on $I \cap [r, \infty)$. On the other hand for every $x \in (a, r)$ there exists some $N = N(x) > 0$ such that for every $n \geq N$, we have $x \in (a_n, r)$, i.e. $L_0(x) \leq L_n(x) \leq \frac{V(x)}{\varphi(x)}$ by Proposition A.3.

Since $\lim_{y \downarrow a} \frac{V(x)}{\varphi(x)} \geq \ell_a$ by (B.8), it is obvious that $L \leq L_0$ on $I \cap (-\infty, r]$, and $L \geq L_0$ on $I \cap [r, \infty)$. Therefore $L \leq L_0 \leq \frac{V}{\varphi}$ on (a, r), and $L \geq L_0 \geq \frac{V}{\varphi}$ on $I \cap [r, \infty)$ following the discussion in the previous paragraph. Using these properties of L, we shall now show that L is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I.

L is nonnegative on $I \cap [r, +\infty)$ since V is nonnegative and $L \geq \frac{V}{\varphi}$ on $I \cap [r, +\infty)$. On the other hand, because $\frac{V(r)}{\varphi(r)} \geq 0$ and $\ell_a \geq 0$, L is obviously nonnegative on $I \cap (-\infty, r]$.

Since L is F–linear, it is obviously F–concave on I. It remains to prove that L majorizes $\frac{h}{\varphi}$ on I. Assume on the contrary that $\frac{h}{\varphi} > L$ somewhere in I. Then

$$\theta \triangleq \sup_{x \in I} \left[\frac{h(x)}{\varphi(x)} - L(x)\right] = \sup_{x \in I \cap (-\infty, r]} \left[\frac{h(x)}{\varphi(x)} - L(x)\right] > 0.$$

We have the equality above since $L \geq \frac{V}{\varphi} \geq \frac{h}{\varphi}$ on $I \cap [r, +\infty)$. Since $\limsup_{x \downarrow a} \frac{h(x)}{\varphi(x)} \leq \limsup_{x \downarrow a} \frac{h^+(x)}{\varphi(x)} = \ell_a < \infty$, there exists some $\widetilde{r} \in (a, r)$ such that $\frac{h}{\varphi} < 1 + \ell_b$ in $I \cap (-\infty, \widetilde{r})$. Because $\frac{h}{\varphi}$ is continuous in $I \supset [\widetilde{r}, r]$, it is bounded on $[\widetilde{r}, r]$. Hence $\frac{h}{\varphi}$ is bounded on $I \cap (-\infty, r]$. Since furthermore $L \geq 0$, it is easy to see that θ is finite. Therefore $\theta + L$ is a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Hence $\theta + L \geq \frac{V}{\varphi}$ on I.

We claim that θ is attained in (a, r). Since $\limsup_{x \downarrow a} \frac{h(x)}{\varphi(x)} - L(x) \leq \limsup_{x \downarrow a} \frac{h^+(x)}{\varphi(x)} -$

$\ell_a = 0$, there exists some $l \in (a, r)$ such that

$$\frac{h(x)}{\varphi(x)} - L(x) \leq \frac{\theta}{2}, \qquad \forall\, x \in I \cap (-\infty, l).$$

Thus $\theta = \sup_{x \in [l,r]} \left[\frac{h(x)}{\varphi(x)} - L(x) \right]$. Since $\frac{h}{\varphi} - L$ is continuous in $I \supseteq [l, r]$, θ is attained in $[l, r]$. In fact, because $r \in \Gamma$, we have $\frac{h(r)}{\varphi(r)} - L(r) = 0 < \theta$, i.e. θ must be attained at some $x_0 \in [l, r) \subseteq I \cap (-\infty, r)$. However

$$\frac{h(x_0)}{\varphi(x_0)} = \theta + L(x_0) \geq \frac{V(x_0)}{\varphi(x_0)} \geq \frac{h(x_0)}{\varphi(x_0)}.$$

Hence $h(x_0) = V(x_0)$ or $x_0 \in \Gamma$. This means $x_0 \in I \cap (-\infty, r) \cap \Gamma \subseteq \mathbf{C} \cap \Gamma = \varnothing$. Contradiction.

This proves that L is indeed a nonnegative F–concave majorant of $\frac{h}{\varphi}$ on I. Therefore $L \geq \frac{V}{\varphi}$. Together with the reverse inequality on $I \cap (-\infty, r]$ proved above, we conclude $L = \frac{V}{\varphi}$ on $I \cap (-\infty, r]$.

The proof of (iii) is very similar to that of (ii). We leave it to the reader. $\qquad\square$

Proposition B.2 shows that $\frac{V}{\psi}$ is G–linear, and $\frac{V}{\varphi}$ is F–linear over the "continuation region" $\mathbf{C}$.

Corollary B.1. *V is continuous on I. Moreover*

$$\lim_{x \to a} \frac{V(x)}{\varphi(x)} = \ell_a \quad and \quad \lim_{x \to b} \frac{V(x)}{\psi(x)} = \ell_b.$$

Proof. F–concavity of $\frac{V}{\varphi}$ and continuity of F on I imply that V is continuous in the interior of I (Proposition A.1). We need to show that V is continuous at the boundaries if they are contained in I.

Suppose $a \in I$. By Proposition B.2 there are three possibilities: (1) $a \in \Gamma$ and there exists some $r \in \Gamma$ such that $(a, r) \subseteq \mathbf{C}$, or there exists some $\varepsilon > 0$ such that (2) $[a, \varepsilon) \subseteq \mathbf{C}$, or (3) $[a, \varepsilon) \subseteq \Gamma$. Proposition B.2 identifies V explicitly in each case. Note that V is continuous at a in all of the cases (Continuity of V follow in cases (1)

and (2) from (i) and (ii) of Proposition B.2 whereas in case (3) it follows from the continuity of h at a).

The results about the limits can also be proved by studying V case by case and by using Proposition B.2 in each case. Consider again a. If for some $r > a$, $I \cap (-\infty, r) \subseteq \mathbf{\Gamma}$, then $\frac{h}{\varphi} = \frac{V}{\varphi} \geq 0$ on $I \cap (-\infty, r)$. Therefore $\lim_{x \to a} \frac{V(x)}{\varphi(x)} = \lim_{x \to a} \frac{h(x)}{\varphi(x)} = \lim_{x \to a} \frac{h^+(x)}{\varphi(x)} = \ell_a$.

If $I \cap (-\infty, r) \subseteq \mathbf{C}$, then Proposition B.2(ii) immediately implies $\lim_{x \to a} \frac{V(x)}{\varphi(x)} = \ell_a$.

If $a \in I$, then there is one more possibility: Suppose $a \in \mathbf{\Gamma}$, and for some $r \in \mathbf{\Gamma}$, $r > a$, $(a, r) \subseteq \mathbf{C}$. By replacing every l in the first expression in (i) of Proposition B.2, we find

$$\frac{V(x)}{\varphi(x)} = \frac{h(a)}{\varphi(a)} \cdot \frac{F(r) - F(x)}{F(r) - F(a)} + \frac{h(r)}{\varphi(r)} \cdot \frac{F(x) - F(a)}{F(r) - F(a)}, \qquad x \in [a, r].$$

Thus

$$0 \leq \lim_{x \to a} \frac{V(x)}{\varphi(x)} = \frac{h(a)}{\varphi(a)} = \frac{h^+(a)}{\varphi(a)} = \lim_{x \to a} \frac{h^+(x)}{\varphi(x)} = \ell_a.$$

The second equality follows since h is nonnegative by the first equality. Third equality follows from continuity of h on $I \ni a$.

Similar arguments for b will give the proof for the other half of the Corollary. $\square$

Bibliography

[1] M. Broadie and J. Detemple. American capped call options on dividend–paying assets. *The Review of Financial Studies*, 8(1):161–191, 1995.

[2] E. B. Dynkin. The optimum choice of the instant of stopping a markov process. *Dokl. Acad. Nauk. SSSR*, 150:238–240, 1963.

[3] E. B. Dynkin and A. A. Yushkevich. *Markov Processes: Theorems and Problems*. Plenum Press, New York, 1969.

[4] X. Guo and L. Shepp. Some optimal stopping problems with nontrivial boundaries for pricing exotic options. *J. Appl. Probab.*, 38(3):647–658, 2001.

[5] K. Ito and H. P. McKean. *Diffusion Processes and their Sample Paths*. Classics in Mathematics. Springer–Verlag, New York, second edition, 1974.

[6] I. Karatzas and D. Ocone. A leavable bounded-velocity stochastic control problem. *Stochastic Processes and Applications*, 2002. To appear.

[7] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer Verlag, second edition, 1991.

[8] I. Karatzas and W. D. Sudderth. Control and stopping of a diffusion process on an interval. *Ann. Appl. Probab.*, 9(1):188–196, 1999.

[9] I. Karatzas and H. Wang. A barrier option of american type. *Appl. Math. Optim.*, 42:259–279, 2000.

[10] B. Oksendal. *Stochastic Differential Equations: An Introduction with Applications.* Universitext. Springer–Verlag, Berlin, fifth edition, 1998.

[11] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *A Series of Comprehensive Studies in Mathematics.* Springer–Verlag, third edition, 1999.

[12] P. Salminen. Optimal stopping of one–dimensional diffusions. *Mathematische Nachrichten*, 124:85–101, 1985.

[13] H. M. Taylor. Optimal stopping in a markov process. *The Annals of Mathematical Statistics*, 39(4):1333–1344, 1968.