WIENER DISORDER PROBLEM WITH OBSERVATIONS AT FIXED DISCRETE TIME EPOCHS

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Abstract. Suppose that a Wiener process gains a known drift rate at some unobservable disorder time with some zero-modified exponential distribution. The process is observed only at known fixed discrete time epochs, which may not always be spaced in equal distances. The problem is to detect the disorder time as quickly as possible by an alarm which depends only on the observations of Wiener process at those discrete time epochs. We show that Bayes optimal alarm times which minimize expected total cost of frequent false alarms and detection delay time always exist. Optimal alarms may in general sound between observation times and when the space-time process of the odds that disorder happened in the past hits a set with a nontrivial boundary. The optimal stopping boundary is piecewise-continuous and explodes as time approaches from left to each observation time. On each observation interval, if the boundary is not strictly increasing everywhere, then it firstly decreases and then increases. It is strictly monotone wherever it does not vanish. Its decreasing portion always coincides with some explicit function. We develop numerical algorithms to calculate nearly-optimal detection algorithms and their Bayes risks, and illustrate their use on numerical examples. The solution of Wiener disorder problem with discretely spaced observation times will help reduce risks and costs associated with disease outbreak and production quality control, where the observations are often collected and/or inspected periodically.

1. Introduction

In Shiryaev's (1963; 1978) classical Bayesian formulation of Wiener disorder problem, a Wiener process gains a constant nonzero known drift rate at some unknown unobserved random time with zero-modified exponential distribution. The objective is to detect the disorder time as soon as after it occurs by means of a stopping time of the *continuously monitored* Wiener process. The solution of Wiener disorder problem is important, because quickest detection of disease outbreak from the number of emergency room visits, machine failures from the measurements of incompliant finished products, sudden shifts in the riskiness and profitability of investment instruments can save lives, reduce maintenance and scrap costs, cut financial losses or enhance financial gains, respectively.

In this paper, we revisit Wiener disorder problem, but assume that Wiener process is observed only *at fixed known discrete time epochs*, which may be separated from each other with unequal distances. In disease outbreak monitoring and production quality control problems, the observations are typically gathered and inspected at the end of shifts, which may sometimes be spaced out in time at different distances from each other because of noon and night breaks, long weekends or national and religious holidays. Even though the observations are now being taken only at discrete time epochs, an alarm may be set at any time—at observation times or any time between observation

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times. Our goal is to solve the continuous-time Bayesian quickest detection problem while the information become available at discrete time epochs.

More precisely, suppose that a Wiener process $X = \{X_t;\ t \geq 0\}$ gains a known drift rate $\mu \neq 0$ at some unknown random time Θ, which either equals zero with some known probability $p \in [0,1)$ or has exponential distribution with some known mean $1/\lambda$ with probability $1-p$. The process X is observed at fixed known time epochs $0 = t_0 < t_1 < \ldots$, and we want to detect the disorder time Θ as quickly as possible, in the sense that the expected total cost of frequent false alarms and detection delay time is minimized, by setting alarm at some *real-valued* stopping time τ of the history $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of observations, where

$$\mathcal{F}_0 = \{\varnothing, \Omega\} \qquad \text{and} \qquad \mathcal{F}_t = \sigma\{X_{t_n};\ t_n \leq t,\ n \geq 0\} \quad \text{for every } t \geq 0. \tag{1.1}$$

We prove that a quickest detection rule always exists. We show that optimal alarms do not always sound at some observation times. One should therefore remain alert at all times for an alarm which may sound at some time strictly between two observations. We also describe how to calculate a nearly-optimal change detection rule.

Because the times between observations may in general be different, the Markov sufficient statistic for the quickest detection problem is the space-time process $\{(\Phi_t, t);\ t \geq 0\}$ of the conditional odds Φ_t at time t of that the disorder happened in the past given the past observations $\mathcal{F}_t$; see (2.2) for the precise definition. As shown in Appendix A.1, the conditional odds-ratio process can be calculated recursively by

$$\Phi_t = \begin{cases} \varphi(t - t_{n-1}, \Phi_{t_{n-1}}), & \text{if } t \in [t_{n-1}, t_n) \text{ for some } n \geq 1, \\ \jmath\Big(\Delta t_n, \Phi_{t_{n-1}}, \dfrac{\Delta X_n}{\sqrt{\Delta t_n}}\Big), & \text{if } t = t_n \text{ for some } n \geq 1, \end{cases} \tag{1.2}$$

where $\Delta t_\ell = t_\ell - t_{\ell-1}$ and $\Delta X_\ell = X_{t_\ell} - X_{t_{\ell-1}}$ for every $\ell \geq 1$, $\varphi(t,\phi) = e^{\lambda t}(\phi+1) - 1$ for every $t \geq 0$ and $\phi \geq 0$, and $\Delta t > 0,\ \phi \geq 0,\ z \in \mathbb{R}$

$$\jmath(\Delta t, \phi, z) = \exp\Big\{\mu z\sqrt{\Delta t} + \Big(\lambda - \frac{\mu^2}{2}\Big)\Delta t\Big\}\phi + \int_0^{\Delta t} \lambda \exp\Big\{\Big(\lambda + \frac{\mu z}{\sqrt{\Delta t}}\Big)u - \frac{\mu^2 u^2}{2\Delta t}\Big\}\mathrm{d}u.$$

If an alarm has not yet been raised until time $t \geq 0$, then an optimal alarm time

$$\sigma_0(t) = \inf\Big\{s \geq t;\ \sum_{n=0}^{\infty} 1_{[t_n, t_{n+1})}(s)\Phi_{t_n} \geq \phi_0(s)\Big\}, \quad t \geq 0$$

is the first time $s \geq t$, when the conditional odds-ratio Φ_{t_n} calculated at the last observation time t_n ($n \geq 0$ such that $t_n \leq s < t_{n+1}$) exceeds the optimal stopping boundary $\phi_0(s)$. For every $n \geq 0$, the optimal stopping boundary $\phi_0(s)$, $s \in [t_n, t_{n+1})$ between the nth and $(n+1)$st observation times is continuous and increases to infinity as $s \nearrow t_{n+1}$; see Figure 1 for a typical optimal stopping boundary. If the boundary is not strictly increasing, then it firstly decreases and then increases. It is strictly monotone wherever it does not vanish. Therefore, it is never optimal to stop as the next observation time nears. If the optimal stopping boundary is strictly increasing and it is not optimal to raise alarm at the last observation, then the same remains true at least until the next observation time. Otherwise an alarm may sound at some time strictly between the last and next observations. In Figure 1, if an alarm has not been raised before times t_1, t_3, or t_4, then optimal alarm may sound at some time strictly inside the intervals $[t_1, t_2)$, $[t_3, t_4)$, or $[t_4, t_5)$, respectively. We also show that

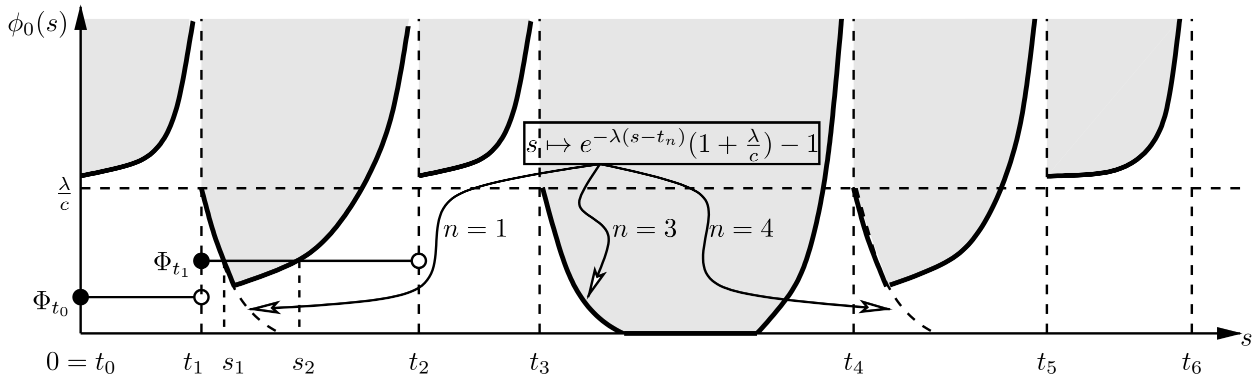


FIGURE 1. A typical optimal stopping boundary $s \mapsto \phi_0(s)$. Shaded is the optimal stopping region. Suppose that an alarm has not been raised before time $t \in [t_n, t_{n+1})$ for some $n \geq 0$. If $[t, t_{n+1}) \cap \{s \in [t_n, t_{n+1});\ \Phi_{t_n} \geq \phi_0(s)\}$ is not empty, then it is optimal to stop at the first time $s \in [t, t_{n+1})$ when $\Phi_{t_n} \geq \phi_0(s)$. Otherwise, it is optimal to wait at least until next observation time t_{n+1}. Suppose that Φ_{t_0} and Φ_{t_1} realized as on the figure. It is then optimal to stop at times s_1 and t, respectively, for every $t \in [0, s_1]$ and $t \in [s_1, s_2]$. If $t \in (s_2, t_2)$, then it is optimal to wait at least until time t_2 and act optimally after Φ_{t_2} is observed.

the strictly decreasing portion of $s \mapsto \phi_0(s)$ always coincides with $s \mapsto e^{-\lambda(s-t_n)}(1+\lambda/c)-1$, while the strictly increasing part has to be calculated numerically.

Continuous-time quickest change detection problems with discretely spaced observation times have recently started to receive attention. Brown and Zacks (2006) studied Bayesian formulation of detecting a change in the arrival of a Poisson process monitored at discrete time epochs, derived one- and two-step ahead stopping rules, and gave conditions under which those myopic stopping rules are optimal. Brown (2008) revisited the same problem, but also assumed that the the arrival rates before and after change are unknown, and developed one- and two-step look-ahead stopping rules, and illustrated their effectiveness on numerical examples. Sezer (2009) has recently solved Bayesian and variational formulations of Wiener disorder problem when the disorder is caused by one of the shocks, which arrive according to an observable Poisson process independent of the Wiener process. The classical Bayesian and variational formulations of Wiener disorder problem were given and solved by Shiryaev (1963; 1978). Wiener disorder problem with finite horizon was solved by Gapeev and Peskir (2006). Hadjiliadis (2005) and Hadjiliadis and Moustakides (2005) developed optimal and asymptotically optimal CUSUM rules for Wiener disorder problems with multiple alternatives. The optimality of the CUSUM algorithm was established under Lorden's criterion by Moustakides (1986) in discrete time and by Shiryaev (1996) and Beibel (1996) for the Wiener process. Asymptotic optimality of Shiryaev's procedure in continuous-time models were proved by Baron and Tartakovsky (2006). Quickest change detection problems were reviewed in the monographs of Basseville and Nikiforov (1993), Peskir and Shiryaev (2006), and Poor and Hadjiliadis (2009).

Let us also mention two important alternative formulations, the *variational formulation* and the *generalized Bayesian formulation* of the Wiener disorder problem with observations at fixed discrete time epochs. In the variational problem, one fixes the probability of false alarm and wants to minimize the expected detection delay cost. The Bayesian formulation in (2.1) can be seen as the Langrange relaxation of the variational formulation. Particularly, the Bayes optimal alarm time is optimal also for the variational formulation if the false alarm probability of the Bayes optimal alarm time exactly matches the requirement. We shall see later that the explicit characterization of the Bayes optimal alarm times allows one to easily calculate their false alarm probabilities, and by a straightforward search over a suitable grid of unit delay time cost c and the observation times $t_1 < t_2 < \ldots$, one can also solve the variational formulation in practice. For the classical Wiener disorder problem, the variational formulation and its solution by means of the Bayesian formulation were studied by Shiryaev (1963; 1978). As the required false alarm probability tends to zero and under some general conditions, Baron and Tartakovsky (2006) and Tartakovsky and Veeravalli (2004) established simple and explicit forms of optimal alarm times for both Bayesian and variational formulations of disorder problems in discrete and continuous times. In the future, we plan to investigate if the asymptotic analysis can be fruitfully extended to Wiener disorder problem with observations at fixed discrete time epochs.

In the generalized Bayesian formulation, instead of an exponentially distributed prior distribution, an uninformed prior distribution is assumed for the unknown and unobserved disorder time. The objective is to find a stopping time $\tau \in \mathcal{S}$ which minimizes

$$\int_0^\infty \mathbb{E}[(\tau - \Theta)^+ \mid \Theta = t]\mathrm{d}t - c\,\mathbb{E}[\tau \mid \Theta = \infty]$$

for some constant $c > 0$, or alternatively $\int_0^\infty \mathbb{E}[(\tau - \Theta)^+ \mid \Theta = t]\mathrm{d}t$ subject to the additional constraint $\mathbb{E}[\tau \mid \Theta = \infty] \geq \gamma$ for some prespecified $\gamma > 0$. Shiryaev (1963; 2002) and Feinberg and Shiryaev (2006) studied both formulations for the classical Wiener disorder problem, and we plan to investigate them for the case of discretely spaced observations in the future.

We conclude the introduction with an outline of the paper and its main results. In Section 2, we start by describing the problem, which is then expressed as an optimal stopping problem of the Markov sufficient statistic, space-time process $\{(\Phi_t, t);\, t \geq 0\}$ of conditional odds-ratio Φ. The process $\Phi = \{\Phi_t,\ t \geq 0\}$ is a continuous-time stochastic process with RCLL sample paths jumping only at deterministic observation times t_n, $n \geq 0$. Therefore, the solution of the optimal stopping problem depends on the explicit characterization of Theorem 3.2 of admissible stopping times, which is of independent interest and should also be useful for stochastic dynamic optimization problems in general. In Section 4, suitable dynamic programming operators are introduced, and the solution of optimal stopping problem is described at observation times. Theorem 4.3 shows how to construct ε-optimal stopping rules for every $\varepsilon \geq 0$ for the optimal stopping problems truncated at observation times, the value functions of which also coincide with successive approximations of the value function of the original infinite-horizon optimal stopping problem. Theorem 4.6 shows that successive approximations converge uniformly at known exponential rates, which are used for efficient numerical solution methods described later in Section 7. Between the observation times, the solution of the optimal stopping problem turns out to depend on nontrivial optimal stopping boundaries, the existence and properties of which are established in Sections 5 and 6, respectively. Theorem 5.1 describes the explicit construction of ε-optimal stopping times for every $\varepsilon \geq 0$. Theorems 5.4 and 5.7 respectively present for truncated and infinite-horizon problems alternative ε-optimal stopping times which can be characterized as the first hitting times of the space-time processes to suitable sets, whose nontrivial boundaries are characterized explicitly by Theorem 6.10. A numerical algorithm to calculate ε-optimal stopping rules is described in Figure 3 and illustrated on examples in Section 7. Section 8 describes how the false alarm probabilities of Bayes optimal alarm times can be accurately calculated. The relation between variational and Bayesian formulations is revisited, and a practical solution for the variational formulation is described and then illustrated on an example. Long proofs are deferred to the appendix.

2. Problem description

On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, suppose that $X = \{X_t;\ t \geq 0\}$ is a Wiener process whose zero drift changes to some known constant $\mu \neq 0$ at some unknown statistically independent time Θ, which has zero-modified exponential distribution $\mathbb{P}\{\Theta = 0\} = p$ and $\mathbb{P}\{\Theta > t\} = (1-p)e^{-\lambda t}$ for every $t \geq 0$ for some known constants $p \in [0,1)$ and $\lambda > 0$.

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n < \ldots$ be an infinite sequence of fixed real numbers, along which the process X may be observed as long as it is desired before an alarm τ is raised to declare that the drift of process X has changed. For each stopping rule τ of the history $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ in (1.1) of observations, we define its Bayes risk as the sum $R_\tau(p) = \mathbb{P}\{\tau < \Theta\} + c\,\mathbb{E}[(\tau - \Theta)^+]$ $p \in [0,1)$ of false alarm probability $\mathbb{P}\{\tau < \Theta\}$ and the expected detection delay penalty $c\,\mathbb{E}[(\tau - \Theta)^+]$. The problem is (i) to calculate the minimum Bayes risk

$$R(p) := \inf_{\tau \in \mathcal{S}} R_\tau(p), \quad p \in [0,1), \tag{2.1}$$

where the infimum is taken over the collection $\mathcal{S}$ of all stopping times of the filtration $\mathbb{F}$, and (ii) to find a stopping time in $\mathcal{S}$ which attains the infimum, if such a stopping time exists. If we define

$$L_t(u, x_0, x_1, \ldots) = \prod_{\ell \geq 1:\, t_\ell \leq t} \frac{1}{\sqrt{2\pi(t_\ell - t_{\ell-1})}} \exp\left\{ \frac{[x_\ell - x_{\ell-1} - \mu(t_\ell - (t_{\ell-1} \vee u))^+]^2}{2(t_\ell - t_{\ell-1})} \right\}, \quad u \geq 0,\ t \geq 0,$$

then we have $\mathbb{P}\left\{X_{t_\ell} \in \mathrm{d}x_\ell \text{ for every } \ell \geq 1 \text{ and } t_\ell \leq t \mid \Theta\right\} = L_t(\Theta, x_0, x_1, \ldots) \prod_{\ell \geq 1:\, t_\ell \leq t} \mathrm{d}x_\ell$ for every $t \geq 0$, and the conditional likelihood of the observations $X_{t_0}, X_{t_1}, \ldots$ given $\Theta = u$ is

$$\begin{aligned} L_t(u) &:= L_t(u, X_{t_0}, X_{t_1}, \ldots) \\ &= \prod_{\ell \geq 1:\, t_\ell \leq t} \frac{1}{\sqrt{2\pi(t_\ell - t_{\ell-1})}} \exp\left\{ \frac{[X_{t_\ell} - X_{t_{\ell-1}} - \mu(t_\ell - (t_{\ell-1} \vee u))^+]^2}{2(t_\ell - t_{\ell-1})} \right\}, \quad u \geq 0,\ t \geq 0. \end{aligned}$$

Model. Let $(\Omega, \mathcal{F}, \mathbb{P}_\infty)$ be a probability space hosting a random variable Θ with zero-modified exponential distribution $\mathbb{P}_\infty\{\Theta = 0\} = p$ and $\mathbb{P}_\infty\{\Theta > t\} = (1-p)e^{-\lambda t}$ for every $t \geq 0$, and an independent Wiener process X. Therefore, $\mathbb{P}_\infty\left\{X_{t_\ell} \in \mathrm{d}x_\ell \text{ for every } \ell \geq 1 \text{ and } t_\ell \leq t \mid \Theta\right\}$ equals

$$L_t(\infty, x_0, x_1, \ldots) \prod_{\ell \geq 1:\, t_\ell \leq t} \mathrm{d}x_\ell = \prod_{\ell \geq 1:\, t_\ell \leq t} \frac{1}{\sqrt{2\pi(t_\ell - t_{\ell-1})}} \exp\left\{ \frac{[x_\ell - x_{\ell-1}]^2}{2(t_\ell - t_{\ell-1})} \right\} \mathrm{d}x_\ell \quad \text{for all } t \geq 0.$$

Let $\mathbb{F}$ be the filtration in (1.1) obtained by observing process X at fixed times $0 = t_0 < t_1 < t_2 < \ldots$, and denote by $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ the augmentation of the filtration $\mathbb{F}$ by the information about Θ; i.e., $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\Theta)$ for every $t \geq 0$, and define $\mathbb{P}$ on $\mathcal{G}_\infty$ locally along the filtration $\mathbb{G}$ by means of

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_\infty}\bigg|_{\mathcal{G}_t} = Z_t(\Theta) := \frac{L_t(\Theta)}{L_t(\infty)}$$
$$= \exp\left\{\sum_{\ell=1}^{\infty} 1_{\{t_\ell \le t\}} \left[\frac{(X_{t_\ell} - X_{t_{\ell-1}})\mu[t_\ell - (\Theta \vee t_{\ell-1})]^+}{t_\ell - t_{\ell-1}} - \frac{\mu^2([t_\ell - (\Theta \vee t_{\ell-1})]^+)^2}{2(t_\ell - t_{\ell-1})}\right]\right\}, \ t \ge 0.$$

Under $\mathbb{P}$, the random variables $X_{t_\ell} - X_{t_{\ell-1}}$, $\ell \ge 1$ are, given Θ, conditionally independent Gaussian random variables with mean $\mu[t_\ell - (\Theta \vee t_{\ell-1})]^+$ and variance $t_\ell - t_{\ell-1}$ for every $\ell \ge 1$. Because $Z_0(\Theta) = 1$, probability measures $\mathbb{P}$ and $\mathbb{P}_0$ are identical on $\mathcal{G}_0 = \sigma(\Theta)$, and $\mathbb{P}\{\Theta \in B\} = \mathbb{P}_\infty\{\Theta \in B\}$; therefore, Θ has also zero-modified exponential distribution with the same parameters p and λ under $\mathbb{P}$. Thus, $\mathbb{P}$ has the same properties as the probability measure in the description of the original problem. In the remainder, we will work with $\mathbb{P}$ constructed as above.

Let us define the conditional odds-ratio process

$$\Phi_t := \frac{\mathbb{P}\{\Theta \le t \mid \mathcal{F}_t\}}{\mathbb{P}\{\Theta > t \mid \mathcal{F}_t\}} = \frac{\mathbb{E}_\infty[Z_t(\Theta)1_{\{\Theta \le t\}} \mid \mathcal{F}_t]}{\mathbb{E}_\infty[Z_t(\Theta)1_{\{\Theta > t\}} \mid \mathcal{F}_t]} = \frac{\mathbb{E}_\infty[Z_t(\Theta)1_{\{\Theta \le t\}} \mid \mathcal{F}_t]}{(1-p)e^{-\lambda t}}, \quad t \ge 0, \tag{2.2}$$

where the second equality follows from Bayes theorem and the third equality from

$$\mathbb{E}_\infty[Z_t(\Theta)1_{\{\Theta > t\}} \mid \mathcal{F}_t] = \mathbb{P}_\infty\{\Theta > t \mid \mathcal{F}_t\} = \mathbb{P}_\infty\{\Theta > t\} = (1-p)e^{-\lambda t}, \ t \ge 0, \tag{2.3}$$

because, on the event $\{\Theta > t\}$, we have $[t_\ell - (\Theta \vee t_{\ell-1})]^+ = (t_\ell - \Theta)^+ = 0$ for every $\ell \ge 1$ and $t_\ell < t$, and therefore,

$$Z_t(\Theta)1_{\{\Theta > t\}} = 1_{\{\Theta > t\}} \qquad \mathbb{P}_\infty\text{-almost surely.} \tag{2.4}$$

In the appendix, we prove that the conditional odds-ratio process $\Phi = \{\Phi_t;\ t \ge 0\}$ has the dynamics (1.2). Because for every $n \ge 1$ and $t_{n-1} \le s < t_n$, we have $\mathcal{F}_s \equiv \mathcal{F}_{t_{n-1}} = \sigma\{X_{t_1}, \ldots, X_{t_n-1}\}$, and $\Delta X_n = X_{t_n} - X_{t_{n-1}}$ is independent of $\mathcal{F}_{t_{n-1}}$ under $\mathbb{P}_\infty$, the dynamics in (1.2) ensure that $\mathbb{E}_\infty[f(\Phi_t, t) \mid \mathcal{F}_s] = \mathbb{E}_\infty[f(\Phi_t, t) \mid \mathcal{F}_{t_{n-1}}] = \mathbb{E}_\infty[f(\Phi_t, t) \mid \Phi_{t_{n-1}}, t_{n-1}] = \mathbb{E}_\infty[f(\Phi_t, t) \mid \Phi_s, s]$ for every $t > s$ and bounded Borel measurable function $f : [0,\infty)\times\mathbb{R} \mapsto \mathbb{R}$, and the process $\{(\Phi_t, t), \mathcal{F}_t;\ t \ge 0\}$ is a (piecewise-deterministic strong) Markov process under $\mathbb{P}_\infty$. Proposition 2.1 below shows that the sequential detection problem reduces to a discounted optimal stopping problem with running cost $\phi \mapsto \phi - \lambda/c$ for the conditional odds-ratio process Φ.

Proposition 2.1. *The Bayes risk equals* $R_\tau(p) = 1 - p + (1-p)c\,\mathbb{E}_\infty\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right)\mathrm{d}t\right]$ *for every* $p \in [0,1)$ *and* $\tau \in \mathcal{S}$. *The minimum Bayes risk equals* $R(p) = 1 - p + (1-p)cV(p/(1-p))$ *for every* $p \in [0,1)$, *where* $V(\cdot)$ *is the value function of the optimal stopping problem*

$$V(\phi) = \inf_{\tau\in\mathcal{S}} \mathbb{E}_\infty^\phi\left[\int_0^\tau e^{-\lambda t}\left(\Phi_t - \frac{\lambda}{c}\right)\mathrm{d}t\right], \qquad \phi \ge 0 \tag{2.5}$$

for piecewise-deterministic strong Markov space-time process $\{(\Phi_t, t);\ t \ge 0\}$ *of conditional odds-ratio process* Φ, *and* $\mathbb{E}_\infty^\phi$ *is the expectation with respect to* $\mathbb{P}_\infty^\phi$, *which is* $\mathbb{P}_\infty$ *s.t.* $\Phi_0 = \phi$ *a.s.*

The proof is similar to that of Bayraktar et al.'s (2005) Proposition 2.1. In the remainder, we solve the optimal stopping problem in (2.5). The solution method reduces the continuous-time optimal stopping problem to a discrete-time optimal stopping problem by means of suitable single-jump operators, which take advantage of the special structure of admissible stopping times. The solution is presented in Sections 4 and 5 after jump operators are introduced. In the next section, we first characterize the stopping times in the collection $\mathcal{S}$.

3. The characterization of admissible stopping times

Recall that every admissible stopping time $\tau \in \mathcal{S}$ is a stopping time of observation filtration $\mathbb{F} = (F_t)_{t\geq 0}$ defined by (1.1). The main result of this section is Theorem 3.2 and implies that every stopping time $\tau \in \mathcal{S}$ is essentially a discrete random variable, and the original optimal stopping problem can essentially be solved in discrete time. Let $\mathcal{F}_\tau = \{A \in \mathcal{F};\ A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for every } t \geq 0\}$ and $\mathcal{H}_\tau := \sigma\left(X_{t_k} 1_{\{t_k \leq \tau\}}, 1_{\{t_k > \tau\}};\ k \geq 0\right)$ generated by those observations $X_{t_0}, X_{t_1}, \ldots$ before time τ.

Proposition 3.1. *We have $\mathcal{F}_\tau = \mathcal{H}_\tau$ for every $\tau \in \mathcal{S}$.*

Proof. ($\supseteq$) Clear. ($\subseteq$) Fix any $A \in \mathcal{F}_\tau$ and write $1_A = \sum_{k=0}^\infty 1_A 1_{\{t_k \leq \tau < t_{k+1}\}} + 1_A 1_{\{\tau = +\infty\}}$. For every $k \geq 0$, $A \cap \{t_k \leq \tau < t_{k+1}\} = \{t_k \leq \tau\} \cap \bigcup_{n=1}^\infty \left[A \cap \left\{\tau \leq t_{k+1} - \frac{1}{n}\right\}\right]$ belongs to $\mathcal{F}_{t_k}$ because $\{t_k \leq \tau\} \in \mathcal{F}_{t_k}$ and $A \cap \left\{\tau \leq t_{k+1} - \frac{1}{n}\right\} \in \mathcal{F}_{t_{k+1}-1/n} = \mathcal{F}_{t_k}$. Then there is a Borel function $f_k : \mathbb{R}_+^{k+1} \mapsto \{0, 1\}$ such that $1_A 1_{\{t_k \leq \tau < t_{k+1}\}} = f_k(X_{t_0}, \ldots, X_{t_k}) 1_{\{t_k \leq \tau < t_{k+1}\}}$ for every $k \geq 0$, which is $\mathcal{H}_\tau$-mble because $f_k(X_{t_0}, \ldots, X_{t_k}) 1_{\{t_k \leq \tau < t_{k+1}\}} = f_k(X_{t_0} 1_{\{t_0 \leq \tau\}}, \ldots, X_{t_k} 1_{\{t_k \leq \tau\}}) 1_{\{t_k \leq \tau\}} 1_{\{t_{k+1} > \tau\}}$ is measurable with respect to $\sigma\left(X_{t_\ell} 1_{\{t_\ell \leq \tau\}}, 1_{\{t_\ell > \tau\}};\ \ell = 0, 1, \ldots, k+1\right) \subseteq \mathcal{H}_\tau$. Similarly, $1_A 1_{\{\tau = +\infty\}} \in \mathcal{H}_\tau$. ☐

Theorem 3.2. *Let τ be an $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$-stopping time. Then there is a nonnegative $\mathcal{F}_{t_n}$-measurable random variable R_n for every $n \geq 0$ such that*

$$(i) \qquad \tau 1_{\{t_n \leq \tau < t_{n+1}\}} = (t_n + R_n) 1_{\{t_n \leq \tau < t_{n+1}\}},$$

$$(ii) \qquad (\tau \wedge t_{n+1}) 1_{\{t_n \leq \tau\}} = [(t_n + R_n) \wedge t_{n+1}] 1_{\{t_n \leq \tau\}},$$

$$(iii) \qquad \{\tau \geq t_{n+1}\} = \{R_0 \geq t_1, t_1 + R_1 \geq t_2, \ldots, t_n + R_n \geq t_{n+1}\},$$

$$(iv) \qquad \{t_n \leq \tau < t_{n+1}\} = \{R_0 \geq t_1, t_1 + R_1 \geq t_2, \ldots, t_{n-1} + R_{n-1} \geq t_n, t_n + R_n < t_{n+1}\}.$$

Let $N := \inf\{n \geq 0;\ t_n + R_n < t_{n+1}\}$. Then N is an $(\mathcal{F}_{t_n})_{n\geq 0}$-stopping time, and

$$(v) \qquad \tau = (t_N + R_N) 1_{\{N < \infty\}} + \infty \cdot 1_{\{N = +\infty\}}.$$

Proof. Let τ be an $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Since $\tau \in \mathcal{F}_\tau = \mathcal{H}_\tau$, there is a Borel function f such that $\tau = f(X_{t_0} 1_{\{t_0 \leq \tau\}}, 1_{\{t_0 > \tau\}}, X_{t_1} 1_{\{t_1 \leq \tau\}}, 1_{\{t_1 > \tau\}}, \ldots, X_{t_n} 1_{\{t_n \leq \tau\}}, 1_{\{t_n > \tau\}}, \ldots)$. For all $n \geq 0$, $\tau 1_{\{t_n \leq \tau < t_{n+1}\}} = f(X_{t_0}, 0, X_{t_1}, 0, \ldots, X_{t_n}, 0, 0, 1, 0, 1 \ldots) 1_{\{t_n \leq \tau < t_{n+1}\}} = [t_n + R_n] 1_{\{t_n \leq \tau < t_{n+1}\}}$ and

$$(\tau \wedge t_{n+1}) 1_{\{t_n \leq \tau\}} = \tau 1_{\{t_n \leq \tau < t_{n+1}\}} + t_{n+1} 1_{\{\tau \geq t_{n+1}\}} = [(t_n + R_n) \wedge t_{n+1}] 1_{\{t_n \leq \tau\}}$$

in terms of $\mathcal{F}_{t_n}$-mble $R_n := [f(X_{t_0}, 0, X_{t_1}, 0, \ldots, X_{t_n}, 0, 0, 1, 0, 1 \ldots) - t_n] 1_{\{t_n \leq \tau < t_{n+1}\}} + \infty \cdot 1_{\{\tau \geq t_{n+1}\}}$. Then $\tau 1_{\{t_n \leq \tau < t_{n+1}\}} = (t_n + R_n) 1_{\{t_n \leq \tau < t_{n+1}\}}$, and (i) and (ii) follow. By (i), $\{t_n \leq \tau < t_{n+1}\} = \{t_n \leq \tau < t_{n+1}, \tau = t_n + R_n\} \subseteq \{t_n \leq \tau\} \cap \{t_n + R_n < t_{n+1}\}$, and since $R_n \geq t_{n+1} - t_n$ on $\{\tau \geq t_{n+1}\}$, we have the converse inclusion $\{t_n \leq \tau\} \cap \{t_n + R_n < t_{n+1}\} = \{t_n \leq \tau\} \cap \{t_n + R_n < t_{n+1}\} \cap \{\tau < t_{n+1}\} \subseteq \{t_n \leq \tau\} \cap \{\tau < t_{n+1}\} \equiv \{t_n \leq \tau < t_{n+1}\}$. Hence, $\{t_n \leq \tau < t_{n+1}\} = \{t_n \leq \tau\} \cap \{t_n + R_n < t_{n+1}\}$, which proves the first equality in (iv). As a consequence, $\{\tau < t_1\} = \{t_0 \leq \tau < t_1\} = \{t_0 \leq \tau\} \cap \{R_0 < t_1\} = \{R_0 < t_1\}$. Therefore, $\{\tau \geq t_1\} = \{R_0 \geq t_1\}$, and (iii) holds for $n = 0$. Suppose that (iii) holds for some $n \geq 0$. Then by the first equality of (iv) (after n is replaced with $n + 1$)

$$\begin{aligned}\{\tau \geq t_{n+2}\} &= \{\tau \geq t_{n+1}\} \setminus \{\tau \geq t_{n+1}, \tau < t_{n+2}\} = \{\tau \geq t_{n+1}\} \setminus \{\tau \geq t_{n+1}, t_{n+1} + R_{n+1} < t_{n+2}\} \\ &= \{\tau \geq t_{n+1}\} \cap \{t_{n+1} + R_{n+1} \geq t_{n+2}\} = \{R_0 \geq t_1, \ldots, t_n + R_n \geq t_{n+1}, t_{n+1} + R_{n+1} \geq t_{n+2}\},\end{aligned}$$

which proves (iii). The first equality in (iv) and (iii) give that $\{t_n \le \tau < t_{n+1}\} = \{\tau \ge t_n\} \cap \{t_n + R_n < t_{n+1}\} = \{R_0 \ge t_1, \ldots, t_{n-1} + R_{n-1} \ge t_n\} \cap \{t_n + R_n < t_{n+1}\}$, which proves (iv).

Since $R_n \in \mathcal{F}_{t_n}$ for all $n \ge 0$, $N = \inf\{n \ge 0;\ t_n + R_n < t_{n+1}\}$ is an $(\mathcal{F}_{t_n})_{n\ge 0}$-stopping time, and $\{N = n\} = \{R_0 \ge t_1, \ldots, t_{n-1} + R_{n-1} \ge t_n, t_n + R_n < t_{n+1}\} = \{t_n \le \tau < t_{n+1}\}$ and $\{N = +\infty\} = \{R_0 \ge t_1, t_1 + R_1 \ge t_2, \ldots\} = \{\tau = +\infty\}$ by (iv), which imply $\tau = \sum_{n=0}^{\infty} \tau 1_{\{t_n \le \tau < t_{n+1}\}} + \tau 1_{\{\tau=\infty\}} = \sum_{n=0}^{\infty} (t_n + R_n) 1_{\{N=n\}} + \infty \cdot 1_{\{N=\infty\}} = (t_N + R_N) 1_{\{N<\infty\}} + \infty \cdot 1_{\{N=\infty\}}$ by (i). This proves (v). ☐

The next proposition shows that (v) of Theorem 3.2 also has a converse.

Proposition 3.3. *For each $n \ge 0$, let R_n be an a.s. nonnegative $\mathcal{F}_{t_n}$-mble r.v. Define $N := \inf\{n \ge 0;\ t_n + R_n < t_{n+1}\}$ and $\tau := (t_N + R_N) 1_{\{N<\infty\}} + \infty \cdot 1_{\{N=+\infty\}}$. Then τ is a $(\mathcal{F}_t)_{t\ge 0}$-stopping time.*

Proof. Fix $t \ge 0$. Then $t_m \le t < t_{m+1}$ for some $m \ge 0$. Since $R_n \in \mathcal{F}_{t_n}$ for $n \ge 0$, $\{\tau \le t\} = \{N < \infty, t_N + R_N \le t\} = \bigcup_{n=0}^{m-1} \{t_0 + R_0 \ge t_1, \ldots, t_{n-1} + R_{n-1} \ge t_n, t_n + R_n \le t_{n+1}\} \bigcup \{t_0 + R_0 \ge t_1, \ldots, t_{m-1} + R_{m-1} \ge t_m, t_m + R_m \le t\} \in \mathcal{F}_{t_m} \equiv \mathcal{F}_t$, and τ is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time. ☐

4. The solution at observation times

Let $\varphi(\cdot,\cdot)$ and $\jmath(\cdot,\cdot,\cdot)$ be as in (1.2) and define for every bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$ operators

$$(J_y w)(\Delta t, \phi) := \inf_{r \ge y} (Jw)(\Delta t, \phi, y, r), \quad \Delta t > 0,\ \phi \ge 0,\ 0 \le y \le \Delta t, \tag{4.1}$$

$$(Jw)(\Delta t, \phi, y, r) := \int_y^{r\wedge\Delta t} e^{-\lambda t} \left(\varphi(t, \phi) - \frac{\lambda}{c}\right) \mathrm{d}t + 1_{[\Delta t, \infty)}(r) e^{-\lambda \Delta t} (Kw)(\Delta t, \phi),\ r \ge y, \tag{4.2}$$

$$(Kw)(\Delta t, \phi) := \int_{-\infty}^{\infty} w(\jmath(\Delta t, \phi, z)) \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}} \mathrm{d}z. \tag{4.3}$$

Let us pretend that we have not raised an alarm until t_n. Suppose also that we are told the value $w(\phi)$ of the optimal policy if Φ has not been stopped until time t_{n+1} and equals ϕ at time t_{n+1}. Given history $\mathcal{F}_{t_n}$ of observations until time t_n, we want to know if stopping before t_{n+1} or waiting at least until t_{n+1} is the best. If τ is an $(\mathcal{F}_t)_{t\ge 0}$-stopping time such that $\tau \ge t_n$ ($\mathbb{P}_\infty$-a.s.), then optimality principle suggests that the conditional expected total remaining cost given $\mathcal{F}_{t_n}$ equals

$$\mathbb{E}_\infty\Big[\int_{t_n}^{\tau\wedge t_{n+1}} e^{-\lambda(t-t_n)} \Big(\Phi_t - \frac{\lambda}{c}\Big) \mathrm{d}t + 1_{\{\tau \ge t_{n+1}\}} e^{-\lambda \Delta t_{n+1}} w(\Phi_{t_{n+1}}) \Big| \mathcal{F}_{t_n}\Big]$$

in time-t_n monetary units. On the one hand, by Theorem 3.2 (ii) and (iii), there is a nonnegative $\mathcal{F}_{t_n}$-mble r.v. R_n such that $\mathbb{P}_\infty$-a.s. $\tau \wedge t_{n+1} = (t_n + R_n) \wedge t_{n+1}$ and $\{\tau \ge t_{n+1}\} = \{t_n + R_n \ge t_{n+1}\}$, since $\tau \ge t_n$ ($\mathbb{P}_\infty$-a.s.). On the other hand, the dynamics in (1.2) of Φ imply $\Phi_t = \varphi(t - t_n, \Phi_{t_n})$ for every $t_n \le t < t_{n+1}$ and $\Phi_{t_{n+1}} = \jmath(\Delta t_{n+1}, \Phi_{t_n}, \frac{\Delta X_{n+1}}{\sqrt{\Delta t_{n+1}}})$. Therefore, the conditional expected total remaining cost given $\mathcal{F}_{t_n}$ can be rewritten as

$$\int_{t_n}^{(t_n+R_n)\wedge t_{n+1}} e^{-\lambda(t-t_n)} \Big(\varphi(t - t_n, \Phi_{t_n}) - \frac{\lambda}{c}\Big) \mathrm{d}t$$

$$
\begin{aligned}
&+ 1_{\{t_n+R_n \geq t_{n+1}\}} e^{-\lambda \Delta t_{n+1}} \mathbb{E}_\infty \Big[w\Big(\jmath\Big(\Delta t_{n+1}, \phi, \frac{\Delta X_{n+1}}{\sqrt{\Delta t_{n+1}}}\Big)\Big)\Big]\Big|_{\phi=\Phi_{t_n}} \\
&= \int_0^{R_n \wedge \Delta t_{n+1}} e^{-\lambda t}\Big(\varphi(t,\Phi_{t_n}) - \frac{\lambda}{c}\Big)\mathrm{d}t + 1_{[\Delta t_{n+1},\infty)}(R_n) e^{-\lambda \Delta t_{n+1}} (Kw)(\Delta t_{n+1}, \Phi_{t_n}) \\
&= (Jw)(\Delta t_{n+1}, \Phi_{t_n}, 0, R_n),
\end{aligned}
$$

because R_n and Φ_n are $\mathcal{F}_{t_n}$-measurable, and $\Delta X_{n+1}/\sqrt{\Delta t_{n+1}}$ has standard Gaussian distribution independent of $\mathcal{F}_{t_n} = \sigma(X_{t_0}, \ldots, X_{t_n})$ under $\mathbb{P}_\infty$. Thus, the minimum conditional expected total remaining cost given $\mathcal{F}_{t_n}$ is obtained by taking the infimum over the collection of all $(\mathcal{F}_t)_{t\geq 0}$-stopping times τ such that $\tau \geq t_n$ ($\mathbb{P}_\infty$-a.s.), or equivalently, over all $\mathcal{F}_{t_n}$-measurable nonnegative r.v.'s R_n:

$$
\begin{aligned}
&\operatorname*{ess\,inf}_{\tau\in\mathcal{S}:\tau\geq t_n \text{ a.s.}} \mathbb{E}_\infty\Big[\int_{t_n}^{\tau\wedge t_{n+1}} e^{-\lambda(t-t_n)}\Big(\Phi_t - \frac{\lambda}{c}\Big)\mathrm{d}t + 1_{\{\tau\geq t_{n+1}\}} e^{-\lambda\Delta t_{n+1}} w\Big(\Phi_{t_{n+1}}\Big)\Big|\mathcal{F}_{t_n}\Big] \\
&= \operatorname*{ess\,inf}_{0\leq R_n\in\mathcal{F}_{t_n}} (Jw)(\Delta t_{n+1}, \Phi_{t_n}, 0, R_n) = \Big[\inf_{r\geq 0} (Jw)(\Delta t_{n+1}, \phi, 0, r)\Big]\Big|_{\phi=\Phi_{t_n}} = (J_0 w)(\Delta t_{n+1}, \Phi_{t_n}).
\end{aligned}
$$

Thus, $(J_0 w)(\Delta t, \phi)$ can be thought as a dynamic programming operator (namely, J_0) applied to a continuation function $w(\cdot)$ in order to determine the best decision, based only on the currently available information ϕ, before Δt, at which time new information arrives.

Let us define optimal stopping problems

$$
\begin{aligned}
\gamma_n &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{S}_n} \mathbb{E}_\infty\Big[\int_{t_n}^{\tau} e^{-\lambda(t-t_n)}\Big(\Phi_t - \frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big], \\
\gamma_n^{(m)} &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{S}_n} \mathbb{E}_\infty\Big[\int_{t_n}^{\tau\wedge t_m} e^{-\lambda(t-t_n)}\Big(\Phi_t - \frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big]
\end{aligned} \tag{4.4}
$$

obtained from the original problem in (2.5) by allowing stopping only in $[t_n, \infty)$ and $[t_n, t_m]$, respectively, based on observation history $\mathcal{F}_{t_n}$ until time t_n for some $0 \leq n \leq m$, where

$$
\mathcal{S}_n := \{\tau \in \mathcal{S};\ \tau \geq t_n,\ \mathbb{P}_\infty\text{-a.s.}\}, \qquad n \geq 0 \quad (\mathcal{S}_0 \equiv \mathcal{S})
$$

is the collection of all $\mathbb{F}$-stopping times which are $\mathbb{P}_\infty$-a.s. greater than or equal to t_n, $n \geq 0$. By Proposition 4.2, for each $n \geq 0$, γ_n can be pathwise approximated well by the elements in the tail of the sequence $(\gamma_n^{(m)})_{m\geq n}$, and by Theorem 4.3 each $\gamma_n^{(m)}$ coincides $\mathbb{P}_\infty$-a.s. with $v_n^{(m)}(\Phi_{t_n})$, where

$$
\begin{aligned}
v_m^{(m)}(\phi) &= 0 && \text{for every } \phi \geq 0 \text{ and } m \geq 0, \\
v_n^{(m)}(\phi) &= \Big(J_0 v_{n+1}^{(m)}\Big)(\Delta t_{n+1}, \phi) && \text{for every } \phi \geq 0 \text{ and } 0 \leq n \leq m-1,
\end{aligned} \tag{4.5}
$$

and $(v_n^{(m)}(\Phi_{t_n}))_{m\geq n}$ gives pathwise a sequence of successive approximations to γ_n for every $n \geq 0$. For the proof of all of the major results in the remainder, we will need Lemma 4.1 about important properties of dynamic programming operator $J_\bullet$, and its proof is in the appendix.

Lemma 4.1. *For every $\Delta t > 0$ and $0 \leq y \leq \Delta t$, the followings are true.*

(i) *If $w(\cdot)$ is bounded and $w(\cdot) \geq -1/c$, then $-1/c \leq e^{\lambda y}(J_y w)(\Delta t, \cdot) \leq 0$. If $w(\cdot)$ is also nondecreasing, concave, and continuous, then so is $(J_y w)(\Delta t, \cdot)$, and there exists some finite $\phi(\Delta t, y)$ such that $(J_y w)(\Delta t, \phi) = 0$ for every $\phi \geq \phi(\Delta t, y)$.*

(ii) *If $w_1(\cdot)$ and $w_2(\cdot)$ are bounded and $w_1(\cdot) \leq w_2(\cdot)$, then $(J_y w_1)(\Delta t, \cdot) \leq (J_y w_2)(\Delta t, \cdot)$.*

(iii) *If $w_3(\cdot)$ and $w_4(\cdot)$ are bounded, then*

$$\sup_{\phi\geq 0}|(J_y w_3)(\Delta t,\phi)-(J_y w_4)(\Delta t,\phi)|\leq e^{-\lambda\Delta t}\sup_{\phi\geq 0}|w_3(\phi)-w_4(\phi)|.$$

(iv) *If $w(\cdot)$ is bounded and nonpositive, then for every $\Delta t>0$, $\phi\geq 0$, and $0\leq y\leq\Delta t$,*

$$y\mapsto (J_y w)(\Delta t,\phi)=\inf_{r\geq y}(Jw)(\Delta t,\phi,y,r)=\min_{r\in[y,\Delta t]}(Jw)(\Delta t,\phi,y,r) \tag{4.6}$$

is continuous, and infimum is attained since $r\mapsto (Jw)(\Delta t,\phi,y,r)$ is lower semi-continuous.

(v) *If for some $0\leq y_0<y_1\leq\Delta t$ and $\phi\geq 0$, we have $(J_y w)(\Delta t,\phi)<0$ for every $y_0\leq y\leq y_1$, then $(J_y w)(\Delta t,\phi)=\int_y^z e^{-\lambda u}(\varphi(u,\phi)-\frac{\lambda}{c})\mathrm{d}u+(J_z w)(\Delta t,\phi)$ $y_0\leq y\leq z\leq y_1$.*

Proposition 4.2. *For every fixed $n\geq 0$, the sequence $(\gamma_n^{(m)})_{m\geq n}$ converges $\mathbb{P}_\infty$-a.s. to γ_n as $m\to\infty$. More precisely, $\mathbb{P}_\infty$-a.s. $0\leq\gamma_n^{(m)}-\gamma_n\leq(1/c)\,e^{-\lambda(t_m-t_n)}$ for every $0\leq n\leq m$.*

Proof. Fix $0\leq n\leq m$. For all $\tau\in\mathcal{S}_n$, $\tau\wedge t_m\in\mathcal{S}_n$ and $\gamma_n\leq\mathbb{E}_\infty[\int_{t_n}^{\tau\wedge t_m}e^{-\lambda(t-t_n)}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t\mid\mathcal{F}_{t_n}]$. Then $\mathbb{P}_\infty$-a.s. $\gamma_n\leq\gamma_n^{(m)}$. But $\mathbb{E}_\infty^\phi[\int_{t_n}^{\tau}e^{-\lambda(t-t_n)}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_n}]\geq\mathbb{E}_\infty^\phi[\int_{t_n}^{\tau\wedge t_m}e^{-\lambda(t-t_n)}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_n}]-\frac{\lambda}{c}\int_{t_m}^\infty e^{-\lambda(t-t_n)}\mathrm{d}t\geq\gamma_n^{(m)}-\frac{1}{c}\,e^{-\lambda(t_m-t_n)}$. Taking the infimum over $\tau\in\mathcal{S}_n$ completes the proof. □

Theorem 4.3. *For every $0\leq n\leq m$, we have*

$$(i)\qquad \gamma_n^{(m)}=v_n^{(m)}(\Phi_{t_n}),\quad \mathbb{P}_\infty\text{-a.s.},$$

$$(ii)\qquad \nu_n^{(m)}:=\inf_{\tau\in\mathcal{S}_n}\mathbb{E}_\infty\Big[\int_{t_n}^{\tau\wedge t_m}e^{-\lambda(t-t_n)}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big]=\mathbb{E}_\infty\gamma_n^{(m)}.$$

For every $\varepsilon\geq 0$, let $R_{m,\varepsilon}^{(m)}\equiv 0$ and $R_{n,\varepsilon}^{(m)}\equiv R_{n,\varepsilon}^{(m)}(\Delta t_{n+1},\Phi_{t_n})$ be a nonnegative real number such that $(Jv_{n+1}^{(m)})(\Delta t_{n+1},\Phi_{t_n},0,R_{n,\varepsilon}^{(m)})\leq(J_0 v_{n+1}^{(m)})(\Delta t_{n+1},\Phi_{t_n})+\varepsilon$ for every $0\leq n\leq m-1$. Then for every $0\leq n\leq m$, $R_{n,\varepsilon}^{(m)}$ is a nonnegative $\mathcal{F}_{t_n}$-measurable random variable, and

$$\tau_{n,\varepsilon}^{(m)}:=\begin{Bmatrix} t_n+R_{n,\varepsilon/2}^{(m)}, & \textit{if}\quad R_{n,\varepsilon/2}^{(m)}<\Delta t_{n+1}\\ \tau_{n+1,\varepsilon/2}^{(m)}, & \textit{if}\quad R_{n,\varepsilon/2}^{(m)}\geq\Delta t_{n+1}\end{Bmatrix}\in\mathcal{S}_n$$

is ε-optimal in the sense that

$$(iii)\qquad \gamma_n^{(m)}+\varepsilon\geq\mathbb{E}_\infty\Big[\int_{t_n}^{\tau_{n,\varepsilon}^{(m)}\wedge t_m}e^{-\lambda(t-t_n)}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big],\quad \mathbb{P}_\infty\text{-a.s.},$$

$$(iv)\qquad \nu_n^{(m)}+\varepsilon\geq\mathbb{E}_\infty\Big[\int_{t_n}^{\tau_{n,\varepsilon}^{(m)}\wedge t_m}e^{-\lambda(t-t_n)}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big].$$

Proof of Theorem 4.3. Note that $\gamma_m^{(m)}=0$ ($\mathbb{P}_\infty$-a.s.), $v_m^{(m)}(\Phi_{t_m})=\nu_m^{(m)}=0$, and $\tau_{m,\varepsilon}^{(m)}=t_m$. Therefore, the theorem holds for $n=m$. Suppose now that the theorem holds for some $0<n\leq m$, and let us prove that it also holds when n is replaced with $n-1$.

(i) Fix any stopping time $\tau\in\mathcal{S}_{n-1}$. By Theorem 3.2 *(ii)* there is a nonnegative $\mathcal{F}_{t_{n-1}}$-measurable r.v. R_{n-1} such that $\tau\wedge t_n=(t_{n-1}+R_{n-1})\wedge t_n$, and the dynamics in (1.2) of Φ implies that $\Phi_t=\varphi(t-t_{n-1},\Phi_{t_{n-1}})$ for every $t_{n-1}\leq t<t_n$. Therefore, $\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau\wedge t_m}e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_{n-1}}]=$

$$\mathbb{E}_\infty\Big[\int_{t_{n-1}}^{\tau\wedge t_n}e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t+1_{\{\tau\geq t_n\}}e^{-\lambda\Delta t_n}\mathbb{E}_\infty\Big\{\int_{t_n}^{(\tau\vee t_n)\wedge t_m}e^{-\lambda(t-t_n)}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big\}\Big|\mathcal{F}_{t_{n-1}}\Big]$$

$$\geq \mathbb{E}_\infty\Big[\int_{t_{n-1}}^{\tau\wedge t_n} e^{-\lambda(t-t_{n-1})}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t + 1_{\{\tau\geq t_n\}}e^{-\lambda\Delta t_n}v_n^{(m)}(\Phi_{t_n})\Big|\mathcal{F}_{t_{n-1}}\Big],$$

because $(\tau\vee t_n)\in\mathcal{S}_n$, and $\mathbb{E}_\infty\{\int_{t_n}^{(\tau\vee t_n)\wedge t_m} e^{-\lambda(t-t_n)}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_n}\}\geq \gamma_n^{(m)}=v_n^{(m)}(\Phi_{t_n})$ by induction hypothesis. By Theorem 3.2 (ii) and (iii), there is a nonnegative $\mathcal{F}_{t_{n-1}}$-mble r.v. R_{n-1} such that $\mathbb{P}_\infty$-a.s. $\tau\wedge t_n=(t_{n-1}+R_{n-1})\wedge t_n$ and $\{\tau\geq t_n\}=\{t_0+R_0\geq t_1,\ldots,t_{n-1}+R_{n-1}\geq t_n\}=\{t_{n-1}+R_{n-1}\geq t_n\}$ because $\tau\in\mathcal{S}_{n-1}$ implies $\mathbb{P}_\infty$-a.s. $\Omega=\{\tau\geq t_{n-1}\}=\{t_0+R_0\geq t_1,\ldots,t_{n-2}+R_{n-2}\geq t_{n-1}\}$. Since $\Phi_{t_n}=\jmath(\Delta t_n,\Phi_{t_{n-1}},\frac{\Delta X_n}{\sqrt{\Delta t_n}})$ by (1.2), $\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau\wedge t_m} e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t\Big|\mathcal{F}_{t_{n-1}}]=$

$$\int_0^{R_{n-1}\wedge\Delta t_n} e^{-\lambda t}\Big(\varphi(t,\Phi_{t_{n-1}})-\frac{\lambda}{c}\Big)\mathrm{d}t + 1_{\{R_{n-1}\geq\Delta t_n\}}e^{-\lambda\Delta t_n}\mathbb{E}_\infty\Big[v_n^{(m)}\Big(\jmath\Big(\Delta t_n,\phi,\frac{\Delta X_n}{\sqrt{\Delta t_n}}\Big)\Big)\Big]\Big|_{\phi=\Phi_{t_{n-1}}}$$

$$= (Jv_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}},0,R_{n-1})\geq (J_0v_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}})=v_{n-1}^{(m)}(\Phi_{t_{n-1}}), \tag{4.7}$$

because R_{n-1} and $\Phi_{t_{n-1}}$ are $\mathcal{F}_{t_{n-1}}$-mble, and $\Delta X_n/\sqrt{\Delta t_n}$ has standard Gaussian distribution independent of $\mathcal{F}_{t_{n-1}}=\sigma(X_{t_0},X_{t_1},\ldots,X_{t_{n-1}})$ under $\mathbb{P}_\infty$. Taking the essential infimum of both sides over $\tau\in\mathcal{S}_{n-1}$ gives that $\mathbb{P}_\infty$-a.s. $\gamma_{n-1}^{(m)}\geq v_{n-1}^{(m)}(\Phi_{t_{n-1}})$. To show the reverse inequality, recall that

$$\tau_{n-1,\varepsilon}^{(m)} := \begin{cases} t_{n-1}+R_{n-1,\varepsilon/2}^{(m)}, & \text{if} \quad R_{n-1,\varepsilon/2}^{(m)}<\Delta t_n,\\ \tau_{n,\varepsilon/2}^{(m)}, & \text{if} \quad R_{n-1,\varepsilon/2}^{(m)}\geq\Delta t_n\end{cases}$$

is in $\mathcal{S}_{n-1}$, where $R_{n-1,\varepsilon/2}^{(m)}\geq 0$ is such that $(Jv_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}},0,R_{n-1,\varepsilon/2}^{(m)})\leq (J_0v_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}})+\varepsilon/2$. Moreover, $\tau_{n-1,\varepsilon}^{(m)}\wedge t_n=(t_{n-1}+R_{n-1,\varepsilon/2}^{(m)})\wedge t_n$ and $\{\tau_{n-1,\varepsilon}^{(m)}\geq t_n\}=\{R_{n-1,\varepsilon/2}^{(m)}\geq\Delta t_n\}$, on which $\tau_{n-1,\varepsilon}^{(m)}=\tau_{n,\varepsilon/2}^{(m)}\in\mathcal{S}_n$. Then $\gamma_{n-1}^{(m)}\leq\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau_{n-1,\varepsilon}^{(m)}\wedge t_m} e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_{n-1}}]=$

$$\int_{t_{n-1}}^{(t_{n-1}+R_{n-1,\varepsilon/2}^{(m)})\wedge t_n} e^{-\lambda(t-t_{n-1})}\Big(\varphi(t-t_{n-1},\Phi_{t_{n-1}})-\frac{\lambda}{c}\Big)\mathrm{d}t$$

$$+1_{\{R_{n-1,\varepsilon/2}^{(m)}\geq\Delta t_n\}}e^{-\lambda\Delta t_n}\mathbb{E}_\infty\Big[\mathbb{E}_\infty\Big\{\int_{t_n}^{\tau_{n,\varepsilon/2}^{(m)}\wedge t_m} e^{-\lambda(t-t_n)}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big\}\Big|\mathcal{F}_{t_{n-1}}\Big]$$

$$\leq\int_0^{R_{n-1,\varepsilon/2}^{(m)}\wedge\Delta t_n} e^{-\lambda t}\Big(\varphi(t,\Phi_{t_{n-1}})-\frac{\lambda}{c}\Big)\mathrm{d}t+1_{[\Delta t_n,\infty)}(R_{n-1,\varepsilon/2}^{(m)})e^{-\lambda\Delta t_n}\mathbb{E}_\infty\Big[v_n^{(m)}(\Phi_{t_n})\Big|\mathcal{F}_{t_{n-1}}\Big]+\frac{\varepsilon}{2},$$

where $\mathbb{E}_\infty\{\int_{t_n}^{\tau_{n,\varepsilon/2}^{(m)}\wedge t_m} e^{-\lambda(t-t_n)}\left(\Phi_t-\frac{\lambda}{c}\right)\mathrm{d}t \mid \mathcal{F}_{t_n}\}\leq\gamma_n^{(m)}+\varepsilon/2=v_n^{(m)}(\Phi_{t_n})+\varepsilon/2$ by induction hypothesis. Since $\Phi_{t_n}=\jmath(\Delta t_n,\Phi_{t_{n-1}},\frac{\Delta X_n}{\sqrt{\Delta t_n}})$ by (1.2), and $\Phi_{t_{n-1}}$ and $R_{n-1,\varepsilon/2}^{(m)}$ are $\mathcal{F}_{t_{n-1}}$-mble,

$$\gamma_{n-1}^{(m)}\leq\int_0^{R_{n-1,\varepsilon/2}^{(m)}\wedge\Delta t_n} e^{-\lambda t}\Big(\varphi(t,\Phi_{t_{n-1}})-\frac{\lambda}{c}\Big)\mathrm{d}t$$

$$+1_{[\Delta t_n,\infty)}\Big(R_{n-1,\varepsilon/2}^{(m)}\Big)e^{-\lambda\Delta t_n}\mathbb{E}_\infty\Big[v_n^{(m)}\Big(\jmath\Big(\Delta t_n,\phi,\frac{\Delta X_n}{\sqrt{\Delta t_n}}\Big)\Big)\Big]\Big|_{\phi=\Phi_{t_{n-1}}}+\frac{\varepsilon}{2}$$

$$=\int_0^{R_{n-1,\varepsilon/2}^{(m)}\wedge\Delta t_n} e^{-\lambda t}\Big(\varphi(t,\Phi_{t_{n-1}})-\frac{\lambda}{c}\Big)\mathrm{d}t+1_{[\Delta t_n,\infty)}(R_{n-1,\varepsilon/2}^{(m)})e^{-\lambda\Delta t_n}(Kv_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}})+\frac{\varepsilon}{2}$$

$$=(Jv_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}},0,R_{n-1,\varepsilon/2}^{(m)})+\frac{\varepsilon}{2}<(J_0v_n^{(m)})(\Delta t_n,\Phi_{t_{n-1}})+\varepsilon=v_{n-1}^{(m)}(\Phi_{t_{n-1}})+\varepsilon.$$

Because $\varepsilon\geq 0$ is arbitrary, we conclude that $\gamma_{n-1}^{(m)}=v_{n-1}^{(m)}(\Phi_{t_{n-1}})$, which proves (i) for $n-1$.

In the meantime, $\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau^{(m)}_{n-1,\varepsilon}\wedge t_m} e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t|\mathcal{F}_{t_{n-1}}]\le v^{(m)}_{n-1}(\Phi_{t_{n-1}})+\varepsilon=\gamma^{(m)}_{n-1}+\varepsilon$, and taking expectations proves (iii) and (iv) for $(n-1)$ and that stopping time $\tau^{(m)}_{n-1,\varepsilon}$ is ε-optimal.

(ii) Let us finally prove (ii) for $n-1$. By (iv) that we have just established for $(n-1)$, we obtain $\nu^{(m)}_{n-1}\le\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau^{(m)}_{n-1,\varepsilon}\wedge t_m} e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t]\le\mathbb{E}_\infty\gamma^{(m)}_{n-1}+\varepsilon$, and because $\varepsilon\ge 0$ is arbitrary, we get $\nu^{(m)}_{n-1}\le\mathbb{E}_\infty\gamma^{(m)}_{n-1}$. For reverse inequality, take expectations in (4.7) and obtain $\mathbb{E}_\infty[\int_{t_{n-1}}^{\tau\wedge t_m} e^{-\lambda(t-t_{n-1})}(\Phi_t-\frac{\lambda}{c})\mathrm{d}t\mid\mathcal{F}_{t_{n-1}}]\ge\mathbb{E}_\infty[v^{(m)}_{n-1}(\Phi_{t_{n-1}})]=\mathbb{E}_\infty\gamma^{(m)}_{n-1}$ for all $\tau\in\mathcal{S}_{n-1}$. Taking infimums over $\tau\in\mathcal{S}_{n-1}$ gives $\nu^{(m)}_{n-1}\ge\mathbb{E}_\infty\gamma^{(m)}_{n-1}$, which proves (ii) for $n-1$, and the theorem. ☐

The next corollary follows immediately from Proposition 4.2 and Theorem 4.3 and shows that the value function $V(\phi)$ in (2.5) can be approximated successively by the elements of the sequence $(v^{(m)}_0(\phi))_{n\ge 0}$. The explicit uniform bound on the approximation error allows one to determine the least number of iterations sufficient to obtain any given level of accuracy.

Corollary 4.4. *The value function $V(\cdot)$ of the original optimal stopping problem in (2.5) can be found in the limit by $V(\Phi_0)=\gamma_0=\lim_{m\to\infty}\gamma^{(m)}_0=\lim_{m\to\infty}v^{(m)}_0(\Phi_0)$, where the convergence is uniform in Φ_0. More precisely, we have $0\le V(\phi)-v^{(m)}_0(\phi)\le\frac{1}{c}e^{-\lambda t_m}$ for every $\phi\ge 0$ and $m\ge 0$. For every $\varepsilon>0$, let $M(\varepsilon):=\min\{m\ge 0;\ t_m\ge\frac{1}{\lambda}\ln\frac{1}{c\varepsilon}\}$. Then the $(\mathcal{F}_t)_{t\ge 0}$-stopping time $\tau^{(M(\varepsilon/2))}_{0,\varepsilon/2}\wedge t_{M(\varepsilon/2)}\in\mathcal{S}_0$ is ε-optimal for the problem in (2.5); namely,*

$$0\le V(\phi)-\mathbb{E}^\phi_\infty\Big[\int_0^{\tau^{(M(\varepsilon/2))}_{0,\varepsilon/2}\wedge t_{M(\varepsilon/2)}} e^{-\lambda t}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big]\le\varepsilon\qquad\textit{for every }\phi\ge 0.$$

Proposition 4.5 shows that, for every $0\le n\le m$ and $\varepsilon>0$, the ε-optimal stopping rule $\tau^{(m)}_{n,\varepsilon}$ of Theorem 4.3 admits a simple characterization of the same form as in the general characterization of all $(\mathcal{F}_t)_{t\ge 0}$-stopping rules described by Theorem 3.2 and Proposition 3.3.

Proposition 4.5. *For every $0\le n\le m-1$ and $\varepsilon\ge 0$, let $\tau^{(m)}_{n,\varepsilon}$ and $R^{(m)}_{n,\varepsilon}$ be as in Theorem 4.3. Define $N^{(m)}_{n,\varepsilon}=\min\{n\le k\le m;\ R^{(m)}_{k,\varepsilon/2^{k+1-n}}<\Delta t_{k+1}\}$. Then $N^{(m)}_{n,\varepsilon}$ is an $\{n,n+1,\dots,m\}$-valued $(\mathcal{F}_{t_k})_{k\ge 0}$-stopping time, $\{t_k\le\tau^{(m)}_{n,\varepsilon}<t_{k+1}\}=\{N^{(m)}_{n,\varepsilon}=k\}$, and*

$$\tau^{(m)}_{n,\varepsilon}=t_{N^{(m)}_{n,\varepsilon}}+R^{(m)}_{N^{(m)}_{n,\varepsilon},\varepsilon/2^{N^{(m)}_{n,\varepsilon}+1-n}}=\Big(t_k+R^{(m)}_{k,\varepsilon/2^{k+1-n}}\Big)\Big|_{k=N^{(m)}_{n,\varepsilon}}.$$

Proof. Since $\tau^{(m)}_{n,\varepsilon}=\tau^{(m)}_{n+1,\varepsilon/2}$ on $\{\tau^{(m)}_{n,\varepsilon}\ge t_{n+1}\}=\{R^{(m)}_{n,\varepsilon/2}\ge\Delta t_{n+1}\}$, we have $\{\tau^{(m)}_{n,\varepsilon}\ge t_k\}=\{R^{(m)}_{n,\varepsilon/2}\ge\Delta t_{n+1}\}\cap\bigcap_{i=n+2}^k\{\tau^{(m)}_{n+1,\varepsilon/2}\ge t_i\}=\ldots=\bigcap_{\ell=n}^{k-1}\{R^{(m)}_{\ell,\varepsilon/2^{\ell+1-n}}\ge\Delta t_{\ell+1}\}$, for $n+1\le k\le m$ and $\{t_k\le\tau^{(m)}_{n,\varepsilon}<t_{k+1}\}=\bigcap_{\ell=n}^{k-1}\{R^{(m)}_{\ell,\varepsilon/2^{\ell+1-n}}\ge\Delta t_{\ell+1}\}\setminus\bigcap_{\ell=n}^{k}\{R^{(m)}_{\ell,\varepsilon/2^{\ell+1-n}}\ge\Delta t_{\ell+1}\}=\{N^{(m)}_{n,\varepsilon}=k\}$ for $n\le k\le m-1$; $\tau^{(m)}_{n,\varepsilon}=t_k+R^{(m)}_{k,\varepsilon/2^{k+1-n}}$ on $\{t_k\le\tau^{(m)}_{n,\varepsilon}<t_{k+1}\}=\{N^{(m)}_{n,\varepsilon}=k\}$ for $n\le k\le m$. ☐

Theorem 4.6 generalizes Corollary 4.4. The theorem shows that the minimum conditional expected remaining Bayes risk at t_n given the past observations $\mathcal{F}_{t_n}$ equals $\mathbb{P}_\infty$-a.s. $\gamma_n=v_n(\Phi_{t_n})$, where $v_n(\cdot)$ is the limit of its successive approximations $(v^{(m)}_n(\cdot))_{m\ge 0}$ as $m\to\infty$. Because the convergence turns out to be uniform, the error in the approximation of $v_n(\phi)$ by $v^{(m)}_0(\phi)$ can be made arbitrarily small simultaneously for every $\phi\ge 0$ if $m\ge 0$ is chosen sufficiently large.

Theorem 4.6. *For every $n \geq 0$ and $\phi \geq 0$, the sequence $(v_n^{(m)}(\phi))_{m\geq 0}$ is decreasing, and the pointwise limit $v_n(\phi) := \lim_{m\to\infty} v_n^{(m)}(\phi)$ exists and is uniform in $\phi \geq 0$. More precisely,*

$$\sup_{\phi\geq 0} |v_n(\phi) - v_n^{(m)}(\phi)| \leq \frac{1}{c}\, e^{-\lambda(t_m - t_n)} \quad \textit{for every } 0 \leq n \leq m.$$

The functions $v_n^{(m)}(\cdot)$, $0 \leq m \leq n$ and $v_n(\cdot)$, $n \geq 0$ are nondecreasing, concave, continuous, and bounded between $-1/c$ and 0. Moreover, for every $n \geq 0$, we have $v_n(\cdot) = (J_0 v_{n+1})(\Delta t_{n+1}, \cdot)$, and

$$\gamma_n = v_n(\Phi_{t_n}),\ \mathbb{P}_\infty\text{-}a.s. \quad \textit{and} \quad \nu_n := \inf_{\tau\in\mathcal{S}_n} \mathbb{E}_\infty\Big[\int_{t_n}^{\tau} e^{-\lambda(t-t_n)}\Big(\Phi_t - \frac{\lambda}{c}\Big)\mathrm{d}t\Big] = \mathbb{E}_\infty \gamma_n.$$

For every $n \geq 0$ and $\varepsilon > 0$, let $M_n(\varepsilon) := \min\Big\{m \geq n;\ t_m - t_n \geq \frac{1}{\lambda}\ln\frac{1}{c\varepsilon}\Big\}$. Then the $(\mathcal{F}_t)_{t\geq 0}$-stopping time $\tau_{n,\varepsilon/2}^{(M_n(\varepsilon/2))} \wedge t_{M_n(\varepsilon/2)} \in \mathcal{S}_n$, defined as in Theorem 4.3, is ε-optimal for the problem $\inf_{\tau\in\mathcal{S}_n} R_\tau(p) = 1 - p + (1-p)c\,\mathbb{E}_\infty[\int_0^{t_n} e^{-\lambda t}(\Phi_t - \frac{\lambda}{c})\mathrm{d}t + e^{-\lambda t_n}\gamma_n]$ of the minimum Bayes risk if an alarm has not yet been raised before time t_n; namely,

$$\gamma_n + \varepsilon > \mathbb{E}_\infty\Big[\int_{t_n}^{\tau_{n,\varepsilon/2}^{(M_n(\varepsilon/2))}\wedge t_{M_n(\varepsilon/2)}} e^{-\lambda(t-t_n)}\Big(\Phi_t - \frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big].$$

Proof. For every $m \geq 0$, because $v_m^{(m)}(\cdot) \equiv 0 \in [-1/c, 0]$ is bounded, nondecreasing, concave, and continuous, Lemma 4.1 (i) implies that $v_n^{(m)}(\cdot)$ is bounded between $-1/c$ and 0, nondecreasing, concave, and continuous for every $0 \leq n \leq m$. Moreover, for every $n \geq 0$, the sequence $(v_n^{(m)}(\cdot))_{m\geq n}$ is decreasing. To see this, note that for every $m < p$ we have $v_m^{(p)}(\cdot) \leq 0 \equiv v_m^{(m)}(\cdot)$. Suppose $v_n^{(p)}(\cdot) \leq v_n^{(m)}(\cdot)$ for some $0 < n \leq m$. Then by Lemma 4.1 (ii) $v_{n-1}^{(p)}(\cdot) = (J_0 v_n^{(p)})(\Delta t_n, \cdot) \leq (J_0 v_n^{(m)})(\Delta t_n, \cdot) = v_{n-1}^{(m)}(\cdot)$, and an induction on $0 \leq n \leq m$ proves that $v_n^{(p)}(\cdot) \leq v_n^{(m)}(\cdot)$ for every $0 \leq n \leq m \leq p$. Thus, the limit $v_n(\phi) := \lim_{m\to\infty} v_n^{(m)}(\phi)$ exists for every $\phi \geq 0$ and is bounded between $-1/c$ and 0, nondecreasing, and concave. For all $0 \leq n \leq m \leq p$, by Lemma 4.1 (iii)

$$\begin{aligned}\sup_{\phi\geq 0}|v_n(\phi) - v_n^{(m)}(\phi)| &\leq \sup_{\phi\geq 0}|v_n^{(p)}(\phi) - v_n^{(m)}(\phi)| = \sup_{\phi\geq 0}|(J_0 v_{n+1}^{(p)})(\Delta t_{n+1}, \phi) - (J_0 v_{n+1}^{(m)})(\Delta t_{n+1}, \phi)| \\ &\leq e^{-\lambda \Delta t_{n+1}} \sup_{\phi\geq 0}|v_{n+1}^{(p)}(\phi) - v_{n+1}^{(m)}(\phi)| = e^{-\lambda(t_m - t_n)}\sup_{\phi\geq 0}|v_m^{(p)}(\phi)| \leq \frac{1}{c}\, e^{-\lambda(t_m - t_n)}.\end{aligned}$$

Hence, the sequence $(v_n^{(m)}(\phi))_{m\geq n}$ of continuous functions converges as $m \to \infty$ to $v_n(\phi)$ uniformly in $\phi \geq 0$, and $v_n(\cdot)$ is also continuous for all $n \geq 0$. Moreover, $v_n(\cdot) = (J_0 v_{n+1})(\Delta t_{n+1}, \cdot)$ for all $n \geq 0$, since $v_n(\phi) = \inf_{m\geq n}(J_0 v_{n+1}^{(m)})(\Delta t_{n+1}, \phi) = \inf_{m\geq n}\ \inf_{r\geq 0}\ (J v_{n+1}^{(m)})(\Delta t_{n+1}, \phi, 0, r) = \inf_{r\geq 0}\ \inf_{m\geq n}\ (J v_{n+1}^{(m)})(\Delta t_{n+1}, \phi, 0, r) = \inf_{r\geq 0}\ (J v_{n+1})(\Delta t_{n+1}, \phi, 0, r) = (J_0 v_{n+1})(\Delta t_{n+1}, \phi)$, by the bounded convergence. Finally, by Proposition 4.2 and Theorem 4.3, $\gamma_n = \lim_{m\to\infty} v_n^{(m)}(\Phi_{t_n}) = v_n(\Phi_{t_n})$. For $0 \leq n \leq m$ and $\tau \in \mathcal{S}_n$, we have $\tau \wedge t_m \in \mathcal{S}_n$ and $\nu_n \leq \mathbb{E}_\infty[\int_{t_n}^{\tau\wedge t_m} e^{-\lambda(t-t_n)}\left(\Phi_t - \frac{\lambda}{c}\right)\mathrm{d}t]$, and taking the infimums gives $\nu_n \leq \nu_n^{(m)} = \mathbb{E}_\infty \gamma_n^{(m)}$ by Theorem 4.3. Taking limits as $m \to \infty$ gives $\nu_n \leq \mathbb{E}_\infty \gamma_n$ since $\gamma_n^{(m)} \to \gamma_n$ as $m \to \infty$, $\mathbb{P}_\infty$-a.s. uniformly across sample-paths by Proposition 4.2. For the reverse, $\gamma_n \leq \mathbb{E}_\infty[\int_{t_n}^{\tau} e^{-\lambda(t-t_n)}\left(\Phi_t - \frac{\lambda}{c}\right)\mathrm{d}t \mid \mathcal{F}_{t_n}]$ for all $\tau \in \mathcal{S}_n$, and taking expectations and infimum over $\tau \in \mathcal{S}_n$ gives $\mathbb{E}_\infty \gamma_n \leq \inf_{\tau\in\mathcal{S}_n} \mathbb{E}_\infty[\int_{t_n}^{\tau} e^{-\lambda(t-t_n)}\left(\Phi_t - \frac{\lambda}{c}\right)\mathrm{d}t] = \nu_n$, and $\nu_n = \mathbb{E}_\infty \gamma_n$.

According to the first parts of Proposition 2.1 and Theorem 4.6, if an alarm has not yet been raised before time t_n, then $\inf_{\tau\in\mathcal{S}_n} R_\tau(p) = 1 - p + (1-p)c\,\mathbb{E}_\infty[\int_0^{t_n} e^{-\lambda t}(\Phi_t - \frac{\lambda}{c})\mathrm{d}t + e^{-\lambda t_n}\gamma_n]$.

Theorem 4.3 (iii) with $m = M_n(\varepsilon/2)$ implies

$$\mathbb{E}_\infty\Big[\int_{t_n}^{\tau_{n,\varepsilon/2}^{(M_n(\varepsilon/2))}\wedge t_{M_n(\varepsilon/2)}} e^{-\lambda(t-t_n)}\Big(\Phi_t-\frac{\lambda}{c}\Big)\mathrm{d}t\Big|\mathcal{F}_{t_n}\Big] \le \gamma_n^{(M_n(\varepsilon/2))}+\frac{\varepsilon}{2}$$
$$= v_n^{(M_n(\varepsilon/2))}(\Phi_{t_n})+\frac{\varepsilon}{2} < v_n(\Phi_{t_n})+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\gamma_n+\varepsilon. \quad \square$$

5. The solution between observation times

If detection alarm has not been raised until time $t \ge 0$, then one faces optimal stopping problems

$$(5.1)\qquad \begin{aligned}\gamma(t) &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{S}(t)} \mathbb{E}_\infty\Big[\int_t^{\tau} e^{-\lambda(u-t)}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u\Big|\mathcal{F}_t\Big], \qquad t\ge 0,\\ \gamma^{(m)}(t) &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{S}(t)} \mathbb{E}_\infty\Big[\int_t^{\tau\wedge t_m} e^{-\lambda(u-t)}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u\Big|\mathcal{F}_t\Big], \quad t\ge 0,\ m\ge 0,\end{aligned}$$

where $\mathcal{S}(t)=\{\tau\in\mathcal{S};\ \tau\ge t, \mathbb{P}_\infty\text{-a.s.}\}$. Note that $\mathcal{S}_n$, $\gamma_n^{(m)}$, and γ_n of Section 4 are the same as, respectively, $\mathcal{S}(t_n)$, $\gamma^{(m)}(t_n)$, and $\gamma(t_n)$ for every $0\le n\le m$. Theorem 5.1 below shows how the solution and ε-optimal stopping rules between observation times can be easily identified after they are first found at observation times as described in Section 4.

Theorem 5.1. *For every $0\le n<m$ and $t_n\le t<t_{n+1}$, we have*

$$(i)\qquad \gamma^{(m)}(t)=e^{\lambda(t-t_n)}(J_{t-t_n}v_{n+1}^{(m)})(\Delta t_{n+1},\Phi_{t_n}),\quad \mathbb{P}_\infty\text{-a.s.},$$
$$(ii)\qquad \nu^{(m)}(t):=\inf_{\tau\in\mathcal{S}(t)}\mathbb{E}_\infty\Big[\int_t^{\tau\wedge t_m} e^{-\lambda(u-t)}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u\Big]=\mathbb{E}_\infty\gamma^{(m)}(t),$$

where $(v_n^{(m)}(\cdot))_{0\le n\le m}$ are the successive approximations calculated by (4.5). For every $m\ge 0$ and $0\le t\le t_m$, we have $\mathbb{P}_\infty$-a.s. $-1/c\le\gamma^{(m)}(t)\le 0$, and $-1/c\le\nu^{(m)}(t)\le 0$.

For every $\varepsilon\ge 0$, $m\ge 0$, and $0\le t\le t_m$, let $R_\varepsilon^{(m)}(t)\equiv 0$ and $R_\varepsilon^{(m)}(t)\equiv R_\varepsilon^{(m)}(t,\Delta t_{n+1},\Phi_{t_n})$ be a real number greater than or equal to $t-t_n$ such that

$$(Jv_{n+1}^{(m)})(\Delta t_{n+1},\Phi_{t_n},t-t_n,R_\varepsilon^{(m)}(t))\le(J_{t-t_n}v_{n+1}^{(m)})(\Delta t_{n+1},\Phi_{t_n})+\varepsilon\cdot e^{-\lambda(t-t_n)},$$

if $t_n\le t<t_{n+1}$ for some $0\le n<m$. For every $\varepsilon\ge 0$, $R_\varepsilon^{(m)}(t)$ is a nonnegative r.v., which is $\mathcal{F}_{t_m}$ measurable if $t=t_m$ and $\mathcal{F}_t\equiv\mathcal{F}_{t_n}$ measurable if $t_n\le t<t_{n+1}$ for some $0\le n<m$. Moreover,

$$\tau_\varepsilon^{(m)}(t):=\left\{\begin{array}{ll} t_n+R_{\varepsilon/2}^{(m)}(t), & \text{if}\quad R_{\varepsilon/2}^{(m)}(t)<\Delta t_{n+1}\\ \tau_{n+1,\varepsilon/2}^{(m)}, & \text{if}\quad R_{\varepsilon/2}^{(m)}(t)\ge\Delta t_{n+1}\end{array}\right\}\in\mathcal{S}(t)$$

is ε-optimal in the sense that, if $t_n\le t<t_{n+1}$ for some $0\le n<m$, then

$$(iii)\qquad \gamma^{(m)}(t)+\varepsilon\ge\mathbb{E}_\infty\Big[\int_t^{\tau_\varepsilon^{(m)}(t)\wedge t_m} e^{-\lambda(u-t)}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u\Big|\mathcal{F}_t\Big],\quad \mathbb{P}_\infty\text{-a.s.},$$
$$(iv)\qquad \nu^{(m)}(t)+\varepsilon\ge\mathbb{E}_\infty\Big[\int_{t_n}^{\tau_\varepsilon^{(m)}(t)\wedge t_m} e^{-\lambda(u-t)}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u\Big].$$

$R_{n,\varepsilon}^{(m)}$ and $\tau_{n,\varepsilon}^{(m)}$ of Theorem 4.3 are the same as $R_\varepsilon^{(m)}(t_n)$ and $\tau_\varepsilon^{(m)}(t_n)$ for all $0\le n\le m$, $\varepsilon>0$.

The proof of Theorem 5.1 is similar to that of Theorem 4.3 and is omitted. As expected from Proposition 4.2 and Theorem 4.6, $\gamma(t)$ is $\mathbb{P}_\infty$-a.s. limit of $\gamma^{(m)}(t)$ as $m \to \infty$ and is related to $v_n(\cdot)$, if $t_n \le t < t_{n+1}$ for some n, through the dynamic programming operator $J_\bullet$. For each $t \ge 0$, the convergence is uniform across the sample path realizations, and the explicit bound on the approximation error helps one determine ε-stopping times.

Proposition 5.2. *For every fixed $n \ge 0$ and $t_n \le t < t_{n+1}$, the sequence $(\gamma^{(m)}(t))_{m>n}$ converges $\mathbb{P}_\infty$-a.s. to $\gamma(t)$ as $m \to \infty$. More precisely, $\mathbb{P}_\infty$-a.s. $0 \le \gamma^{(m)}(t) - \gamma(t) \le \frac{1}{c}\, e^{-\lambda(t_m - t_n)}$ for every $0 \le n < m$ and $t_n \le t < t_{n+1}$. For every $n \ge 0$ and $t_n \le t < t_{n+1}$,*

$$\gamma(t) = e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1})(\Delta t_{n+1}, \Phi_{t_n}), \quad \mathbb{P}_\infty\text{-a.s.},$$
$$\nu(t) := \inf_{\tau \in \mathcal{S}(t)} \mathbb{E}_\infty\Big[\int_t^\tau e^{-\lambda(u-t)}\Big(\Phi_u - \frac{\lambda}{c}\Big)\mathrm{d}u\Big] = \mathbb{E}_\infty \gamma(t).$$

If $M_n(\varepsilon)$ is defined for every $\varepsilon > 0$ and $n \ge 0$ as in Theorem 4.6, then for every $t_n \le t < t_{n+1}$ the $\mathbb{F}$-stopping time $\tau_{\varepsilon/2}^{(M_n(\varepsilon/2))}(t) \wedge t_{M_n(\varepsilon/2)} \in \mathcal{S}(t)$ defined as in Theorem 5.1 is ε-optimal for

$$\inf_{\tau \in \mathcal{S}(t)} R_\tau(p) = 1 - p + (1-p)c\, \mathbb{E}_\infty\Big[\int_0^t e^{-\lambda u}\Big(\Phi_u - \frac{\lambda}{c}\Big)\mathrm{d}u + e^{-\lambda t}\gamma(t)\Big] \tag{5.2}$$

of the minimum Bayes risk if an alarm has not been raised before time t; namely,

$$\gamma(t) + \varepsilon > \mathbb{E}_\infty\Big[\int_t^{\tau_{\varepsilon/2}^{(M_n(\varepsilon/2))}(t) \wedge t_{M_n(\varepsilon/2)}} e^{-\lambda(u-t)}\Big(\Phi_u - \frac{\lambda}{c}\Big)\mathrm{d}u \Big| \mathcal{F}_t\Big].$$

Proof. Fix $n \ge 0$ and $t_n \le t < t_{n+1}$. For every $\tau \in \mathcal{S}(t)$ and $m > n$, we have $\tau \wedge t_m \in \mathcal{S}(t)$ and $\gamma(t) \le \mathbb{E}_\infty[\int_t^{\tau\wedge t_m} e^{-\lambda(u-t)}\left(\Phi_u - \frac{\lambda}{c}\right)\mathrm{d}u \mid \mathcal{F}_t]$. Hence, $\mathbb{P}_\infty$-a.s. $\gamma(t) \le \gamma^{(m)}(t)$. We also have $\mathbb{E}_\infty[\int_t^\tau e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t] \ge \mathbb{E}_\infty[\int_t^{\tau\wedge t_m} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t] - \frac{1}{c}\int_{t_m}^\infty \lambda e^{-\lambda(u-t)}\mathrm{d}u \ge \gamma^{(m)}(t) - \frac{1}{c}\, e^{-\lambda(t_m - t)} \ge \gamma^{(m)}(t) - \frac{1}{c}\, e^{-\lambda(t_m-t_n)}$. Taking essential infimums over $\tau \in \mathcal{S}(t)$ gives the first inequality of the proposition, which shows that $\gamma^{(m)}(t)$ converges uniformly and $\mathbb{P}_\infty$-a.s. to $\gamma(t)$ as $m \to \infty$. By Theorem 5.1 (i), $\mathbb{P}_\infty$-a.s. $\gamma(t) = \lim_{m\to\infty} \gamma^{(m)}(t) = \lim_{m\to\infty} e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n}) = e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1})(\Delta t_{n+1}, \Phi_{t_n})$ by the bounded convergence and Theorem 4.6. Since for every $\tau \in \mathcal{S}(t)$, we have $\gamma(t) \le \mathbb{E}_\infty\left[\int_t^\tau e^{-\lambda(u-t)}\left(\Phi_u - \frac{\lambda}{c}\right)\mathrm{d}u \middle| \mathcal{F}_t\right]$, taking expectations and infimums over $\tau \in \mathcal{S}(t)$ gives $\mathbb{E}_\infty\gamma(t) \le \nu(t)$. Since $(\gamma^{(m)}(t))_{m\ge 0}$ converges uniformly to $\gamma(t)$ as $m \to \infty$, we have $\mathbb{E}_\infty\gamma(t) = \lim_{m\to\infty}\mathbb{E}_\infty\gamma^{(m)}(t) = \lim_{m\to\infty}\nu^{(m)}(t) \ge \nu(t)$ by Theorem 4.6 (ii). This proves (*ii*).

By the first parts of Proposition 2.1 and Theorem 4.6, if an alarm has not yet been raised before t_n, then minimum expected risk becomes $\inf_{\tau\in\mathcal{S}_n} R_\tau(p) = 1 - p + (1-p)c\,\mathbb{E}_\infty[\int_0^t e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda t}\gamma(t)]$. Theorem 5.1 (iii) with $m = M_n(\varepsilon/2)$ implies that $\mathbb{E}_\infty[\int_t^{\tau_{\varepsilon/2}^{(M_n(\varepsilon/2))}(t)\wedge t_{M_n(\varepsilon/2)}} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t] \le \gamma^{(M_n(\varepsilon/2))}(t) + \frac{\varepsilon}{2} < \gamma(t) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \gamma(t) + \varepsilon$, where the last inequality follows from the first part of the proposition. Taking expectations gives the last inequality of the proposition. $\square$

Remark 5.3. We can write more compactly that $\mathbb{P}_\infty$-a.s.

$$\begin{aligned}
\gamma(t) &= \sum_{n=0}^\infty 1_{[t_n, t_{n+1})}(t) e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1})(\Delta t_{n+1}, \Phi_{t_n}), \quad t \ge 0,\\
\gamma^{(m)}(t) &= \sum_{n=0}^{m-1} 1_{[t_n, t_{n+1})}(t) e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n}), \quad 0 \le t < t_m,\ m \ge 1.
\end{aligned} \tag{5.3}$$

For every $0 \le n \le m$, because the functions $v_{n+1}(\cdot)$ and $v_{n+1}^{(m)}(\cdot)$ are bounded and nonpositive, the mappings $t \mapsto (J_{t-t_n} v_{n+1})(\Delta t_{n+1}, \Phi_{t_n})$ and $t \mapsto (J_{t-t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n})$ are continuous on the interval $t \in [t_n, t_{n+1}]$ by Lemma 4.1 (iv). Therefore, the processes in (5.3) are RCLL versions of $\{\gamma(t);\ t \ge 0\}$ and $\{\gamma^{(m)}(t);\ 0 \le t \le t_m\}$, $m \ge 1$, and we work with those in the remainder.

The next theorem introduces alternative ε-optimal stopping rules, which will later be characterized as simple first hitting times of process Φ to suitable regions.

Theorem 5.4. *The stopping times*

$$\sigma_\varepsilon^{(m)}(t) := \inf\{s \ge t;\ \gamma^{(m)}(s) \ge -\varepsilon\}, \quad \varepsilon \ge 0,\ 0 \le t \le t_m,\ m \ge 1 \tag{5.4}$$

belong to $\mathcal{S}(t)$, are $\mathbb{P}_\infty$-a.s. less than or equal to t_m, and are ε-optimal in the sense that $\gamma^{(m)}(t)+\varepsilon \ge \mathbb{E}_\infty[\int_t^{\sigma_\varepsilon^{(m)}(t)\wedge t_m} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t]$. Particularly, $\sigma_0^{(m)}(t)$, $0 \le t \le t_m$, $m \ge 1$ are optimal in the sense that $\gamma^{(m)}(t) = \mathbb{E}_\infty[\int_t^{\sigma_0^{(m)}(t)\wedge t_m} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t]$.

For the proof of Theorem 5.4, we need the following proposition and its corollary, which are proved in the appendix.

Proposition 5.5. *For every $m \ge 1$, let $M^{(m)}(t) := \int_0^t e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda t}\gamma^{(m)}(t)$ for every $0 \le t \le t_m$. Then $M^{(m)}(t)$ is integrable for every $0 \le t \le t_m$ under $\mathbb{P}_\infty$, and $\mathbb{E}_\infty[M^{(m)}(\tau \wedge \sigma_\varepsilon^{(m)}(t))] = \mathbb{E}_\infty[M^{(m)}(t)]$ for every $m \ge 1$, $0 \le t \le t_m$, $\tau \in \mathcal{S}(t)$, and $\varepsilon \ge 0$.*

Corollary 5.6. *The stopped process $\{M^{(m)}(s \wedge \sigma_\varepsilon^{(m)}(t)), \mathcal{F}_s;\ t \le s \le t_m\}$ is a RCLL martingale under $\mathbb{P}_\infty$ for every $m \ge 1$, $0 \le t \le t_m$, and $\varepsilon \ge 0$.*

Proof of Theorem 5.4. Because $\gamma^{(m)}(t_m) \equiv 0$, we have $t \le \sigma_\varepsilon^{(m)}(t) \le t_m$ for every $m \ge 1$, $0 \le t \le t_m$, and $\varepsilon \ge 0$. Moreover, optional sampling theorem and Corollary 5.6 imply that $\int_0^t e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda t}\gamma^{(m)}(t) = M^{(m)}(t) = \mathbb{E}_\infty[M^{(m)}(\sigma_\varepsilon^{(m)}(t)) \mid \mathcal{F}_t] = \mathbb{E}_\infty[\int_0^{\sigma_\varepsilon^{(m)}(t)} e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda\sigma_\varepsilon^{(m)}(t)}\gamma^{(m)}(\sigma_\varepsilon^{(m)}(t))| \mathcal{F}_t]$, which leads to $\gamma^{(m)}(t) \ge \mathbb{E}_\infty[\int_t^{\sigma_\varepsilon^{(m)}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u| \mathcal{F}_t] - \varepsilon$, since $\gamma^{(m)}(\sigma_\varepsilon^{(m)}(t)) \ge -\varepsilon$ and $\sigma_\varepsilon^{(m)}(t) - t \ge 0$. Finally, taking the expectations of both sides and Theorem 5.1 (ii) give $\nu^{(m)}(t) = \mathbb{E}_\infty\gamma^{(m)}(t) \ge \mathbb{E}_\infty[\int_t^{\sigma_\varepsilon^{(m)}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u] - \varepsilon$. □

The stopping time $\sigma_\varepsilon(t) := \inf\{s \ge t;\ \gamma(s) \ge -\varepsilon\}$ is ε-optimal in infinite-horizon for all $\varepsilon \ge 0$, $t \ge 0$ by Theorem 5.7, Proposition 5.8, and Corollary 5.9, whose very similar proofs are omitted.

Theorem 5.7. *The stopping times*

$$\sigma_\varepsilon(t) := \inf\{s \ge t;\ \gamma(s) \ge -\varepsilon\}, \quad \varepsilon \ge 0,\ t \ge 0 \tag{5.5}$$

belong to $\mathcal{S}(t)$ and are ε-optimal in the sense that $\gamma(t) + \varepsilon \ge \mathbb{E}_\infty[\int_t^{\sigma_\varepsilon(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t]$. Particularly, $\sigma_0(t)$, $t \ge 0$ are optimal; namely, $\gamma(t) = \mathbb{E}_\infty[\int_t^{\sigma_0(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t]$, $t \ge 0$.

Proposition 5.8. *$M(t) := \int_0^t e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda t}\gamma(t)$ is integrable for every $t \ge 0$ under $\mathbb{P}_\infty$, and $\mathbb{E}_\infty[M(t_m \wedge \tau \wedge \sigma_\varepsilon(t))] = \mathbb{E}_\infty[M(t)]$ for every $m \ge 0$, $0 \le t \le t_m$, $\tau \in \mathcal{S}(t)$, and $\varepsilon \ge 0$.*

Corollary 5.9. *$\{M(s \wedge \sigma_\varepsilon(t)), \mathcal{F}_s;\ s \ge t\}$ is a RCLL martingale under $\mathbb{P}_\infty$ for all $t \ge 0$, and $\varepsilon \ge 0$.*

The process $\gamma(\cdot) = \lim_{m\to\infty} \gamma^{(m)}(\cdot)$ can be obtained only in the limit, and optimal stopping times $\sigma_0(t)$ of Theorem 5.7 are impractical. We can use successive approximations $\gamma^{(m)}(\cdot)$ to define, in light of Proposition 5.2 and Theorem 5.4, practical ε-optimal stopping rules of Proposition 5.10.

Proposition 5.10. *If $M_n(\varepsilon)$ is defined for all $\varepsilon > 0$ and $n \geq 0$ as in Theorem 4.6, then for all $t_n \leq t < t_{n+1}$ the $\mathbb{F}$-stopping time $\sigma_{\varepsilon/2}^{(M_n(\varepsilon/2))}(t) \in \mathcal{S}(t)$ defined as in Theorem 5.4 is ε-optimal for the problem of the minimum Bayes risk in (5.2) if an alarm was not raised before time t; namely,*

$$\gamma(t) + \varepsilon > \mathbb{E}_\infty\Big[\int_t^{\sigma_{\varepsilon/2}^{(M_n(\varepsilon/2))}(t)\wedge t_{M_n(\varepsilon/2)}} e^{-\lambda(u-t)}\Big(\Phi_u - \frac{\lambda}{c}\Big)\mathrm{d}u\Big|\mathcal{F}_t\Big].$$

6. The structure of ε-optimal stopping rules

Here, we shall characterize ε-optimal stopping time $\sigma_\varepsilon^{(m)}(t)$ of (5.4) for arbitrary but fixed $m \geq 1$, $\varepsilon \geq 0$, $0 \leq t \leq t_m$ and ε-optimal stopping time $\sigma_\varepsilon(t)$ of (5.5) for arbitrary but fixed $\varepsilon \geq 0$ and $t \geq 0$. Remark 5.3 implies that $\gamma^{(m)}(s) = e^{\lambda(s-t_\ell)}(J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \Phi_{t_\ell})$ for every $0 \leq \ell \leq m-1$, $s \in [t_\ell, t_{\ell+1})$ and $\gamma(s) = e^{\lambda(s-t_\ell)}(J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \Phi_{t_\ell})$ for $\ell \geq 0$, $t \geq 0$. Then

$$\begin{aligned} \gamma^{(m)}(s) \geq -\varepsilon \quad &\Leftrightarrow \quad (J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \Phi_{t_\ell}) \geq -\varepsilon e^{-\lambda(s-t_\ell)}, \quad s \in [t_\ell, t_{\ell+1}),\ 0 \leq \ell < m, \\ \gamma(s) \geq -\varepsilon \quad &\Leftrightarrow \quad (J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \Phi_{t_\ell}) \geq -\varepsilon e^{-\lambda(s-t_\ell)}, \quad s \in [t_\ell, t_{\ell+1}),\ \ell \geq 0. \end{aligned} \tag{6.1}$$

By Theorem 4.6, $\phi \mapsto v_{\ell+1}^{(m)}(\phi)$ and $\phi \mapsto v_{\ell+1}(\phi)$ are nondecreasing, concave, continuous, bounded between $-1/c$ and 0. Then $(J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \phi) = 0 \geq -\varepsilon e^{-\lambda(s-t_\ell)}$ and $(J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \phi) = 0 \geq -\varepsilon e^{-\lambda(s-t_\ell)}$ for every large $\phi \geq 0$ by Lemma 4.1 (i), and the sets $\{\phi \geq 0;\ (J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \phi) \geq -\varepsilon e^{-\lambda(s-t_\ell)}\}$ and $\{\phi \geq 0;\ (J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \phi) \geq -\varepsilon e^{-\lambda(s-t_\ell)}\}$ are not empty. Therefore,

$$\begin{aligned} \phi_\varepsilon^{(m)}(s) &:= \sum_{\ell=0}^{m-1} 1_{[t_\ell, t_{\ell+1})}(s) \inf\{\phi \geq 0;\ (J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \phi) \geq -\varepsilon e^{-\lambda(s-t_\ell)}\}, s \in [0, t_m], \\ \phi_\varepsilon(s) &:= \sum_{\ell=0}^{\infty} 1_{[t_\ell, t_{\ell+1})}(s) \inf\{\phi \geq 0;\ (J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \phi) \geq -\varepsilon e^{-\lambda(s-t_\ell)}\},\ s \geq 0 \end{aligned} \tag{6.2}$$

are finite. Because $\phi \mapsto (J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \phi)$ and $\phi \mapsto (J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \phi)$ are continuous, we have $(J_{s-t_\ell} v_{\ell+1}^{(m)})(\Delta t_{\ell+1}, \phi_\varepsilon^{(m)}(s)) \geq -\varepsilon e^{-\lambda(s-t_\ell)}$ and $(J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \phi_\varepsilon(s)) \geq -\varepsilon e^{-\lambda(s-t_\ell)}$ if $s \in [t_\ell, t_{\ell+1})$ for some $\ell \geq 0$. Moreover, (6.1) becomes

$$\begin{aligned} \gamma^{(m)}(s) \geq -\varepsilon \quad &\Leftrightarrow \quad \Phi_{t_\ell} \geq \phi_\varepsilon^{(m)}(s), \qquad && s \in [t_\ell, t_{\ell+1}),\ 0 \leq \ell \leq m-1, \\ \gamma(s) \geq -\varepsilon \quad &\Leftrightarrow \quad \Phi_{t_\ell} \geq \phi_\varepsilon(s), && s \in [t_\ell, t_{\ell+1}),\ \ell \geq 0, \end{aligned}$$

which imply that ε-optimal stopping rules $\sigma_\varepsilon^{(m)}(t)$ in (5.4) and $\sigma_\varepsilon(t)$ in (5.5) can be written as

$$\begin{aligned} \sigma_\varepsilon^{(m)}(t) &= \min\Big\{t \leq s \leq t_m;\ \sum_{\ell=0}^{m-1} 1_{[t_\ell, t_{\ell+1})}(s)\Phi_{t_\ell} \geq \phi_\varepsilon^{(m)}(s)\Big\}, \qquad 0 \leq t \leq t_m, \\ \sigma_\varepsilon(t) &= \min\Big\{s \geq t;\ \sum_{\ell=0}^{\infty} 1_{[t_\ell, t_{\ell+1})}(s)\Phi_{t_\ell} \geq \phi_\varepsilon(s)\Big\}, \qquad t \geq 0. \end{aligned} \tag{6.3}$$

Proposition 6.1. *For every $m \geq 1$, $\varepsilon \geq 0$, and $0 \leq s \leq t_m$, the sequence $(\phi_\varepsilon^{(m)}(s))_{m\geq 1}$ is increasing. Moreover, $\phi_\varepsilon(s) = \lim_{m\to\infty} \uparrow \phi_\varepsilon^{(m)}(s)$ for every $\varepsilon \geq 0$ and $s \geq 0$.*

Proof of Proposition 6.1. Since $(v_\ell^{(m)})_{m\geq 1}$ is a decreasing sequence, which converges uniformly to v_ℓ for $\ell \geq 0$, we have $\phi_\varepsilon^{(k)}(s) \leq \phi_\varepsilon^{(m)}(s) \leq \phi_\varepsilon(s)$ for $0 \leq k \leq m-1$ and $t_\ell \leq s < t_{\ell+1}$. Hence, $(\phi_\varepsilon^{(m)}(s))_{m\geq 1}$ is increasing, and $\lim_{m\to\infty} \phi_\varepsilon^{(m)}(s) \leq \phi_\varepsilon(s)$ for $\varepsilon \geq 0$, $s \geq 0$. For the reverse inequality, $(J_{s-t_\ell} v_{\ell+1})(\Delta t_{\ell+1}, \lim_{m\to\infty} \phi_\varepsilon^{(m)}(s)) = \lim_{k\to\infty} (J_{s-t_\ell} v_{\ell+1}^{(k)})(\Delta t_{\ell+1}, \lim_{m\to\infty} \phi_\varepsilon^{(m)}(s))$ by dominated convergence. Since $\phi \mapsto (J_{s-t_\ell} v_{\ell+1}^{(k)})(\Delta t_{\ell+1}, \phi)$ is increasing and $\lim_{m\to\infty} \phi_\varepsilon^{(m)}(s) \geq \phi_\varepsilon^{(k)}(s)$, the righthand side is greater than or equal to $\lim_{k\to\infty} (J_{s-t_\ell} v_{\ell+1}^{(k)})(\Delta t_{\ell+1}, \phi_\varepsilon^{(k)}(s)) \geq -\varepsilon e^{-\lambda(s-t_\ell)}$, and $\phi_\varepsilon(s) \leq \lim_{m\to\infty} \phi_\varepsilon^{(m)}(s)$. This proves that $\phi_\varepsilon(s) = \lim_{m\to\infty} \phi_\varepsilon^{(m)}(s)$. $\square$

Next we characterize optimal stopping boundaries $\phi_0^{(m)}(s)$, $s \geq 0$ for all $m \geq 0$ and $\phi_0(s)$, $s \geq 0$. For all fixed $\ell \geq 0$ and $m > \ell$, we show that $\lim_{s\uparrow t_{\ell+1}} \phi_0^{(m)}(s) = \lim_{s\uparrow t_{\ell+1}} \phi_0(s) = +\infty$. Moreover, $s \mapsto \phi_0^{(m)}(s)$ and $s \mapsto \phi_0(s)$ on $s \in [t_\ell, t_{\ell+1})$ either strictly increase or first decrease and then increase; in the latter case, they are strictly monotone wherever they do not vanish.

Assumption 6.2. *Let $\Delta t > 0$ be a finite real number and $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be a continuous concave nondecreasing function, which is between $-1/c$ and 0, but does not identically vanish.*

By Theorem 4.6, $v_\ell^{(m)}(\cdot)$, $0 < \ell \leq m-1$ and $v_\ell(\cdot)$, $\ell > 0$ satisfy Assumption 6.2. Define

$$\phi(\Delta t, y, w) = \inf\{\phi \geq 0;\ (J_y w)(\Delta t, \phi) \geq 0\}, \qquad 0 \leq y < \Delta t.$$

Then $\phi_0^{(m)}(s) = \phi(\Delta t_{\ell+1}, s - t_\ell, v_{\ell+1}^{(m)})$ for $s \in [t_\ell, t_{\ell+1})$, $0 \leq \ell \leq m-1$ and $\phi_0(s) = \phi(\Delta t_{\ell+1}, s - t_\ell, v_{\ell+1})$ for $s \in [t_\ell, t_{\ell+1})$ and $\ell \geq 0$, and the analysis of $y \mapsto \phi(\Delta t, y, w)$ on $y \in [0, \Delta t)$ applies to optimal stopping boundaries $s \mapsto \phi_0^{(m)}(s)$, $m > \ell$ and $s \mapsto \phi_0(s)$ on $s \in [t_\ell, t_{\ell+1})$ for $\ell \geq 0$.

Proposition 6.3. *Let $\Delta t > 0$ and $w : \mathbb{R}_+ \mapsto \mathbb{R}$ be as in Assumption 6.2. Then, for every $\phi \geq 0$ and $0 \leq y < \Delta t$, we have $(J_y w)(\Delta t, \phi) \geq 0$ if and only if*

$$\left\{ \begin{array}{c} \phi \geq e^{-\lambda y}\left(1 + \dfrac{\lambda}{c}\right) - 1 \\[2ex] (1+\phi)(\Delta t - y) - \left(\dfrac{1}{\lambda} + \dfrac{1}{c}\right)(e^{-\lambda y} - e^{-\lambda \Delta t}) + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi) \geq 0 \end{array} \right\}. \tag{6.4}$$

Therefore, for every $0 \leq y < \Delta t$, the critical boundary $\phi(\Delta t, y, w)$ equals $\inf\{\phi \geq [e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1]^+;\ (1+\phi)(\Delta t - y) - \left(\frac{1}{\lambda} + \frac{1}{c}\right)(e^{-\lambda y} - e^{-\lambda \Delta t}) + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi) \geq 0\}$, *and* $\underline{\phi}(\Delta, y) \leq \phi(\Delta t, y, w) \leq \overline{\phi}(\Delta t, y)$, *where* $\underline{\phi}(\Delta t, y) = [e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1]^+$ *and* $\overline{\phi}(\Delta t, y) = \max\{[e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1]^+, (\frac{1}{\lambda} + \frac{1}{c})\frac{e^{-\lambda y} - e^{-\lambda \Delta t}}{\Delta t - y} + \frac{1}{c}\frac{e^{-\lambda \Delta t}}{\Delta t - y} - 1\}$.

Remark 6.4. One can find $\phi(\Delta t, y, w)$ by a binary search on $\left[\underline{\phi}(\Delta t, y), \overline{\phi}(\Delta t, y)\right]$ for $y \in [0, \Delta t)$.

Proof of Proposition 6.3. $0 \leq (J_y w)(\Delta t, \phi)$ implies that (i) $\int_y^r e^{-\lambda u}(\varphi(u, \phi) - \frac{\lambda}{c})\mathrm{d}u \geq 0$ for every $y \leq r < \Delta t$ and (ii) $0 \leq \int_y^{\Delta t} e^{-\lambda u}(\varphi(u, \phi) - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi) = (1+\phi)(\Delta t - y) - (\frac{1}{\lambda} + \frac{1}{c})(e^{-\lambda y - e^{-\lambda \Delta t}}) + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi)$. Dividing both sides of (i) by $r - y$ and letting $r \downarrow y$ give $\phi \geq e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1$, and the inequalities in (6.4) must hold. If ϕ satisfies (6.4), then since $u \mapsto \varphi(u, \phi) \geq \lambda/c$ is increasing, $\int_y^r e^{-\lambda u}(\varphi(u, \phi) - \frac{\lambda}{c})\mathrm{d}u \geq 0$ for every $y \leq r < \Delta t$. Together with (ii), we conclude $(J_y w)(\Delta t, \phi) \geq 0$. The equivalent form of $\phi(\Delta t, y, w)$ follows from (6.4). The lower bound $\underline{\phi}(\Delta t, y)$ on $\phi(\Delta t, y, w)$ follows from alternative form. Note that since $w(\cdot) \geq -1/c$, $\phi(\Delta t, y, w) \leq \inf\{\phi \geq [e^{-\lambda y}(1+\frac{\lambda}{c})-1]^+;\ (\phi+1)(\Delta t-y)-(\frac{1}{\lambda}+\frac{1}{c})e^{-\lambda y}+\frac{1}{\lambda}e^{-\lambda \Delta t} \geq 0\} = \overline{\phi}(\Delta t, y)$. ☐

Lemma 6.5. *Let $\Delta t > 0$ and $w(\cdot)$ be as in Assumption 6.2. Then $(Kw)(\Delta t, \phi) < 0$ for $\phi \geq 0$.*

Proof. Recall that $\jmath(\Delta t, \phi, z)$ in (4.3) is given by (1.2), and $\lim_{|z|\to\infty, \mu z>0} \jmath(\Delta t, \phi, z) = \infty$ and $\lim_{|z|\to\infty, \mu z<0} \jmath(\Delta t, \phi, z) = 0$ by the monotone convergence and bounded convergence theorems, respectively. Since $w \not\equiv 0$ is increasing, there is some $\bar{\phi} > 0$ such that $w(\phi) \leq w(\bar{\phi}) < 0$ for every $\phi < \bar{\phi}$. Then for all $\phi \geq 0$, there is some $\bar{z} = \bar{z}(\Delta t, \phi)$ such that $\jmath(\Delta t, \phi, z) < \bar{\phi}$ for $|z| > \bar{z}$ and $z\mu < 0$, and $(Kw)(\Delta t, \phi) \leq (\int_{-\infty}^{-\bar{z}} + \int_{\bar{z}}^{\infty}) w(\jmath(\Delta t, \phi, z)) \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}} \mathrm{d}z \leq w(\bar{\phi}) \int_{-\infty}^{-\bar{z}} \frac{\exp\{-z^2/2\}}{\sqrt{2\pi}} \mathrm{d}z < 0$. ☐

Lemma 6.6. *Let $\Delta t > 0$ and $w(\cdot)$ be as in Assumption 6.2. For every $\phi \geq 0$, there is some $y(\phi) \in [0, \Delta t)$ such that $y \in [y(\phi), \Delta t)$ implies*

$$\phi < \left(\frac{1}{\lambda} + \frac{1}{c}\right) \frac{e^{-\lambda y} - e^{-\lambda \Delta t}}{\Delta t - y} - e^{-\lambda \Delta t} \frac{(Kw)(\Delta t, \phi)}{\Delta t - y} - 1 \quad \textit{and} \quad (J_y w)(\Delta t, \phi) < 0.$$

Proof. Assume that there is some $\phi \geq 0$ and some sequence $(y_n)_{n\geq 1}$ in $[0, \Delta t)$ increasing to Δt such that $\phi \geq \left(\frac{1}{\lambda} + \frac{1}{c}\right) \frac{e^{-\lambda y_n} - e^{-\lambda \Delta t}}{\Delta t - y_n} - e^{-\lambda \Delta t} \frac{(Kw)(\Delta t, \phi)}{\Delta t - y_n} - 1$ for every $n \geq 1$. Note that $\lim_{n\to\infty} \frac{e^{-\lambda y_n} - e^{-\lambda \Delta t}}{\Delta t - y_n} = \lambda e^{-\lambda \Delta t}$ is finite, and Lemma 6.5 implies $(Kw)(\phi) < 0$ and $\lim_{n\to\infty} \frac{(Kw)(\Delta t, \phi)}{\Delta t - y_n} = -\infty$. Then taking limits in the last inequality as $n \to \infty$ gives $\phi \geq \left(\frac{1}{\lambda} + \frac{1}{c}\right) \lambda e^{-\lambda \Delta t} + \infty - 1 = \infty$, which contradicts with the finiteness of ϕ. Finally, $(J_y w)(\Delta t, \phi) < 0$ follows from the first part of Proposition 6.3. ☐

Corollary 6.7. *Let $\Delta t > 0$ and $w(\cdot)$ be as in Assumption 6.2. Then $\lim_{y\uparrow\Delta t} \phi(\Delta t, y, w) = \infty$.*

Proof. For every $\phi \geq 0$, there is some $y(\phi) \in [0, \Delta t)$ such that $(J_y w)(\Delta t, \phi) < 0$ for all $y \in [y(\phi), \Delta t)$ by Lemma 6.6, and therefore, $\underline{\lim}_{y\uparrow\Delta t} \phi(\Delta t, y, w) \geq \phi$. Letting $\phi \uparrow \infty$ proves the result. ☐

Recall from (4.2) that $\int_y^{\Delta t} e^{-\lambda u}(\varphi(u, \phi) - \frac{\lambda}{c}) \mathrm{d}u + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi) = (Jw)(\Delta t, \phi, y, \Delta t)$ for every $\phi \geq 0$ and $0 \leq y < \Delta t$. Since $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) = -e^{-\lambda y}(\varphi(y, \phi) - \frac{\lambda}{c}) = -1 - \phi + (1 + \frac{\lambda}{c})e^{-\lambda y}$, we can rewrite the equivalent form of $\phi(\Delta t, y, \phi)$ given by Proposition 6.3 as

$$\phi(\Delta t, y, \phi) = \inf\left\{\phi \geq 0;\ \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) \leq 0 \text{ and } (Jw)(\Delta t, \phi, y, \Delta t) \geq 0\right\}.$$

Remark 6.8. The mappings $(y, \phi) \mapsto (Jw)(\Delta t, \phi, y, \Delta t)$ and $(y, \phi) \mapsto \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)$ are jointly continuous in $(y, \phi) \in [0, \Delta t) \times \mathbb{R}_+$. Because $\phi(\Delta t, y, w)$ is finite by Proposition 6.3, we have for every $y \in [0, \Delta t)$ that $\phi(\Delta t, y, w) \geq 0$,

$$\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)\bigg|_{\phi=\phi(\Delta t, y, w)} \leq 0 \quad \text{and} \quad (Jw)(\Delta t, \phi(\Delta t, y, w), y, \Delta t) \geq 0.$$

Therefore, $\{(y, \phi(\Delta t, y, w));\ y \in [0, \Delta t)\}$ belongs to the boundary of the closed set

$$\left\{(y, \phi) \in [0, \Delta t);\ \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) \leq 0,\ (Jw)(\Delta t, \phi, y, \Delta t) \geq 0\right\}.$$

Fix $\phi \in \mathbb{R}$. Since $y \mapsto \varphi(y, \phi)$ is strictly increasing, $y \mapsto \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) = -e^{-\lambda y}\left(\varphi(y, \phi) - \frac{\lambda}{c}\right)$ on $y \in \mathbb{R}$ changes its sign exactly once and from positive to negative at $y = y_*(\phi)$ satisfying

$$\varphi(y_*(\phi), \phi) - \frac{\lambda}{c} = 0 \qquad \text{or} \qquad y_*(\phi) = \frac{1}{\lambda} \ln \frac{1 + \frac{\lambda}{c}}{1 + \phi} \in \mathbb{R}.$$

Hence, $y \mapsto (Jw)(\Delta t, \phi, y, \Delta t)$ is strictly increasing on $(-\infty, y_*(\phi)]$ and strictly decreasing on $[y_*(\phi), \infty)$ and has global maximum at $y = y_*(\phi)$. Since $\frac{\partial^2}{\partial y^2}(Jw)(\Delta t, \phi, y, \Delta t) = -\left(1 + \frac{\lambda}{c}\right) \lambda e^{-\lambda y} < 0$, the mapping $y \mapsto (Jw)(\Delta t, \phi, y, \Delta t)$ is also strictly concave.

Because $\phi \mapsto \varphi(u, \phi)$ and $\phi \mapsto (Kw)(\Delta t, \phi)$ are strictly increasing, $\phi \mapsto (Jw)(\Delta t, \phi, y, \Delta t)$ is strictly increasing for every fixed $y \in (-\infty, \Delta t]$. Note $(Jw)(\Delta t, \phi, \Delta t, \Delta t) = e^{-\lambda \Delta t}(Kw)(\Delta t, \phi)$ for every $\phi \in \mathbb{R}$, and the locations of the maximums $\phi \mapsto y_*(\phi)$ form a decreasing function, which is negative for every $\phi < \lambda/c$, and $y_*(\lambda/c) = 0$.

Remark 6.9. Let $\Delta t > 0$ and $w(\cdot)$ be as in Assumption 6.2. The followings will later be useful.

(i) Suppose that $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi_0, y, \Delta t)|_{y=y_0} \leq 0$ for some $y_0 \in (-\infty, \Delta t]$ and $\phi_0 \in \mathbb{R}$. Then $(Jw)(\Delta t, \phi_0, y_0, \Delta t) > (Jw)(\Delta t, \phi_0, y, \Delta t) > (Jw)(\Delta t, \phi, y, \Delta t)$ for every $y \in (y_0, \Delta t)$ and $\phi < \phi_0$, and since $y_*(\phi) < y_*(\phi_0)$ for all $\phi > \phi_0$, $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) < 0$ for all $y \in [y_0, \Delta t)$ and $\phi > \phi_0$.

(ii) Suppose that $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi_0, y, \Delta t)|_{y=y_0} = 0$ for some $y_0 \in (-\infty, \Delta t]$ and $\phi_0 \in \mathbb{R}$. Then $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) > 0$ for every $y \in (-\infty, y_0)$ and $\phi \leq \phi_0$, since $y_*(\phi) > y_*(\phi_0) \equiv y_0$ for every $\phi < \phi_0$ and $y \mapsto (Jw)(\Delta t, \phi, y, \Delta t)$ is strictly increasing on $y \in (-\infty, y_0)$ for $\phi \leq \phi_0$.

Theorem 6.10. *Suppose $\Delta t > 0$ and $w : \mathbb{R}_+ \mapsto \mathbb{R}$ are as in Assumption 6.2. Then $y \mapsto \phi(\Delta t, y, w)$ on $y \in [0, \Delta t)$ is either strictly increasing everywhere or first decreases and then increases. It is strictly monotone at every $y \in [0, \Delta t)$ where $\phi(\Delta t, y, w) > 0$. At every $y \in [0, \Delta t)$ where $y \mapsto \phi(\Delta t, y, w)$ is decreasing, it coincides with $y \mapsto e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1$. The mapping $y \mapsto \phi(\Delta t, y, w)$ is strictly increasing on a nonempty open neighborhood in $[0, \Delta t)$ of Δt and is continuous everywhere.*

Proof. Fix some $y_0 \in [0, \Delta t)$ and suppose that $\phi(\Delta t, y_0, w) > 0$. Note that we always have $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\substack{y=y_0 \\ \phi=\phi(\Delta t, y_0, w)}} \leq 0$.

Case I. Suppose $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\substack{y=y_0 \\ \phi=\phi(\Delta t, y_0, w)}} < 0$. Then $(Jw)(\Delta t, \phi(\Delta t, y_0, w), y_0, \Delta t) = 0$; otherwise, $\phi(\Delta t, y_0, w)$ can be further lowered. By Remark 6.9 (i) with $\phi_0 = \phi(\Delta t, y, w)$, $0 = (Jw)(\Delta t, \phi(\Delta t, y_0, w), y_0, \Delta t) > (Jw)(\Delta t, \phi, y, \Delta t)$ for $y \in (y_0, \Delta t)$ and $\phi \leq \phi(\Delta t, y_0, w)$. Thus, $\phi(\Delta t, y, w) > \phi(\Delta t, y_0, w)$ for every $y \in (y_0, \Delta t)$, which, in the meantime, implies that

$$\phi(\Delta t, y, w) > 0 \text{ and } \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)\Big|_{\phi=\phi(\Delta t, y, w)} < 0 \quad \text{for every } y \in (y_0, \Delta t) \tag{6.5}$$

by the second part of Remark 6.9 (i) with $\phi_0 = \phi(\Delta t, y_0, w)$. Now (6.5) implies that Case I applies to every $y \in (y_0, \Delta t)$ and that $y \mapsto \phi(\Delta t, y, w)$ is strictly increasing on $y \in [y_0, \Delta t)$.

Case II. Suppose now that $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\substack{y=y_0 \\ \phi=\phi(\Delta t, y_0, w)}} = 0$. Then by Remark 6.9 (ii) with $\phi_0 = \phi(\Delta t, y_0, w)$, we have $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t) > 0$ for every $y < y_0$ and $\phi \leq \phi(\Delta t, y_0, w)$. Therefore, $\phi(\Delta t, y, w) > \phi(\Delta t, y_0, w)$ for every $y < y_0$. This implies $\phi(\Delta t, y, w) > 0$ for every $y < y_0$, and by Case I $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\phi=\phi(\Delta t, y, w)} = 0$ for every $y < y_0$; otherwise, $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\phi=\phi(\Delta t, y, w)} < 0$ and Case I would imply that $\phi(\Delta t, y, w) < \phi(\Delta t, y_0, w)$ and $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\substack{y=y_0 \\ \phi=\phi(\Delta t, y_0, w)}} < 0$, which contradicts with the starting assumption of Case II. Since now $\phi(\Delta t, y, w) > 0$ and $\frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\phi=\phi(\Delta t, y, w)} = 0$ for every $y < y_0$, Case II applies to every $y < y_0$, and we conclude that $y \mapsto \phi(\Delta t, y, w)$ is strictly decreasing on $y \in [0, y_0]$. For every $y \in [0, y_0]$, $0 = \frac{\partial}{\partial y}(Jw)(\Delta t, \phi, y, \Delta t)|_{\phi=\phi(\Delta t, y, w)} = e^{-\lambda y}(\varphi(y, \phi(\Delta t, y, w)) - \frac{\lambda}{c})$ implies $\lambda/c = \varphi(y, \phi(\Delta t, y, w)) = e^{\lambda y}[\phi(\Delta t, y, w) + 1] - 1$ or $\phi(\Delta t, y, w) = e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1$.

Corollary 6.7 implies that $\lim_{y\uparrow\Delta t}\phi(\Delta t,y,w)=+\infty>\sup_{y\in[0,\Delta t)}e^{-\lambda y}(1+\frac{\lambda}{c})-1=\frac{\lambda}{c}$, which implies that $\phi(\Delta t,y,w)>0$ and $\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)|_{\phi=\phi(\Delta t,y,w)}<0$ for some $y\in[0,\Delta t)$; otherwise, we would have $\phi(\Delta t,y,w)>0$ and $\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)|_{\phi=\phi(\Delta t,y,w)}=0$ for every $y\in[0,\Delta t)$, which would imply that $\phi(\Delta t,y,w)=e^{-\lambda y}(1+\frac{\lambda}{c})-1\le\lambda/c$ for every $y\in[0,\Delta t)$, which contradicts with $\lim_{y\uparrow\Delta t}\phi(\Delta t,y,w)=+\infty$. Therefore, Case I implies that $y\mapsto\phi(\Delta t,y,w)$ is strictly increasing in some nonempty neighborhood in $[0,\Delta t)$ of Δt. Let us now define

$$D=\Big\{y\in[0,\Delta t);\ \phi(\Delta t,y,w)>0\text{ and }\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)\Big|_{\phi=\phi(\Delta t,y,w)}=0\Big\},$$
$$E=\Big\{y\in[0,\Delta t);\ \phi(\Delta t,y,w)>0\text{ and }\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)\Big|_{\phi=\phi(\Delta t,y,w)}<0\Big\}.$$

$E\neq\varnothing$ by the previous paragraph. We also know that $\phi(\Delta t,y,w)\ge[e^{-\lambda y}(1+\frac{\lambda}{c})-1]^+|_{y=0}=\frac{\lambda}{c}>0$, and $\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)|_{\substack{y=0\\ \phi=\phi(\Delta t,0,w)}}\le 0$. If $0\in E$, then $E=[0,\Delta t)$ by Case I above, and $y\mapsto(Jw)(\Delta t,\phi,y,\Delta t)$ is strictly increasing on $y\in[0,\Delta t)$. If $y\mapsto(Jw)(\Delta t,\phi,y,\Delta t)$ is not strictly increasing on $y\in[0,\Delta t)$, then we must have $0\notin E$ and $\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)|_{\substack{y=0\\ \phi=\phi(\Delta t,0,w)}}=0$ and $0\in D$. Hence, D is not empty in this case, either. Define $y_\ell:=\sup D$ and $y_r:=\inf E$. There is a sequence $(y_\ell^{(n)})_{n\ge1}$ in D increasing to y_ℓ, and on every $[0,y_\ell^{(n)})\ni y$, the mapping $y\mapsto\phi(\Delta t,y,w)$ is strictly decreasing. Therefore, $y\mapsto\phi(\Delta t,y,w)$ is strictly decreasing on $y\in[0,y_\ell)$. Similarly, there is a sequence $(y_r^{(n)})_{n\ge1}$ in E decreasing to y_r, and on every $[y_r^{(n)},\Delta t)\ni y$, the mapping $y\mapsto\phi(\Delta t,y,w)$ is strictly increasing. Therefore, $y\mapsto\phi(\Delta t,y,w)$ is strictly increasing on $y\in(y_r,\Delta t)$.

We have $0\le y_\ell\le y_r<\Delta t$, since otherwise $y\mapsto\phi(\Delta t,y,w)$ would be both strictly increasing and strictly decreasing on a nonempty set. If $y_\ell=y_r$, then $y\mapsto\phi(\Delta t,y,w)$ firstly strictly decreases and then strictly increases on $[0,\Delta t)\ni y$. Suppose $y_\ell<y_r$. Take any $y_\ell<y_0<y_r$. We claim that $\phi(\Delta t,y_0,w)=0$. Otherwise, $\phi(\Delta t,y_0,w)>0$, and since $\frac{\partial}{\partial y}(Jw)(\Delta t,\phi,y,\Delta t)|_{\substack{y=y_0\\ \phi=\phi(\Delta t,y_0,w)}}\le 0$, we must have either $y_0\in D$ or $y_0\in E$. If $y_0\in D$, then $y_\ell<y_0$ contradicts with the definition of y_ℓ. If $y_0\in E$, then $y_0<y_r$ contradicts with the definition y_r. Thus, $\phi(\Delta t,y,w)=0$ for every $y\in(y_\ell,y_r)$.

Finally, $y\mapsto\phi(\Delta t,\phi,y,\Delta t)$ is continuous by Remark 6.9. Hence, if $y_\ell=y_r$, then $y_\ell=\lim_{y\uparrow y_\ell}\downarrow\phi(\Delta t,\phi,y,\Delta t)=\lim_{y\downarrow y_r}\downarrow\phi(\Delta t,\phi,y,\Delta t)=y_r$ is the unique global minimum of $y\mapsto\phi(\Delta t,\phi,y,\Delta t)$. If $y_\ell<y_r$, then $\phi(\Delta t,\phi,y,\Delta t)=0$ for every $y\in[y_\ell,y_r]$, $D=[0,y_\ell]$ and $E=[y_r,\Delta t)$. In all of the cases, $y\mapsto\phi(\Delta t,\phi,y,\Delta t)$ is strictly monotone at every $y\in[0,\Delta t)$ where $\phi(\Delta t,\phi,y,\Delta t)>0$. ☐

Remark 6.11. The mapping $y\mapsto\phi(\Delta t,y,w)$ is either strictly increasing, or decreases first and then increases. In the latter case, it is strictly monotone at every point where it is strictly positive. Its decreasing part coincides with $y\mapsto e^{-\lambda y}(1+\frac{\lambda}{c})-1$; see the drawing on the left in Figure 2.

Finally, $\lim_{\Delta t\to\infty}\downarrow\phi(\Delta t,y,w)=[e^{-\lambda y}(1+\frac{\lambda}{c})-1]^+$ for $y\in[0,\Delta t)$, since, for $\phi\ge0$ and $0\le y\le y_0<\Delta t$, $(Jw)(\Delta t,\phi,y,\Delta t)\ge\Delta t-y_0-(\frac{1}{\lambda}+\frac{1}{c})-\frac{1}{c}$, and $\inf_{0\le y\le y_0,\phi\ge0}(Jw)(\Delta t,\phi,y,\Delta t)\ge\Delta t-y_0-\frac{1}{\lambda}-\frac{2}{c}\to\infty$ as $\Delta t\to\infty$; see the drawing on the right in Figure 2.

7. Numerical methods and their illustrations

For any $t\ge0$, if a change-detection alarm has not yet been raised before t, then minimum Bayes risk $\inf_{\tau\in\mathcal{S}(t)}R_\tau(p)$ is given in terms of $\gamma(t)=\sum_{n=0}^\infty 1_{[t_n,t_{n+1})}(t)e^{\lambda(t-t_n)}(J_{t-t_n}v_{n+1})(\Delta t_{n+1},\Phi_{t_n})$ by (5.2) and an optimal alarm after time t may be raised at the stopping time $\sigma_0(t)$ of (6.3), where

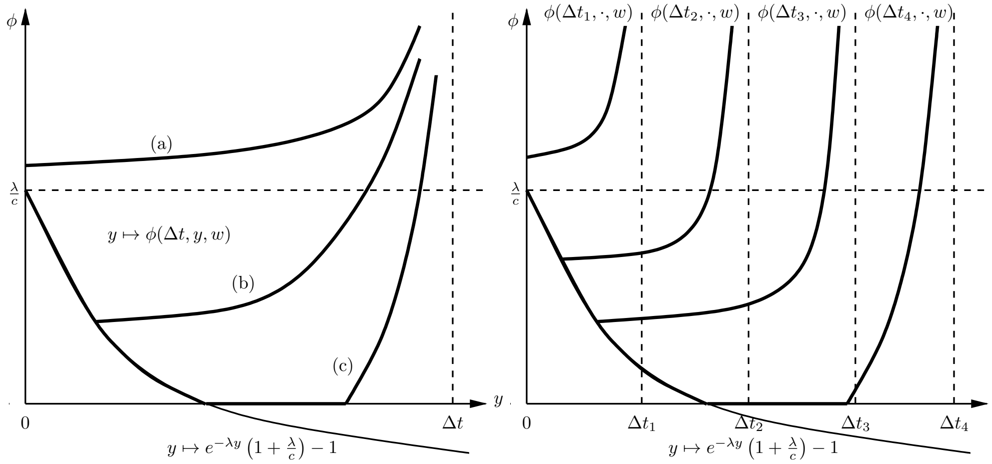


FIGURE 2. The mapping $y \mapsto \phi(\Delta t, y, w)$ on $y \in [0, \Delta t)$ either (a) strictly increases, or (b) firstly decreases strictly and then increases strictly, with unique maximum, or (c) firstly decreases strictly until it hits zero, stays there for a while, and increases strictly afterwards. The drawing on the righthand side illustrates that $\lim_{\Delta t \to \infty} \downarrow \phi(\Delta t, y, w) = [e^{-\lambda y}(1 + \frac{\lambda}{c}) - 1]^+$.

$\phi_0(s) = \inf\{\phi \geq 0;\ (J_{s-t_n} v_{n+1})(\Delta t_{n+1}, \phi) = 0\}$ for every $s \in [t_n, t_{n+1})$ and $n \geq 0$. For the evaluation of the minimum Bayes risks and implementation of optimal alarm times, one needs to calculate the limit $v_n(\cdot) = \lim_{m\to\infty} v_n^{(m)}(\cdot)$ on $\mathbb{R}_+$ for $n \geq 0$ of successive approximations $(v_n^{(m)}(\cdot))_{0 \leq n \leq m}$, $m \geq 0$ defined by (4.5), and functions $(J_{s-t_n} v_{n+1})(\Delta t_{n+1}, \cdot)$ for $s \in [t_n, t_{n+1})$ and $n \geq 0$.

In practice, $v_n(\cdot)$ cannot be calculated exactly, but can be approximated by $v_n^{(m)}(\cdot)$ for some $m \geq n$ with any desired uniform error margin $\varepsilon > 0$ for every $n \geq 0$. Indeed, if $M_n(\varepsilon) = \min\{m \geq n;\ t_m - t_n \geq \frac{1}{\lambda} \ln \frac{1}{c\varepsilon}\}$ for every $n \geq 0$ and $\varepsilon > 0$, then Theorem 4.6 guarantees $\sup_{\phi \geq 0} |v_n(\phi) - v_n^{(m)}(\phi)| \leq \varepsilon$ for every $m \geq M_n(\varepsilon)$, and by Lemma 4.1 (iii)

$$\sup_{\phi \geq 0,\ s \in [t_n, t_{n+1})} \left| (J_{s-t_n} v_{n+1})(\Delta t_{n+1}, \phi) - (J_{s-t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \phi) \right| \\ \leq e^{-\lambda \Delta t_{n+1}} \sup_{\phi \geq 0} |v_{n+1}(\phi) - v_{n+1}^{(m)}(\phi)| \leq e^{-\lambda \Delta t_{n+1}} \varepsilon \quad \text{for every } m \geq M_{n+1}(\varepsilon),$$

which also leads for every $n \geq 0$ to the uniform approximation of $\gamma(t)$, $t \in [t_k, t_{k+1})$ with $\sum_{k=0}^{\infty} 1_{[t_k, t_{k+1})}(t) e^{\lambda(t-t_k)} (J_{t-t_k} v_{k+1}^{(m_{k+1})})(\Delta t_{k+1}, \Phi_{t_k})$ as in

$$\sup_{t \in [t_n, t_{n+1})} \left| \gamma(t) - \sum_{k=0}^{\infty} 1_{[t_k, t_{k+1})}(t) e^{\lambda(t-t_k)} (J_{t-t_k} v_{k+1}^{(m_{k+1})})(\Delta t_{k+1}, \Phi_{t_k}) \right| \leq \varepsilon,$$
$$\text{where } m_k \geq M_k(\varepsilon) \text{ is any fixed finite integer for every } k \geq 0.$$

By replacing $v_{n+1}(\cdot)$ in the definition of the optimal stopping boundary $\phi_0(s)$ for every $s \in [t_n, t_{n+1})$ and $n \geq 0$ with $v_{n+1}^{(m_{n+1})}(\cdot)$ for any fixed $m_{n+1} \geq M_{n+1}(\varepsilon)$, one also gets, instead of impractical optimal alarm times $\sigma_0(t)$ for $t \geq 0$, implementable nearly-optimal alarm times $\sigma_{\varepsilon,\delta}(t)$, $t \geq 0$.

Input. Fix any $\varepsilon > 0$, $\delta \geq 0$, and $n \geq 0$.
Step 1. Let $m_{n+1} = \lceil M_{n+1}(\varepsilon) \rceil$ be the smallest integer $m \geq n+1$ such that $t_m - t_n \geq -(1/\lambda)\ln(c\varepsilon)$.
Step 2. Find $v_n^{(m_{n+1})}(\cdot)$ by calculating $v_{m_{n+1}}^{(m_{n+1})}(\cdot) \equiv 0$ and $v_k^{(m_{n+1})}(\cdot) = (J_0 v_{k+1}^{(m_{n+1})})(\Delta t_{n+1}, \cdot)$ for $k = m_{n+1}-1, \ldots, n+1, n$ successively.
Step 3. Calculate $(J_{t-t_n} v_n^{(m_{n+1})})(\Delta t_{n+1}, \phi)$ for all $t \in [t_n, t_{n+1}), 0 \leq \phi \leq \overline{\phi}(\Delta t_{n+1}, t - t_n)$, where

$$\overline{\phi}(\Delta t, y) = \max\left\{\left[e^{-\lambda y}\left(1 + \frac{\lambda}{c}\right) - 1\right]^+, \left(\frac{1}{\lambda} + \frac{1}{c}\right)\frac{e^{-\lambda y} - e^{-\lambda \Delta t_{n+1}}}{\Delta t_{n+1} - y} + \frac{1}{c}\frac{e^{-\lambda \Delta t}}{\Delta t - y} - 1\right\},$$

and we know that $(J_{t-t_n} v_n^{(m_{n+1})})(\Delta t_{n+1}, \phi) = 0$ for $\phi \geq \overline{\phi}(\Delta t_{n+1}, t - t_n)$.
Step 4. Find $\phi_\delta^{(m_{n+1})}(s) = \min\{\phi \geq 0;\ (J_{s-t_n} v_n^{(m_{n+1})})(\Delta t_{n+1}, \phi) \geq -\delta e^{-\lambda(s-t_n)}\}$ by a binary search on $[0, \overline{\phi}(\Delta t_{n+1}, s - t_n)]$ for every $s \in [t_n, t_{n+1})$.
Output. For every $t \in [t_n, t_{n+1})$, we obtain $|\gamma(t) - e^{\lambda(t-t_n)}(J_{t-t_n} v_{n+1}^{(m_{n+1})})(\Delta t_{n+1}, \Phi_{t_n})| \leq \varepsilon$, and $\phi_\delta^{(m_{n+1})}(t)$, $t \in [t_n, t_{n+1})$ is the critical boundary of $(\varepsilon + \delta)$-optimal rule $\sigma_{\varepsilon,\delta}(t)$, $t \in [t_n, t_{n+1})$.

FIGURE 3. A numerical algorithm to calculate the minimum Bayes risk and the critical boundary of a nearly-optimal optimal stopping rule between observation times.

Proposition 7.1. *Let $M_n(\varepsilon)$ and $\phi_\varepsilon^{(m)}(s)$ be defined as in Theorem 4.6 and (6.2), respectively, for every $n, m \geq 0$, $\varepsilon > 0$, and $0 \leq s \leq t_m$. Fix any $\varepsilon > 0$, $\delta \geq 0$, and $m_n \geq M_n(\varepsilon)$. Define*

$$\phi_{\varepsilon,\delta}(s) := \sum_{n=0}^\infty 1_{[t_n, t_{n+1})}(s)\phi_\delta^{(m_{n+1})}(s), \quad s \geq 0,$$

$$\sigma_{\varepsilon,\delta}(t) := \inf\left\{s \geq t;\ \sum_{n=0}^\infty 1_{[t_n,t_{n+1})}(s)\Phi_{t_n} \geq \phi_{\varepsilon,\delta}(s)\right\}, \quad t \geq 0.$$

Then for every $t \geq 0$, stopping time $\sigma_{\varepsilon,\delta}(t) \in \mathcal{S}(t)$ is $(\varepsilon+\delta)$-optimal for $\inf_{\tau \in \mathcal{S}(t)} R_\tau(\cdot)$ in (5.2) if an alarm has not been raised before time t; namely, $\gamma(t) + \varepsilon + \delta \geq \mathbb{E}_\infty[\int_t^{\sigma_{\varepsilon,\delta}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u | \mathcal{F}_t]$.

Proof. Since $\phi_\delta^{(m)}(s) \leq \phi_\delta(s)$ for $m \geq 0$, $\delta \geq 0$, and $0 \leq s \leq t_m$ by Proposition 6.1, we have $\sigma_{\varepsilon,\delta}(t) \leq \sigma_\delta(t)$ for every $t \geq 0$. Then Corollary 5.9 and the optional sampling imply that the stopped process $\{M(s \wedge \sigma_{\varepsilon,\delta}(t)) \equiv M(s \wedge \sigma_{\varepsilon,\delta}(t) \wedge \sigma_\delta(t)), \mathcal{F}_s;\ s \geq t\}$ is a RCLL $\mathbb{P}_\infty$-martingale, and $\int_0^t e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-rt}\gamma(t) = M(t) = \mathbb{E}_\infty[M(s \wedge \sigma_{\varepsilon,\delta}(t))|\mathcal{F}_t] = \mathbb{E}_\infty[\int_0^{s\wedge\sigma_{\varepsilon,\delta}(t)} e^{-\lambda u}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda(s\wedge\sigma_{\varepsilon,\delta}(t))}\gamma(s \wedge \sigma_{\varepsilon,\delta}(t))|\mathcal{F}_t]$, which gives $\gamma(t) = \mathbb{E}_\infty[\int_t^{s\wedge\sigma_{\varepsilon,\delta}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + e^{-\lambda[(s\wedge\sigma_{\varepsilon,\delta}(t))-t]}\gamma(s \wedge \sigma_{\varepsilon,\delta}(t))|\mathcal{F}_t]$. As $s \uparrow \infty$, $\gamma(t) \geq \mathbb{E}_\infty[\int_t^{\sigma_{\varepsilon,\delta}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u + 1_{\{\sigma_{\varepsilon,\delta}(t)<\infty\}}e^{-\lambda[\sigma_{\varepsilon,\delta}(t)-t]}\gamma(\sigma_{\varepsilon,\delta}(t))|\mathcal{F}_t]$ by Fatou.

Since $m_{n+1} \geq M_{n+1}(\varepsilon)$, we have $\gamma(\sigma_{\varepsilon,\delta}(t)) \geq \gamma^{(m_{n+1})}(\sigma_{\varepsilon,\delta}(t)) - \varepsilon$ on $\{\sigma_{\varepsilon,\delta}(t) \in [t_n, t_{n+1})\}$ by Proposition 5.2, and $1_{\{\sigma_{\varepsilon,\delta}(t)<\infty\}}\gamma(\sigma_{\varepsilon,\delta}(t)) \geq \sum_{n=0}^\infty 1_{[t_n,t_{n+1})}(\sigma_{\varepsilon,\delta}(t))\left[\gamma^{(m_{n+1})}(\sigma_{\varepsilon,\delta}(t)) - \varepsilon\right] \geq -\delta - \varepsilon$, because the definition of $\sigma_{\varepsilon,\delta}(t)$ implies that $\Phi_{t_n} \geq \phi_{\varepsilon,\delta}(\sigma_{\varepsilon,\delta}(t)) = \phi_\delta^{(m_{n+1})}(\sigma_{\varepsilon,\delta}(t))$ on $\{\sigma_{\varepsilon,\delta}(t) \in [t_n, t_{n+1})\}$, which leads to $\gamma^{(m_{n+1})}(\sigma_{\varepsilon,\delta}(t)) \geq -\delta$ for every $n \geq 0$ by definition of $\phi_\delta^{(m_{n+1})}(\cdot)$ in (6.2). Thus, $\gamma(t) \geq \mathbb{E}_\infty[\int_t^{\sigma_{\varepsilon,\delta}(t)} e^{-\lambda(u-t)}(\Phi_u - \frac{\lambda}{c})\mathrm{d}u|\mathcal{F}_t] - \varepsilon - \delta$, which completes the proof. □

Figure 3 gives a numerical algorithm to find the nearly optimal stopping rules of Proposition 7.1. For the numerical examples, we suppose that the lengths of successive observation intervals cycle through some $p \geq 1$ positive constants $\Delta t_1, \ldots, \Delta t_p$. Figures 4 and 5 demonstrate the outputs of the algorithm in Figure 3 applied to Wiener disorder problems with $p = 1$, namely, equally Δt-spaced observation intervals. Four columns of Figure 4 display for $\Delta t = 1, 10, 20, 32$ the successive

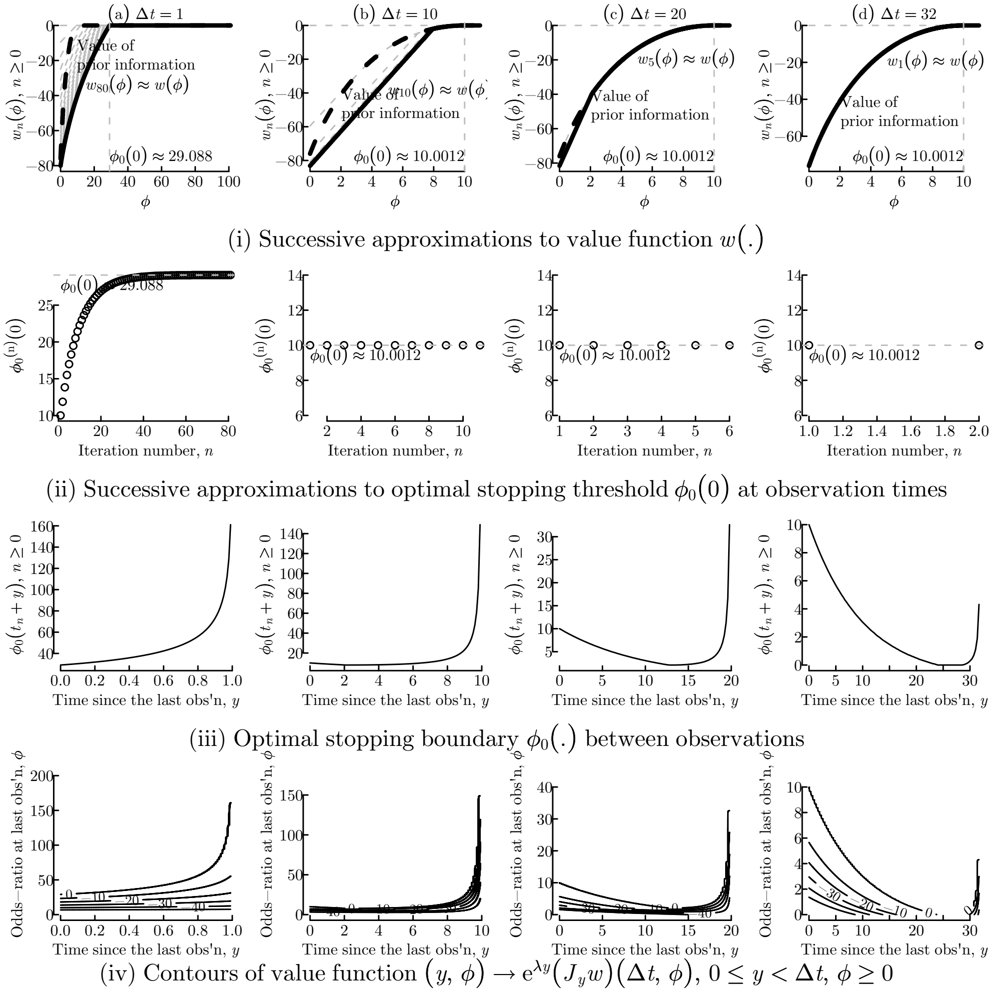


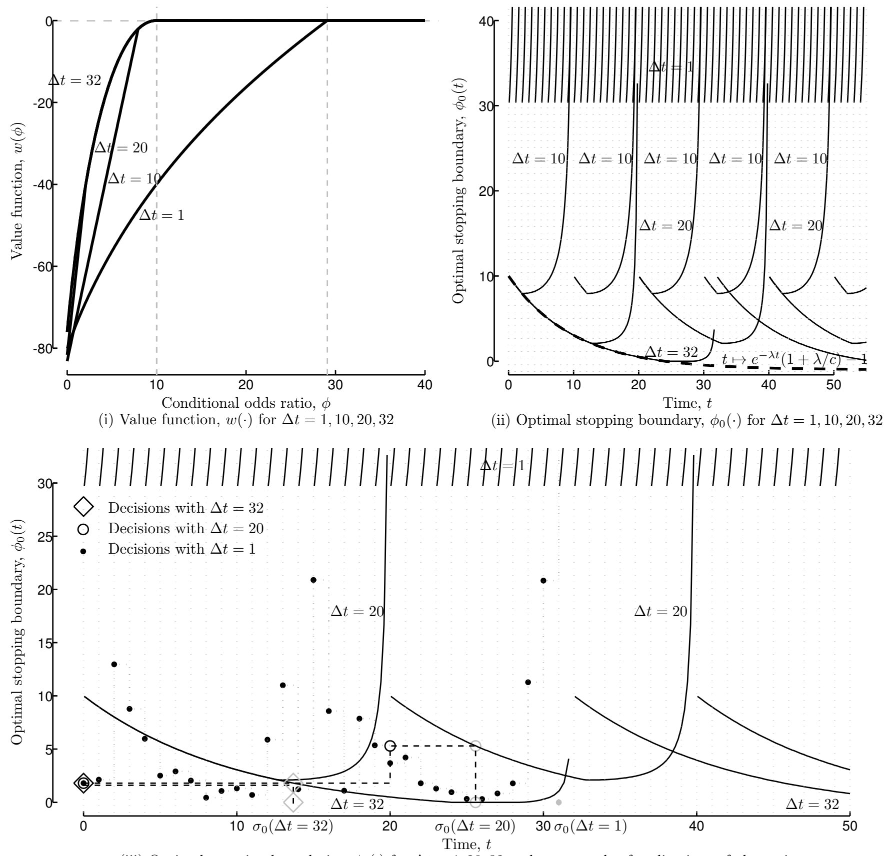
FIGURE 4. Wiener disorder problem with equally Δt-spaced observation intervals, $\lambda = 0.1$, $\mu = 1$, and $c = 0.01$.

approximations of value function $w(\cdot)$, successive approximations of optimal stopping threshold $\phi_0(0)$ at observation times, optimal stopping boundary $\phi_0(y)$, $y \in [0, \Delta t)$ between observations, and the contours of value function $(y, \phi) \mapsto e^{\lambda y} J_y(\Delta t, \phi)$ where y is the time since the last observation and ϕ is the conditional odds-ratio calculated at the last observation time.

As time Δt between observations increases, the number of iterations needed for an accurate value-function approximation decreases, the value functions increase pointwise, the optimal stopping regions expand, and optimal continuation regions shrink; compare the graphs along the row in Figure 4 (i) and the superposition of the value functions in Figure 5 (i).

It is never optimal to raise an alarm at any observation time when the conditional odds-ratio is less than $\lambda/c = 10$. If the conditional odds-ratio is greater than or equal to λ/c, waiting may still be favorable (with the hope that the conditional odds-ratio will jump into the advantageous region $[0, \lambda/c)$ after a favorable observation), but this possibility vanishes rapidly as time Δt between observations is increased; compare the graphs in Figure 4 (ii).

As pointed out by Theorem 6.10, optimal stopping boundary between two adjacent observation times either increases strictly or first decreases and then increases; it is strictly monotone whenever



(i) Value function, $w(\cdot)$ for $\Delta t = 1, 10, 20, 32$

(ii) Optimal stopping boundary, $\phi_0(\cdot)$ for $\Delta t = 1, 10, 20, 32$

(iii) Optimal stopping boundaries, $\phi_0(\cdot)$ for $\Delta t = 1, 20, 32$ and an example of realizations of alarm times

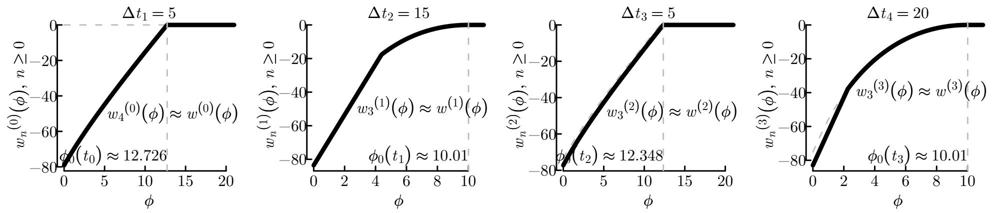
FIGURE 5. Wiener disorder problem with equally Δt-spaced observation intervals, $\lambda = 0.1$, $\mu = 1$, and $c = 0.01$.

it does not vanish. As time Δt between observations increases, optimal stopping boundary tends to decrease more with the passing time, and this encourages early stopping in order to curb the increasing risk of failing to detect the disorder time; see the graphs in Figure 4 (iii) for numerical evidence and Remark 6.11 for rigorous justification.

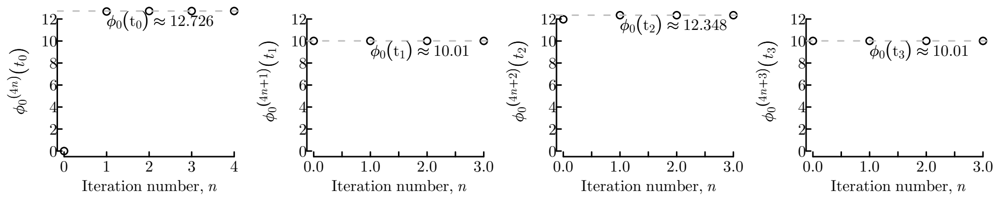
In order to forgo the contribution of a very near new observation in resolving the ambiguity about the unobservable disorder time, the odds of that the disorder must have already happened must intuitively be very large. Therefore, one expects that the optimal stopping boundary increases to infinity as time of next observation is nearing. All of the graphs in Figure 4 (iii) confirm this intuition, which was also analytically established in Corollary 6.7.

Finally, the approximate contours of the value function $(y, \phi) \mapsto e^{\lambda y}(J_y w)(\Delta t, \phi)$ in Figure 4 (iv) help visualize the changes in the process $\gamma(t) = e^{\lambda(t-t_n)}(J_{t-t_n} w)(\Delta t, \Phi_{t_n})$, $t \in [t_n, t_{n+1})$ for every $n \geq 0$, which is fundamental for the calculation of minimum Bayes risks $\inf_{\tau \in \mathcal{S}(t)} R_\tau(\cdot)$ in (5.2) for every $t \geq 0$ and is essentially the conditionally minimum Bayes risk given the past observations if an alarm has not yet been raised before time $t \geq 0$

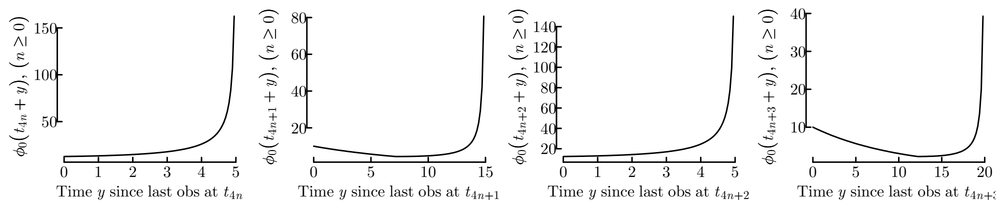
Suppose that the optimal stopping boundary is strictly increasing. Then $t \mapsto \gamma(t)$ is strictly decreasing on $t \in [t_n, t_{n+1})$ if $\gamma(t_n) = (J_0 w)(\Delta t, \Phi_{t_n}) = w(\Phi_{t_n}) < 0$ or $\Phi_{t_n} < \phi_0(0)$. Otherwise, it remains at zero for a while before it starts to decrease; see Figure 4 (iv) for $\Delta t = 1$. Suppose now that the optimal stopping boundary first decreases and then increases. If $\Phi_{t_n} \leq \lambda/c$, then $t \mapsto \gamma(t)$,



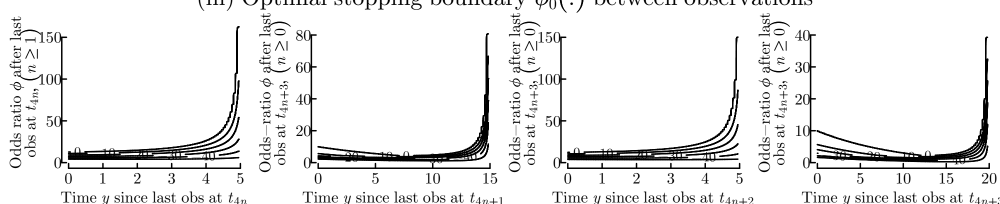
(i) Successive approximations to value function $w^{(q-1)}(.) = v_{4n+q-1}(.)$, $1 \leq q \leq 4$, $n \geq 0$



(ii) Successive approximations to $\phi_0(t_{q-1})$ at observation times t_{4n+q-1}, $1 \leq q \leq 4$



(iii) Optimal stopping boundary $\phi_0(.)$ between observations



(iv) Contours of value function $(y, \phi) \to e^{\lambda y}(J_y w)(\Delta t_q, \phi)$, $0 \leq y < \Delta t_q$, $\phi \geq 0$, $1 \leq q \leq 4$

FIGURE 6. Wiener disorder problem with unequally spaced observation intervals, the lengths of which cycle trough $\Delta t_1 = 5$, $\Delta t_2 = 15$, $\Delta t_3 = 5$, $\Delta t_4 = 20$, and $\lambda = 0.1$, $\mu = 1$, and $c = 0.01$.

$t \in [t_n, t_{n+1})$ strictly increases; if it reaches to zero, it may stay there for a while, but it always eventually starts to strictly decrease. Otherwise, it remains at zero for a while before it starts to strictly decrease; see Figure 4 (iv) for $\Delta t = 10, 20, 32$.

Figures 4 (iii) and 5 (ii) show that the following three cases are possible:

(i) An optimal alarm may sound only at some observation time. If the optimal stopping boundary is increasing, then, whenever postponing an alarm is optimal, it remains so at least until after the next observation. If $\Delta t = 1$, then the optimal stopping boundary is increasing, and an alarm may be raised only at observation times $t_n = n\Delta t$, $n \geq 0$.

(ii) An optimal alarm may sometimes sound strictly between some observation times. If the optimal stopping boundary is not strictly increasing, then it must firstly decrease and then increase, and it is strictly monotone wherever it does not vanish. Moreover, it starts from level $\lambda/c > 0$ and its decreasing portion always coincides with $t \mapsto e^{-\lambda t}(1+\lambda/c)-1$ independently of time Δt between observations; see Figure 5 (ii) and Theorem 6.10. Therefore, an optimal alarm time falls strictly between two observation times, if the conditional odds-ratio calculated at the last observation lies between the minimum of the optimal stopping boundary and its initial value, λ/c. Postponing an alarm at least until after the next observation is still optimal if the conditional odds-ratio at the last observation is below the minimum of the optimal stopping boundary. If it is at or above λ/c, then it is optimal to raise the alarm immediately after the last observation time. If $\Delta t = 10$ or 20, all three optimal alarm types may appear with positive probability.

(iii) An optimal alarm will always be set by the next observation time. This is a special case of (ii), which occurs if the optimal stopping boundary vanishes some time between two observations. If $\Delta t = 32$, then optimal alarm will always sound before the next observation.

It is important to remember that one can always tell with certainty if optimal alarm will sound before the next observation, and its precise time if it will. Figure 5 (iii) shows the sample paths of conditional odds-ratio processes Φ and optimal alarm times for different times between observations, $\Delta t = 1, 20, 32$. Observe also that if optimal stopping boundary is not strictly increasing, then it is not differentiable at its minimum, since its left derivative at the minimum is the derivative of the strictly decreasing function $t \mapsto e^{-\lambda t}(1 + \frac{\lambda}{c}) - 1$, which is always strictly negative.

Finally Figure 6 illustrates the outcome of the numerical algorithm described in Figure 3 for the Wiener disorder problem with unequal observation intervals, the lengths of which cycle through $\Delta t_1 = 5$, $\Delta t_2 = 15$, $\Delta t_3 = 5$, $\Delta_4 = 20$. Optimal stopping boundaries between observations are strictly increasing over $[t_{4n}, t_{4n+1}) \cup [t_{4n+2}, t_{4n+3})$, but firstly decreases and then increases strictly over $[t_{4n+1}, t_{4n+2}) \cup [t_{4n+3,4n+4})$ for every $n \geq 0$. Thus, if the alarm is not set before or at time t_{4n} (respectively, t_{4n+2}), then it is optimal to wait at least until time t_{4n+1} (respectively, t_{4n+3}) for every $n \geq 0$. However, an optimal alarm may sound some time strictly between t_{4n+1} and t_{4n+2} or strictly between t_{4n+3} and t_{4n+4} for some $n \geq 0$.

8. Calculation of false alarm probabilities, variational and general Bayesian formulations

In this section, we shall show how one can calculate the probability of false alarm

$$\text{pfa}(p) = \mathbb{P}\left\{\sigma_0(0) < \Theta \,\middle|\, \Phi_0 = \frac{p}{1-p}\right\}, \quad 0 \leq p < 1 \tag{8.1}$$

for the optimal alarm time $\sigma_0(0) = \min\{s \geq 0;\ \sum_{\ell=0}^{\infty} 1_{[t_\ell, t_{\ell+1})}(s)\Phi_{t_\ell} \geq \phi_0(s)\}$ of (6.3), which is by Proposition 2.1 and Theorem 5.7 an optimal stopping time for the problem in (2.5) and has the smallest Bayes risk $R(p)$ of (2.1) for every $0 \leq p < 1$.

Because $\phi_0(s)$ equals $\phi(\Delta t_{\ell+1}, s - t_\ell, v_{\ell+1})$ for $s \in [t_\ell, t_{\ell+1})$ and $\ell \geq 0$, recall from Remark 6.11 and Figures 1 and 2 that the critical boundary $s \mapsto \phi_0(s)$ is continuous on every observation interval $[t_\ell, t_{\ell+1})$, $\ell \geq 0$, either increases strictly everywhere or first decreases along $s \mapsto e^{-\lambda(s-t_\ell)}(1 + \frac{\lambda}{c}) - 1$ and then increases. Let us define the minimum

$$\phi_{0,\ell} := \min\{\phi_0(s);\ s \in [t_\ell, t_{\ell+1})\}, \quad \ell \geq 0 \tag{8.2}$$

of $\phi_0(\cdot)$ on the observation interval $[t_\ell, t_{\ell+1})$ for every $\ell \geq 0$. Note that, when at time $t = 0$ the surveillance starts, one can determine the exact time $\sigma_0(0)$ of the optimal alarm by only knowing the values $\phi_{0,\ell}$, $\ell \geq 0$. Indeed, we have

(8.3)
$$\sigma_0(0) = \sigma_0(t_\ell) = \left\{ \begin{array}{ll} t_\ell, & \Phi_{t_\ell} \geq \left(\dfrac{\lambda}{c} \vee \phi_{0,\ell}\right) \\ t_\ell - \dfrac{1}{\lambda} \log \dfrac{1+\Phi_{t_\ell}}{1+\frac{\lambda}{c}}, & \left(\dfrac{\lambda}{c} \wedge \phi_{0,\ell}\right) \leq \Phi_{t_\ell} < \dfrac{\lambda}{c} \\ \sigma_0(t_{\ell+1}), & \Phi_{t_\ell} < \phi_{0,\ell} \end{array} \right\} \text{ on } \{\sigma_0(0) \geq t_\ell\},\ \forall \ell \geq 0,\ \mathbb{P}\text{-a.s.}$$

Let us introduce the "conditional probability of false alarm" process

$$\begin{aligned} \text{(8.4)} \quad \mathrm{CPFA}_n &:= \mathbb{P}\{\sigma_0(t_n) < \Theta \mid \mathcal{F}_{t_n},\, \Theta > t_n\} \\ &= \frac{\mathbb{E}_\infty\left[Z_{t_n \wedge \Theta} 1_{\{\sigma_0(t_n) < \Theta\}} \mid \mathcal{F}_{t_n},\ \Theta > t_n\right]}{\mathbb{E}_\infty\left[Z_{t_n \wedge \Theta} \mid \mathcal{F}_{t_n},\ \Theta > t_n\right]} = \mathbb{P}_\infty\{\sigma_0(t_n) < \Theta \mid \mathcal{F}_{t_n},\, \Theta > t_n\}, \quad n \geq 0, \end{aligned}$$

where $Z_{t_n \wedge \Theta} = 1$ $\mathbb{P}$-a.s. on $\{\Theta > t_n\}$. Note that $\mathrm{CPFA}_0 = \mathbb{P}\{\sigma_0(0) < \Theta \mid \mathcal{F}_0, \Theta > 0\} = \frac{\mathrm{pfa}(p)}{1-p}\big|_{p=\frac{\Phi_0}{1+\Phi_0}}$. We shall show that $\mathbb{P}$-a.s. $\mathrm{CPFA}_n = \mathrm{cpfa}_n(\Phi_{t_n})$ for every $n \geq 0$ for some sequence $(\mathrm{cpfa}_n(\cdot))_{n\geq 0}$ of $[0,1]$-valued functions, each element $\mathrm{cpfa}_n(\cdot)$ of which is the pointwise uniform limit of some suitable successive approximations $\left(\mathrm{cpfa}_n^{(m)}(\cdot)\right)_{m\geq 0}$. We will then be able calculate the probability of false alarm by

$$\mathrm{pfa}(p) = (1-p)\,\mathrm{cpfa}_0\left(\frac{p}{1-p}\right) = (1-p) \lim_{m\to\infty} \mathrm{cpfa}_0^{(m)}(p) \quad \text{for every } 0 \leq p < 1.$$

To calculate the conditional probability in (8.4), we shall need the following lemma.

Lemma 8.1. *Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, X be a bounded random variable, $\mathcal{F}$ be a sub σ-algebra of $\mathcal{H}$, and A be an $\mathcal{F}$-measurable event. Then*

$$\mathbb{E}[X \mid \mathcal{F} \wedge \sigma(A)] = \frac{\mathbb{E}[X 1_A \mid \mathcal{F}]}{\mathbb{P}(A \mid \mathcal{F})} 1_A + \frac{\mathbb{E}[X 1_{\Omega\setminus A} \mid \mathcal{F}]}{\mathbb{P}(\Omega \setminus A \mid \mathcal{F})} 1_{\Omega\setminus A}.$$

Therefore, $\mathbb{E}[X \mid \mathcal{F}, A] = \mathbb{E}[X 1_A \mid \mathcal{F}]/\mathbb{P}(A \mid \mathcal{F})$.

Proof. Take a bounded $\mathcal{F}$-measurable random variable Y and constants a and b. Then

$$\begin{aligned} \mathbb{E}\left[Y(a1_A + b1_{\Omega\setminus A})\,\mathbb{E}[X \mid \mathcal{F} \vee \sigma(A)]\right] &= \mathbb{E}[Y(a1_A + b1_{\Omega\setminus A})X] = a\mathbb{E}[YX1_A] + b\mathbb{E}[YX1_{\Omega\setminus A}] \\ &= a\mathbb{E}[Y\mathbb{E}[X1_A \mid \mathcal{F}]] + b\mathbb{E}[Y\mathbb{E}[X1_{\Omega\setminus A} \mid \mathcal{F}]] \\ &= a\mathbb{E}\left[Y\frac{\mathbb{E}[X1_A \mid \mathcal{F}]}{\mathbb{P}(A \mid \mathcal{F})}\mathbb{P}(A \mid \mathcal{F})\right] + b\mathbb{E}\left[Y\frac{\mathbb{E}[X1_{\Omega\setminus A} \mid \mathcal{F}]}{P(\Omega \setminus A \mid \mathcal{F})}P(\Omega \setminus A \mid \mathcal{F})\right] \\ &= a\mathbb{E}\left[\mathbb{E}\left(Y\frac{\mathbb{E}[X1_A \mid \mathcal{F}]}{\mathbb{P}(A \mid \mathcal{F})}1_A \middle| \mathcal{F}\right)\right] + b\mathbb{E}\left[\mathbb{E}\left(Y\frac{\mathbb{E}[X1_{\Omega\setminus A} \mid \mathcal{F}]}{P(\Omega \setminus A \mid \mathcal{F})}1_{\Omega\setminus A} \middle| \mathcal{F}\right)\right] \\ &= \mathbb{E}\left[Y\frac{\mathbb{E}[X1_A \mid \mathcal{F}]}{\mathbb{P}(A \mid \mathcal{F})}a1_A + \frac{\mathbb{E}[X1_{\Omega\setminus A} \mid \mathcal{F}]}{P(\Omega \setminus A \mid \mathcal{F})}b1_{\Omega\setminus A}\right] \\ &= \mathbb{E}\left[Y(a1_A + b1_{\Omega\setminus A})\left(\frac{\mathbb{E}[X1_A \mid \mathcal{F}]}{\mathbb{P}(A \mid \mathcal{F})}1_A + \frac{\mathbb{E}[X1_{\Omega\setminus A} \mid \mathcal{F}]}{P(\Omega \setminus A \mid \mathcal{F})}1_{\Omega\setminus A}\right)\right], \end{aligned}$$

which completes the proof of the lemma. ☐

Because $\mathcal{F}_{t_n}$ and Θ are independent under $\mathbb{P}_\infty$, and $\sigma_0(t_n)$ in (8.3) depends on $\mathcal{F}_{t_n}$ through the future values of the Markov process $(\Phi_t, t)_{t\geq t_n}$, Lemma 8.1 implies that

$$\begin{aligned}\mathrm{CPFA}_n &= \mathbb{P}_\infty\{\sigma_0(t_n) < \Theta \mid \mathcal{F}_{t_n},\, \Theta > t_n\} = \frac{\mathbb{P}_\infty\{\sigma_0(t_n) < \Theta \mid \mathcal{F}_{t_n}\}}{\mathbb{P}_\infty\{\Theta > t_n \mid \mathcal{F}_{t_n}\}} = \frac{\mathbb{P}_\infty\{\sigma_0(t_n) < \Theta \mid \Phi_{t_n}\}}{\mathbb{P}_\infty\{\Theta > t_n\}} \\ &= \frac{(1+\Phi_0)^{-1}\mathbb{E}_\infty\left[e^{-\lambda\sigma_0(t_n)} \mid \Phi_{t_n}\right]}{(1+\Phi_0)^{-1}e^{-\lambda t_n}} = \mathbb{E}_\infty\left[e^{-\lambda(\sigma_0(t_n)-t_n)}\middle|\Phi_{t_n}\right] = \mathrm{cpfa}_n(\Phi_{t_n}),\end{aligned}$$

where for every $\phi \geq 0$ and $n \geq 0$ we define

$$\begin{aligned}\mathrm{cpfa}_n(\phi) := \mathbb{E}_\infty\left[e^{-\lambda(\sigma_0(t_n)-t_n)}\middle|\Phi_{t_n}=\phi\right] &= 1_{[(\frac{\lambda}{c}\vee\phi_{0,n}),\infty)}(\phi) + \frac{1+\phi}{1+\frac{\lambda}{c}}1_{[(\frac{\lambda}{c}\wedge\phi_{0,n}),\frac{\lambda}{c})}(\phi) \\ &\quad + 1_{[0,\phi_{0,\phi_{0,n}})}(\phi)e^{-\lambda\Delta t_{n+1}}\mathbb{E}_\infty[\mathrm{cpfa}_{n+1}(\Phi_{t_{n+1}}) \mid \Phi_{t_n}=\phi],\end{aligned}$$

and the second equality follows from the second equality in (8.3). Using the explicit dynamics in (1.2) of Φ and the definition in (4.3) of K operator, we can evaluate the expectation

$$\begin{aligned}\mathbb{E}_\infty[\mathrm{cpfa}_{n+1}(\Phi_{t_{n+1}}) \mid \Phi_{t_n}=\phi] &= \mathbb{E}_\infty\left[\mathrm{cpfa}_{n+1}\left(\jmath\left(\Delta t_{n+1},\phi,\frac{\Delta X_{n+1}}{\sqrt{\Delta t_{n+1}}}\right)\right)\middle|\,\Phi_{t_n}=\phi\right] \\ &= \int_\infty^\infty \mathrm{cpfa}_{n+1}\big(\jmath(\Delta t_{n+1},\phi,z)\big)\frac{e^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}z = (K\mathrm{cpfa}_{n+1})(\Delta t_{n+1},\phi).\end{aligned}$$

The next proposition summarizes our findings up to now.

Proposition 8.2. *Let L be the operator on bounded functions $w:\mathbb{R}^+\mapsto[0,1]$ defined by*

$$(Lw)(y,\Delta t,\phi) = 1_{[(\frac{\lambda}{c}\vee y),\infty)}(\phi) + \frac{1+\phi}{1+\frac{\lambda}{c}}1_{[(\frac{\lambda}{c}\wedge y),\frac{\lambda}{c})}(\phi) + 1_{[0,y)}(\phi)e^{-\lambda\Delta t}(Kw)(\Delta t,\phi)$$

for every $y,\Delta t,\phi \geq 0$. Then the probability of false alarm $\mathrm{pfa}(p)$ *in (8.1) equals* $(1-p)\mathrm{cpfa}_0(\frac{p}{1-p})$ *for every $0 \leq p < 1$, where* $\mathrm{cpfa}_n(\cdot)$*, $n \geq 0$ are unique $[0,1]$-valued functions satisfying* $\mathrm{cpfa}_n(\phi) = (L\mathrm{cpfa}_{n+1})(\phi_{0,n},\Delta t_{n+1},\phi)$ *for every $\phi \geq 0$ and $n \geq 0$, and each $\phi_{0,n}$ is defined by (8.2).*

To prove the uniqueness of $\mathrm{cpfa}_n(\cdot)$, $n \geq 0$, suppose that $f_n(\cdot)$, $n \geq 0$ be a sequence of $[0,1]$-valued functions satisfying $f_n(\phi) = (Lf_{n+1})(\phi_{0,n},\Delta t_n,\phi)$ for every $\phi \geq 0$ and $n \geq 0$. Then we have $\mathrm{cpfa}_n(\phi) - f_n(\phi) = 1_{[0,\phi_{0,\phi_{0,n}})}(\phi)e^{-\lambda\Delta t_{n+1}}(K(\mathrm{cpfa}_{n+1}-f_{n+1}))(\Delta t_{n+1},\phi) \leq e^{-\lambda\Delta t_{n+1}}\|\mathrm{cpfa}_{n+1}-f_{n+1}\|$ for every $\phi \geq 0$ and $n \geq 0$. Similarly, $f_n(\phi) - \mathrm{cpfa}_n(\phi) =\leq e^{-\lambda\Delta t_{n+1}}\|\mathrm{cpfa}_{n+1} - f_{n+1}\|$ for every $\phi \geq 0$ and $n \geq 0$. Therefore, $\|\mathrm{cpfa}_n - f_n\| \leq e^{-\lambda\Delta t_{n+1}}\|\mathrm{cpfa}_{n+1} - f_{n+1}\|$ for every $n \geq 0$. Reiterating this inequality $m \geq 1$ times leads to $\|\mathrm{cpfa}_n - f_n\| \leq e^{-\lambda(\Delta t_{n+1}+\ldots+\Delta t_{n+m})}\|\mathrm{cpfa}_{n+m} - f_{n+m}\| \leq e^{-\lambda(\Delta t_{n+1}+\ldots+\Delta t_{n+m})}$ for every $n \geq 0$. Letting $m \uparrow \infty$ implies that $\|\mathrm{cpfa}_n - f_n\| = 0$ for every $n \geq 0$.

To calculate $\mathrm{cpfa}_n(\cdot)$ for every $n \geq 0$, we define the successive approximations

$$\text{(8.5)}\quad \mathrm{cpfa}_m^{(m)}(\cdot) \equiv 1, \quad \mathrm{cpfa}_n^{(m)}(\phi) := (L\mathrm{cpfa}_{n+1}^{(m)})(\phi_{0,n},\Delta t_{n+1},\phi), \;\; \phi \geq 0, \;\; 0 \leq n \leq m-1.$$

Proposition 8.3. *For every $\phi \geq 0$ and $n \geq 0$, the sequence $(\mathrm{cpfa}_n^{(m)}(\phi))_{m\geq n}$ is decreasing and its limit as $m \to \infty$ coincides with $\mathrm{cpfa}_n(\phi)$. Moreover, the convergence is uniform in $\phi \geq 0$; more precisely, $\|\mathrm{cpfa}_n^{(m)} - \mathrm{cpfa}_n\| \leq e^{-\lambda(\Delta t_{n+1}+\ldots+\Delta t_m)}$ for every $m > n$.*

Proof. For every $m \geq 0$ and $\phi \geq 0$, we have $1 = \mathrm{cpfa}_m^{(m)}(\phi) \geq \mathrm{cpfa}_m^{(m+1)}(\phi)$. Suppose that $\mathrm{cpfa}_n^{(m)}(\cdot) \geq \mathrm{cpfa}_n^{(m+1)}(\cdot)$ for every $1 \leq n \leq m$. Then

$$\mathrm{cpfa}_{n-1}^{(m)}(\phi) = (L\mathrm{cpfa}_n^{(m)})(\phi_{0,n}, \Delta t_{n+1}, \phi) \geq (L\mathrm{cpfa}_n^{(m+1)})(\phi_{0,n}, \Delta t_{n+1}, \phi) = \mathrm{cpfa}_{n-1}^{(m+1)}(\phi).$$

Hence by induction on $n = m, m-1, \ldots, 0$, we conclude that $\{\mathrm{cpfa}_n^{(m)}(\phi);\ m \geq n\}$ is decreasing for every fixed $n \geq 0$ and $\phi \geq 0$. Therefore, $\lim_{m\to\infty} \mathrm{cpfa}_n^{(m)}(\phi)$ exists, and by the bounded convergence theorem it satisfies $\lim_{m\to\infty} \mathrm{cpfa}_{n-1}^{(m)}(\phi) = (L \lim_{m\to\infty} \mathrm{cpfa}_n^{(m)})(\phi_{0,n}, \Delta t_{n+1}, \phi)$ for every $\phi \geq 0$ and $n \geq 0$. Because by Proposition 8.2 the $[0,1]$-valued functions $\mathrm{cpfa}_n(\cdot)$, $n \geq 0$ uniquely satisfy $\mathrm{cpfa}_n(\phi) = (L\mathrm{cpfa}_{n+1})(\phi_{0,n}, \Delta t_{n+1}, \phi)$ for every $\phi \geq 0$ and $n \geq 0$, we conclude that $\mathrm{cpfa}_n(\phi) = \lim_{m\to\infty} \mathrm{cpfa}_n^{(m)}(\phi)$ for every $\phi \geq 0$ and $n \geq 0$. Moreover,

$$\begin{aligned}\mathrm{cpfa}_n(\phi) - \mathrm{cpfa}_n^{(m)}(\phi) &= 1_{[0,\phi_{0,n})}(\phi) e^{-\lambda \Delta t_{n+1}} \left(K(\mathrm{cpfa}_{n+1} - \mathrm{cpfa}_{n+1}^{(m)}) \right) (\Delta t_{n+1}, \phi) \\ &\leq e^{-\lambda \Delta t_{n+1}} \|\mathrm{cpfa}_{n+1} - \mathrm{cpfa}_{n+1}^{(m)}\| \quad \text{for every } \phi \geq 0 \text{ and } m > n.\end{aligned}$$

Therefore, $\mathrm{cpfa}_n^{(m)}(\phi) - \mathrm{cpfa}_n(\phi) \leq e^{-\lambda \Delta t_{n+1}} \|\mathrm{cpfa}_{n+1} - \mathrm{cpfa}_{n+1}^{(m)}\|$ for every $\phi \geq 0$ and $m > n$. Then $\|\mathrm{cpfa}_n - \mathrm{cpfa}_n^{(m)}\| \leq e^{-\lambda \Delta t_{n+1}} \|\mathrm{cpfa}_{n+1} - \mathrm{cpfa}_{n+1}^{(m)}\| \leq \ldots \leq e^{-\lambda(\Delta t_{n+1} + \ldots + \Delta t_m)} \|\mathrm{cpfa}_m - \mathrm{cpfa}_m^{(m)}\| \leq e^{-\lambda(\Delta t_{n+1} + \ldots + \Delta t_m)}$ for every $m > n \geq 0$. Hence, $(\mathrm{cpfa}_n^{(m)}(\phi))_{m\geq 1}$ decreases to $\mathrm{cpfa}_n(\phi)$ as $m \uparrow \infty$ uniformly in $\phi \geq 0$. $\square$

Remark 8.4. *For every $\varepsilon > 0$, let $M(\varepsilon) := \min\left\{m \geq 1;\ \Delta t_1 + \ldots + \Delta t_m \geq -\frac{1}{\lambda} \log \varepsilon\right\}$. Because $\|\mathrm{cpfa}_0 - \mathrm{cpfa}_0^{(M(\varepsilon))}\| \leq \varepsilon$ by Proposition 8.3, we have $\sup_{p\in[0,1]} \left|(1-p)\mathrm{cpfa}_0^{(M(\varepsilon))}(\frac{p}{1-p}) - \mathrm{pfa}(p)\right| \leq \sup_{p\in[0,1]} (1-p) \left|\mathrm{cpfa}_0^{(M(\varepsilon))}(\frac{p}{1-p}) - \mathrm{cpfa}(\frac{p}{1-p})\right| \leq \|\mathrm{cpfa}_0^{(M(\varepsilon))} - \mathrm{cpfa}_0\| \leq \varepsilon$. Hence, we can approximate the probability of false alarm $\mathrm{pfa}(\cdot)$ in (8.1) uniformly in p with $(1-p)\mathrm{cpfa}_0^{(M(\varepsilon))}(\frac{p}{1-p})$, which can easily be calculated with successive approximations in (8.5).*

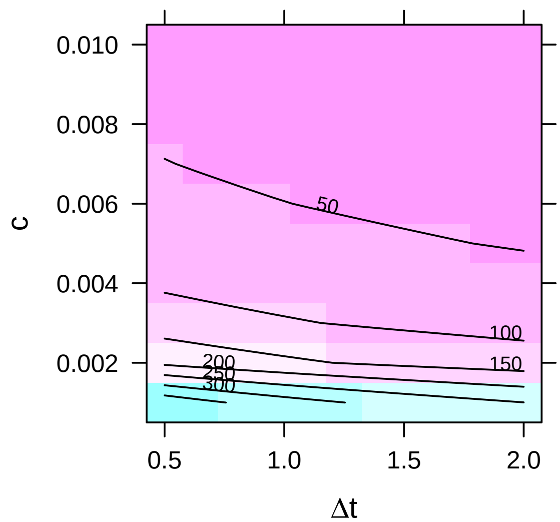
Remark 8.5. *Suppose that $\Delta t_n = \Delta t > 0$ for every $n \geq 1$; namely, all observation intervals have the same length Δt. Then $\phi_{0,n} \equiv \phi_{0,0}$ and $\mathrm{cpfa}_n(\cdot) \equiv \mathrm{cpfa}(\cdot)$ are the same for all $n \geq 0$. Moreover, $\mathrm{cpfa}(\cdot)$ is the unique $[0,1]$-valued function satisfying $\mathrm{cpfa}(\phi) = (L\mathrm{cpfa})(\phi_{0,0}, \Delta t, \phi)$ for every $\phi \geq 0$ and is the limit of successive approximations*

$$\mathrm{cpfa}^{(0)}(\cdot) \equiv 1, \qquad \mathrm{cpfa}^{(n)}(\phi) = (L\mathrm{cpfa}^{(n-1)})(\phi_{0,0}, \Delta t, \phi), \quad \phi \geq 0,\ n \geq 1$$

with $\|\mathrm{cpfa} - \mathrm{cpfa}^{(n)}\| \leq e^{-n\lambda\Delta t}$ for every $n \geq 0$. For every $\varepsilon > 0$, we now have $M(\varepsilon) = \lceil -\frac{\log \varepsilon}{\lambda \Delta t} \rceil$, and $\sup_{p\in[0,1]} \left|(1-p)\mathrm{cpfa}^{(M(\varepsilon))}(\frac{p}{1-p}) - \mathrm{pfa}(p)\right| \leq \varepsilon$.

8.1. **Variational formulation.** In certain applications, one seeks a strict and explicit control on the probability of false alarms. For example, one may not want the probability of false alarm to exceed a prespecified low number $0 < \alpha < 1$. If $\mathcal{S}(\alpha) = \{\tau \in \mathcal{S};\ \mathbb{P}\{\tau < \Theta\} \leq \alpha\}$ denotes the collection of all $\mathbb{F}$-stopping times with false alarm probabilities less than or equal to α, then in the *variational formulation* of the Wiener disorder problem one seeks an alarm time τ in $\mathcal{S}(\alpha)$ which has the smallest expected detection delay time $\mathbb{E}[(\tau - \Theta)^+]$.

The solutions of the variational and Bayesian formulations are closely related. For every $c > 0$ and every sequence of observation times $t_1 < t_2 < \ldots$, the Bayes optimal alarm time $\sigma_0(0)$ for the



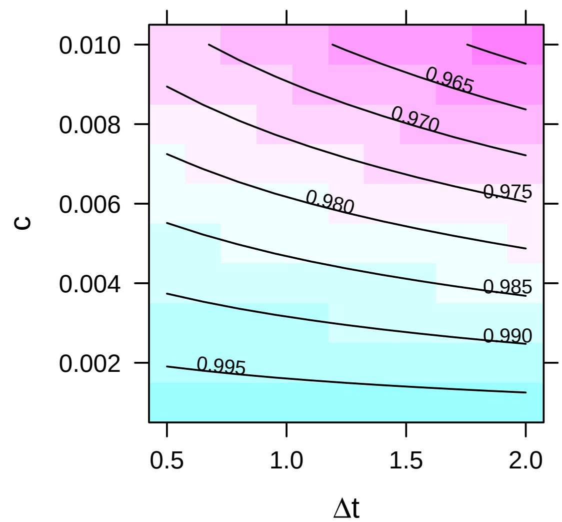


FIGURE 7. On the left $\phi_{0,0} = \min\{\phi_0(s);\ s \in [0, \Delta t)\}$ and on the right $\phi_{0,0}/(1+\phi_{0,0})$ are plotted for a range of unit cost c of detection delay and common length Δt of all observation intervals ($\lambda = 0.1$ and $\mu = 1$).

problem in (2.1) is also optimal for the variational formulation when α equals $\mathbb{P}\{\sigma_0(0) < \Theta\}$, which can be numerically calculated by Remark 8.4 or 8.5. Indeed, for every $\tau \in \mathcal{S}(\alpha) \subseteq \mathcal{S}$, the inequality

$$\mathbb{P}\{\sigma_0(0) < \Theta\} + c\mathbb{E}[(\sigma_0(0) - \Theta)^+] \leq \mathbb{P}\{\tau < \Theta\} + c\mathbb{E}[(\tau - \Theta)^+]$$

implies that $\mathbb{E}[(\sigma_0(0) - \Theta)^+] \leq (1/c)(\mathbb{P}\{\tau < \Theta\} - \mathbb{P}\{\sigma_0(0) < \Theta\}) + \mathbb{E}[(\tau - \Theta)^+] = (1/c)(\mathbb{P}\{\tau < \Theta\} - \alpha) + \mathbb{E}[(\tau - \Theta)^+] \leq \mathbb{E}[(\tau - \Theta)^+]$ or $\mathbb{E}[(\sigma_0(0) - \Theta)^+] \leq \mathbb{E}[(\tau - \Theta)^+]$, and since $\sigma_0(0) \in \mathcal{S}(\alpha)$, we conclude that

$$\inf_{\tau \in \mathcal{S}(\alpha)} \mathbb{E}[(\tau - \Theta)^+] = \mathbb{E}[(\sigma_0(0) - \Theta)^+] = \left[1 - p + (1-p)c\, V\left(\frac{p}{1-p}\right) - \alpha\right] \frac{1}{c}$$

where the second equation follows from Proposition 2.1.

It is unclear if for every $0 < \alpha < 1$ there are always some $c > 0$ and $t_1 < t_2 < \ldots$ such that the optimal alarm time of the Bayesian formulation has the probability of false alarm exactly equal to α. A quick and effective solution of the variational formulation will be to tabulates the probability of false alarms of Bayes optimal alarm times on a fine grid of cost $c > 0$ and the lengths Δt_n, $n \geq 1$ of observation intervals.

Figures 7 and 8 illustrate this practical approach when observation intervals have some common length Δt. In those numerical illustrations, we set $\lambda = 0.1$ and $\mu = 1$. For every fixed $c > 0$ and $\Delta t > 0$, we solve the Bayesian formulation and find for the Bayes optimal alarm time $\sigma_0(0)$ the minimum threshold $\phi_{0,0} = \min\{\phi_0(s);\ s \in [0, \Delta t)\}$; see Figure 7. We then calculate the probability of false alarm of $\sigma_0(0)$ as described in Remark 8.5. Figure 8 display the contourplots of false alarm probabilities and expected detection delay times of Bayes optimal alarm times for every pair of c and Δt values. One can in principle spot the solution of the variational formulation by an inspection of the pictures in Figures 7 and 8. For example, if we are certain that the disorder has not happened

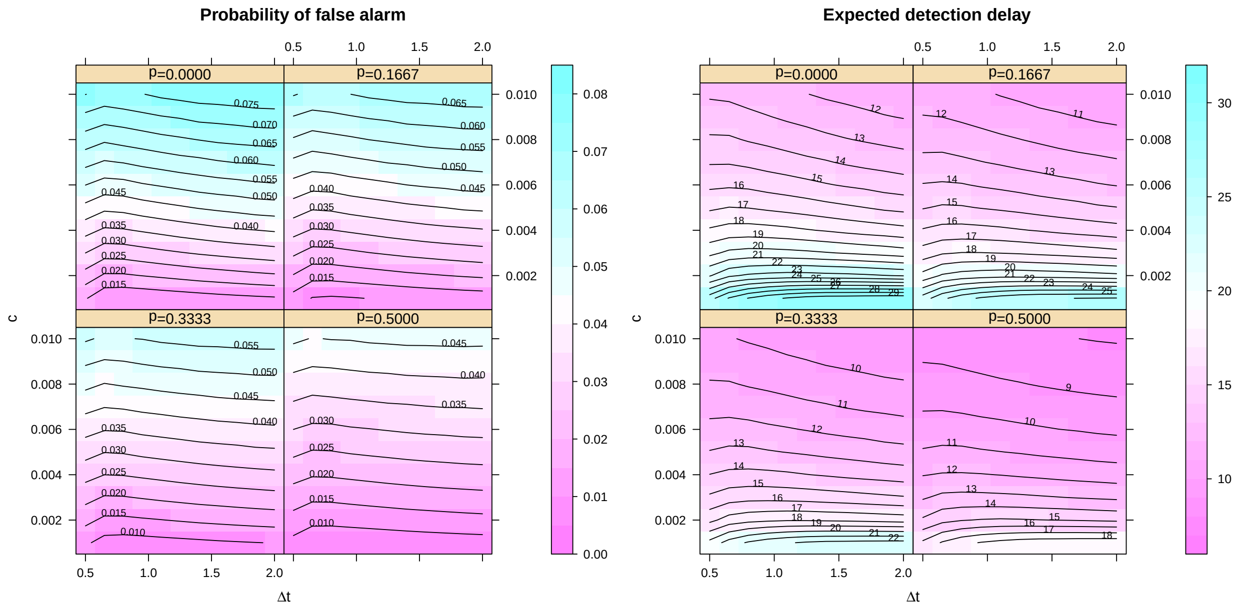


FIGURE 8. The probability of false alarms (on the left) and expected detection delay times (on the right) of Bayes optimal alarm times for prior probabilities $p = 0, 1/6, 1/3, 1/2$ of a change at or before time zero and for a range of unit cost c of detection delay and equal observation interval length Δt ($\lambda = 0.1$ and $\mu = 1$).

yet (namely, $p = 0$), and if we want the probability of false alarm to be less than or equal to $1/50$, then we can choose any pair $(\Delta t, c)$ located on the contour labeled with "0.02" in the upper left corner of the picture on the left in Figure 8. For the pair $(\Delta t, c)$ we picked, we can read from the upper left corner of the picture on the right in the same figure the minimum expected detection delay time and find the minimum critical threshold $\phi_{0,0}$ from the picture on the left in Figure 7.

Acknowledgment

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Appendix A. Selected proofs

A.1. **Derivation of the dynamics in (1.2) of the conditional odds-ratio process Φ.** Because Θ is independent of X and has zero-modified exponential distribution with parameters $p \in [0, 1)$ and rate $\lambda > 0$ under $\mathbb{P}_\infty$, we have

$$\Phi_t = \frac{e^{\lambda t}}{1-p} \mathbb{E}_\infty \left[Z_t(\Theta) 1_{\{\Theta \le t\}} \mid \mathcal{F}_t \right] = \frac{e^{\lambda t}}{1-p} \left[p Z_t(0) + (1-p) \int_0^t \lambda e^{-\lambda u} Z_t(u) \mathrm{d}u \right]$$

Suppose that $t_{n-1} \le t < t_n$ for some $n \ge 1$. Since $Z_t(u) = Z_{t_{n-1}}(u)$ for every $u \ge 0$ and $Z_{t_{n-1}}(u) = 1$ for every $t_{n-1} \le u < t_n$, we have Φ_t equals

$$\begin{aligned} \frac{e^{\lambda t}}{1-p} \Big[p Z_{t_{n-1}}(0) + (1-p) \int_0^t \lambda e^{-\lambda u} Z_{t_{n-1}}(u) \mathrm{d}u \Big] &= \frac{e^{\lambda t}}{1-p} \Big[\frac{1-p}{e^{\lambda t_{n-1}}} \Phi_{t_{n-1}} + (1-p) \Big(e^{-\lambda t_{n-1}} - e^{-\lambda t} \Big) \Big] \\ &= e^{\lambda (t - t_{n-1})} \Phi_{t_{n-1}} + e^{\lambda (t - t_{n-1})} - 1 = e^{\lambda (t - t_{n-1})} (\Phi_{t_n - 1} + 1) - 1 = \varphi(t - t_{n-1}, \Phi_{t_{n-1}}). \end{aligned}$$

On the other hand, $\Phi_{t_{n-1}} = \frac{e^{\lambda t_{n-1}}}{1-p}[pZ_{t_{n-1}}(0)+(1-p)\int_0^{t_{n-1}}\lambda e^{-\lambda u} Z_{t_{n-1}}(u)\mathrm{d}u]$. Because $Z_{t_{n-1}}(u) = 1$ for every $u \geq t_{n-1}$, we have

$$Z_{t_n}(u) = Z_{t_{n-1}}(u)\exp\Big\{\frac{\Delta X_n \mu[t_n-(u\vee t_{n-1})]^+}{t_n-t_{n-1}} - \frac{\mu^2([t_n-(u\vee t_{n-1})]^+)^2}{2(t_n-t_{n-1})}\Big\}, \quad u\geq 0$$

and

$$\begin{aligned}\Phi_{t_n} &= \frac{e^{\lambda t_n}}{1-p}\Big[\Big(pZ_{t_{n-1}}(0)+(1-p)\int_0^{t_{n-1}}\lambda e^{-\lambda u}Z_{t_{n-1}}(u)\mathrm{d}u\Big)\exp\Big\{\Delta X_n\mu - \frac{\mu^2}{2}\Delta t_n\Big\}\\ &\quad + (1-p)\int_{t_{n-1}}^{t_n}\lambda e^{-\lambda u}\exp\Big\{\frac{\Delta X_n\mu(t_n-u)}{t_n-t_{n-1}} - \frac{\mu^2(t_n-u)^2}{2(t_n-t_{n-1})}\Big\}\mathrm{d}u\Big]\\ &= \exp\Big\{\mu\Delta X_n - \frac{\mu^2}{2}\Delta t_n\Big\}e^{\lambda(t_n-t_{n-1})}\Phi_{t_{n-1}}\\ &\quad + \int_{t_{n-1}}^{t_n}\lambda e^{\lambda(t_n-u)}\exp\Big\{\frac{\Delta X_n\mu(t_n-u)}{t_n-t_{n-1}} - \frac{\mu^2(t_n-u)^2}{2(t_n-t_{n-1})}\Big\}\mathrm{d}u,\end{aligned}$$

which gives (1.2) after a change of variable in the integral on the righthand side.

A.2. **Proof of Lemma 4.1.** (i) If $w(\cdot) \geq -1/c$, then $(Jw)(\Delta t,\phi,y,r) \geq -\frac{1}{c}\int_y^{\Delta t}\lambda e^{-\lambda t}\mathrm{d}t - \frac{1}{c}e^{-\lambda\Delta t} = -\frac{1}{c}\left(e^{-\lambda y}-e^{-\lambda\Delta t}\right) - \frac{1}{c}e^{-\lambda\Delta t} = -\frac{1}{c}e^{-\lambda y}$ for every $\Delta t>0$, $\phi\geq 0$, $0\leq y\leq\Delta t$, and $r\geq 0$. Then $(J_y w)(\Delta t,\phi) = \inf_{r\geq y}(Jw)(\Delta t,\phi,y,r) \geq -(1/c)e^{-\lambda y}$, and $(J_y w)(\Delta t,\phi)\leq (Jw)(\Delta t,\phi,y,y) = 0$; therefore, $-1/c \leq e^{\lambda y}(J_y w)(\Delta,\phi)\leq 0$ for every $\Delta t>0$, $\phi\geq 0$, and $0\leq y\leq \Delta t$.

Both $\phi\mapsto\varphi(\Delta t,\phi)$ and $\phi\mapsto \jmath(\Delta t,\phi,z)$ are increasing affine functions for every fixed $\Delta t > 0$ and $z\in\mathbb{R}$. If $w(\cdot)$ is nondecreasing, concave, and continuous, then so are $(Kw)(\Delta t,\cdot)$ and $(Jw)(\Delta t,\cdot,y,r)$ for every fixed $\Delta t>0$, $0\leq y\leq\Delta t$, and $r\geq 0$ by the dominated convergence. Therefore, $(J_y w)(\Delta t,\cdot) = \inf_{r\geq y}(Jw)(\Delta t,\cdot,y,r)$ is also nondecreasing and concave. The continuity on $(0,\infty)$ of $(J_y w)(\Delta t,\cdot)$ follows from its concavity on $[0,\infty)$. It is also continuous at $\phi=0$, because $\lim_{\phi\searrow 0}(J_y w)(\Delta t,\phi) = \inf_{\phi>0}\inf_{r\geq y}(Jw)(\Delta t,\phi,y,r) = \inf_{r\geq y}\inf_{\phi>0}(Jw)(\Delta t,\phi,y,r) = \inf_{r\geq y}(Jw)(\Delta t,0,y,r) = (J_y w)(\Delta t,0)$, since $(J_y w)(\Delta t,\cdot)$ and $(Jw)(\Delta t,\cdot,y,r)$ are nondecreasing.

Let us now prove that $(J_y w)(\Delta t,\phi)$ vanishes for large $\phi\geq 0$. For every $\phi>\lambda/c$ and $u\geq 0$, note that $\varphi(u,\phi)>\lambda/c$ and $\int_y^r e^{-\lambda u}(\varphi(u,\phi)-\lambda/c)\mathrm{d}u > 0$ for every $r>y$. Moreover, there is some finite $\phi(\Delta t,y)>\lambda/c$ such that $\int_y^{\Delta t}e^{-\lambda u}(\varphi(u,\phi)-\frac{\lambda}{c})\mathrm{d}u + e^{-\lambda\Delta t}(Kw)(\Delta t,\phi) \geq \int_y^{\Delta t}(\phi+1-e^{-\lambda u}(1-\frac{\lambda}{c}))\mathrm{d}u - \frac{e^{-\lambda\Delta t}}{c} = (\phi+1)(\Delta t-y) + (1+\frac{\lambda}{c})\frac{1}{\lambda}(e^{-\lambda y}-e^{-\lambda\Delta t}) - \frac{1}{c}e^{-\lambda\Delta t} > 0$ for every $\phi>\phi(\Delta t,y)$ and $(J_y w)(\Delta t,\phi) = \{\inf_{r\in[y,\Delta t]}\int_y^r e^{-\lambda u}(\varphi(u,\phi)-\frac{\lambda}{c})\mathrm{d}u\}\wedge[\int_y^{\Delta t}e^{-\lambda u}(\varphi(u,\phi)-\frac{\lambda}{c})\mathrm{d}u + e^{-\lambda\Delta t}(Kw)(\Delta t,\phi)] = 0$ for every $\phi>\phi(\Delta t,y)$.

(ii) Clearly, if $w_1(\cdot)\leq w_2(\cdot)$, then $(Kw_1)(\Delta t,\phi)\leq (Kw_2)(\Delta t,\phi)$ for every $\Delta t>0$ and $\phi\geq 0$, which implies that $(Jw_1)(\Delta t,\phi,y,r)\leq (Jw_2)(\Delta t,\phi,y,r)$ for every $\Delta t>0$, $\phi\geq 0$, $0\leq y\leq\Delta t$, and $r\geq 0$, and taking infimum of both sides over $r\geq y$ yields the result.

(iii) Let $w_3(\cdot)$ and $w_4(\cdot)$ be two bounded functions. Fix $\Delta t>0$ and $0\leq y\leq\Delta t$. Then for every $\phi\geq 0$, $(J_y w_3)(\Delta t,\phi)$ and $(J_y w_4)(\Delta t,\phi)$ are finite, and there is $r_i(\phi,\varepsilon)\geq y$ such that $(J_y w_i)(\Delta t,\phi)+\varepsilon \geq (Jw_i)(\Delta t,\phi,y,r_i(\phi,\varepsilon))$ for $\phi\geq 0$, $\varepsilon>0$, $i=3,4$. Therefore,

$$\begin{aligned}(J_y w_3)(\Delta t,\phi) - (J_y w_4)(\Delta t,\phi) &\leq (Jw_3)(\Delta t,\phi,y,r_4(\phi,\varepsilon)) - (Jw_4)(\Delta t,\phi,y,r_4(\phi,\varepsilon))+\varepsilon\\ &= 1_{[\Delta t,\infty)}(r_4(\phi,\varepsilon))e^{-\lambda\Delta t}\left[(Kw_3)(\Delta t,\phi)-(Kw_4)(\Delta t,\phi)\right]+\varepsilon \leq e^{-\lambda\Delta t}|w_3(\phi)-w_4(\phi)|+\varepsilon.\end{aligned}$$

Because $\varepsilon > 0$ is arbitrary, this leads to $(J_y w_3)(\Delta t, \phi) - (J_y w_4)(\Delta t, \phi) \le e^{-\lambda \Delta t}|w_3(\phi) - w_4(\phi)|$. Changing the order of $w_3(\cdot)$ and $w_4(\cdot)$ and replacing $r_4(\Delta t, \phi)$ with $r_3(\Delta t, \phi)$ in the last displayed equation similarly gives $(J_y w_4)(\Delta t, \phi) - (J_y w_3)(\Delta t, \phi) \le e^{-\lambda \Delta t}|w_3(\phi) - w_4(\phi)| + \varepsilon$, and we conclude that $|(J_y w_4)(\Delta t, \phi) - (J_y w_3)(\Delta t, \phi)| \le e^{-\lambda \Delta t}|w_3(\phi) - w_4(\phi)|$ for every $\Delta t > 0$, $0 \le y \le \Delta t$, and $\phi \ge 0$. Taking the supremum of both sides over $\phi \ge 0$ proves (iii).

(iv) Because $(Kw)(\Delta t, \phi) \le 0$, the mapping

$$r \mapsto (Jw)(\Delta t, \phi, y, r) = \left\{ \begin{array}{ll} \displaystyle\int_y^r e^{-\lambda u}\Big(\varphi(u,\phi) - \frac{\lambda}{c}\Big)\mathrm{d}u, & y \le r < \Delta t \\ \displaystyle\int_y^{\Delta t} e^{-\lambda u}\Big(\varphi(u,\phi) - \frac{\lambda}{c}\Big)\mathrm{d}u + e^{-\lambda \Delta t}(Kw)(\Delta t, \phi), & r \ge \Delta t \end{array} \right\}$$

is lower semi-continuous, and its infimums over $r \in [y, \infty)$ and the compact interval $r \in [y, \Delta t]$ are the same. Since the mapping is lower semi-continuous, (4.6) follows.

Since $(Jw)(\Delta t, \phi, y, r) = (Jw)(\Delta t, \phi, 0, r) - \int_0^y e^{-\lambda u}(\varphi(u,\phi) - \frac{\lambda}{c})\mathrm{d}u$, we have $(J_y w)(\Delta t, \phi) = \inf_{r \in [y, \Delta t]}(Jw)(\Delta t, \phi, 0, r) - \int_0^y e^{-\lambda u}(\varphi(u,\phi) - \frac{\lambda}{c})\mathrm{d}u$. Because $y \mapsto \int_0^y e^{-\lambda u}(\varphi(u,\phi) - \lambda/c)\mathrm{d}u$ is continuous, we only need to establish that $y \mapsto \inf_{r \in [y, \Delta t]}(Jw)(\Delta t, \phi, 0, r)$ is continuous.

For convenience, let us define $f(r) := (Jw)(\Delta t, \phi, 0, r)$ and $F(y) := \min_{r \in [y, \Delta t]} f(r)$ for every $r, y \in [0, \Delta t]$. Since $r \mapsto f(r)$ is lower semi-continuous, we will show that $y \mapsto F(y)$ is left-continuous. Take any $0 \le u < v \le \Delta t$. Then

$$0 \ge F(u) - F(v) = \min\{\min_{s \in [u,v]} f(s) - F(v), 0\} \ge \min_{s \in [u,v]} f(s) - f(v). \tag{A.1}$$

Fix $v \in (0, \Delta t]$ and show that $F(\cdot)$ is left-continuous at v. Since $f(\cdot)$ is lower semi-continuous at v, for every $\varepsilon > 0$ there is some $\delta > 0$ such that $|x - v| \le \delta$ implies $f(x) > f(v) - \varepsilon$. Therefore, for every $(v - \delta)^+ \le u < v$ we have $\min_{s \in [u,v]} f(s) \ge f(v) - \varepsilon$ and $0 \ge F(u) - F(v) \ge \min_{s \in [u,v]} f(s) - f(v) \ge f(v) - \varepsilon - f(v) = -\varepsilon$; namely, $F(\cdot)$ is left-continuous at v.

Since $r \mapsto f(r)$ is right-continuous, $y \mapsto F(y)$ is also right-continuous. In (A.1), fix $u \in [0, \Delta t)$ and show that $F(\cdot)$ is right-continuous at u. Since $f(\cdot)$ is right-continuous at u, for all $\varepsilon > 0$ there is $\delta > 0$ such that $u < x < (u + \delta) \wedge \Delta t$ implies $|f(x) - f(u)| < \varepsilon/2$. Thus for all $u < v < (u + \delta) \wedge \Delta t$ we have $\min_{s \in [u,v]} f(s) \ge f(u) - \varepsilon/2$ and $f(v) < f(u) + \varepsilon/2$; therefore, $0 \ge F(u) - F(v) \ge \min_{s \in [u,v]} f(s) - f(v) \ge f(u) - \frac{\varepsilon}{2} - [f(u) + \frac{\varepsilon}{2}] = -\varepsilon$. Hence, $y \mapsto F(y)$ is continuous.

(v) For $y_0 \le y \le r \le z \le y_1$, adding $\int_y^r e^{-\lambda u}\,(\varphi(u,\phi) - \lambda/c)\,\mathrm{d}u$ to $0 > (J_r w)(\Delta t, \phi)$ gives

$$\begin{aligned} \int_y^r e^{-\lambda u}\Big(\varphi(u,\phi) - \frac{\lambda}{c}\Big)\mathrm{d}u &> \inf_{\widetilde{r} \ge r}\Big[\int_y^{\widetilde{r} \wedge \Delta t} e^{-\lambda u}\Big(\varphi(u,\phi) - \frac{\lambda}{c}\Big)\mathrm{d}u + 1_{[\Delta t, \infty)}(\widetilde{r})e^{-\lambda \Delta t}(Kw)(\Delta t, \phi)\Big] \\ &\ge \inf_{\widetilde{r} \ge y}\Big[\int_y^{\widetilde{r} \wedge \Delta t} e^{-\lambda u}\Big(\varphi(u,\phi) - \frac{\lambda}{c}\Big)\mathrm{d}u + 1_{[\Delta t, \infty)}(\widetilde{r})e^{-\lambda \Delta t}(Kw)(\Delta t, \phi)\Big] = (J_y w)(\Delta t, \phi). \end{aligned}$$

Since $r \mapsto \int_y^r e^{-\lambda u}\left(\varphi(u,\phi) - \frac{\lambda}{c}\right)\mathrm{d}u$ is continuous on $r \in [y, z]$, its minimum on $r \in [y, z]$ is attained, say at some $r_0 \in [y, z]$. Because the inequalities also hold for $r = r_0$, we have

$$\inf_{r \in [y,z]} \int_y^{r \wedge \Delta t} e^{-\lambda u}\left(\varphi(u,\phi) - \frac{\lambda}{c}\right)\mathrm{d}u = \min_{r \in [y,z]} \int_y^r e^{-\lambda u}\left(\varphi(u,\phi) - \frac{\lambda}{c}\right)\mathrm{d}u > (J_y w)(\Delta t, \phi),$$

and $(J_y w)(\Delta t, \phi) = \inf_{r \ge z}[\int_y^{r \wedge \Delta t} e^{-\lambda u}\left(\varphi(u,\phi) - \frac{\lambda}{c}\right)\mathrm{d}u + 1_{[\Delta t, \infty)}(r)e^{-\lambda \Delta t}(Kw)(\Delta t, \phi)]$ equals

$$\int_y^z e^{-\lambda u}\Big(\varphi(u,\phi)-\frac{\lambda}{c}\Big)\mathrm{d}u+\inf_{r\geq z}\Big[\int_z^{r\wedge\Delta t} e^{-\lambda u}\Big(\varphi(u,\phi)-\frac{\lambda}{c}\Big)\mathrm{d}u+1_{[\Delta t,\infty)}(r)e^{-\lambda\Delta t}(Kw)(\Delta t,\phi)\Big]$$

which is $\int_y^z e^{-\lambda u}\Big(\varphi(u,\phi)-\frac{\lambda}{c}\Big)\mathrm{d}u+(J_z w)(\Delta t,\phi)$ and completes the proof of (v) and the lemma.

A.3. **Proof of Proposition 5.5.** By Theorem 5.1, $-1/c\leq\gamma^{(m)}(t)\leq 0$ is $\mathbb{P}_\infty$-a.s. bounded, and since

$$\mathbb{E}_\infty\Phi_u=\mathbb{E}_\infty\left[\frac{\mathbb{E}_\infty[Z_u 1_{\{\Theta\leq u\}}\mid\mathcal{F}_u]}{\mathbb{E}_\infty[Z_u 1_{\{\Theta>u\}}\mid\mathcal{F}_u]}\right]=\frac{\mathbb{E}_\infty[Z_u 1_{\{\Theta\leq u\}}]}{(1-p)e^{-\lambda u}}\leq\frac{\mathbb{E}_\infty[Z_u]}{(1-p)e^{-\lambda u}}=\frac{e^{\lambda u}}{1-p},$$

we have $\mathbb{E}_\infty|M^{(m)}(t)|\leq\mathbb{E}_\infty[\int_0^t e^{-\lambda u}\Phi_u\mathrm{d}u]+\frac{1}{c}\int_0^t\lambda e^{-\lambda u}\mathrm{d}u+\frac{1}{c}e^{-\lambda t}\leq\int_0^t\frac{1}{1-p}\mathrm{d}u+\frac{1}{c}=\frac{t}{1-p}+\frac{1}{p}<\infty$, and $M^{(m)}(t)$ is integrable under $\mathbb{P}_\infty$ for every $0\leq t\leq t_m$.

Fix some $m\geq 1$, $0\leq t\leq t_m$, $\tau\in\mathcal{S}(t)$, and $\varepsilon\geq 0$. Then there exists some $0\leq k\leq m-1$ such that $t_k\leq t<t_{k+1}$. Let us prove by induction on $n=k+1,k+2,\ldots,m$ that $\mathbb{E}_\infty[M^{(m)}(t_n\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t))]=\mathbb{E}_\infty[M^{(m)}(t)]$ for every $k+1\leq n\leq m$, $\varepsilon\geq 0$.

Basis case. Because $\mathbb{E}_\infty[M^{(m)}(t_{k+1}\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t))]-\mathbb{E}_\infty[M^{(m)}(t)]$ equals

$$\mathbb{E}_\infty\Big[\int_t^{t_{k+1}\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t)} e^{-\lambda u}\Big(\Phi_u-\frac{\lambda}{c}\Big)\mathrm{d}u+e^{-\lambda(t_{k+1}\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t))}\gamma^{(m)}(t_{k+1}\wedge\tau\wedge\sigma^{(m)}(t))-e^{-\lambda t}\gamma^{(m)}(t)\Big],$$

the basis case $n=k+1$ will be established if the displayed expectation equals zero. Since $\gamma^{(m)}(t_{k+1})=(J_0 v_{k+2}^{(m)})(\Delta t_{k+2},\Phi_{t_{k+1}})=v_{k+1}^{(m)}(\Phi_{t_{k+1}})$, $\gamma^{(m)}(t)=e^{\lambda(t-t_k)}(J_{t-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$, and $\gamma^{(m)}(\tau\wedge\sigma_\varepsilon^{(m)}(t))=e^{\lambda([\tau\wedge\sigma_\varepsilon^{(m)}(t)]-t_k)}(J_{[\tau\wedge\sigma_\varepsilon^{(m)}(t)]-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ on $\{\tau\wedge\sigma_\varepsilon^{(m)}(t)<t_{k+1}\}$ because $\mathbb{P}_\infty$-a.s. $\tau\wedge\sigma_\varepsilon^{(m)}(t)\geq t\geq t_k$, we can rewrite the displayed expectation as

$$\begin{aligned}\mathbb{E}_\infty\Big[&\int_t^{t_{k+1}\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t)} e^{-\lambda u}\Big(\varphi(u-t_k,\Phi_{t_k})-\frac{\lambda}{c}\Big)\mathrm{d}u+1_{\{\tau\wedge\sigma_\varepsilon^{(m)}(t)\geq t_{k+1}\}}e^{-\lambda t_{k+1}}v_{k+1}^{(m)}(\Phi_{t_{k+1}})\\&+1_{\{\tau\wedge\sigma_\varepsilon^{(m)}(t)<t_{k+1}\}}e^{-\lambda t_k}(J_{(\tau\wedge\sigma^{(m)}(t))-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})-e^{-\lambda t_k}(J_{t-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})\Big].\end{aligned}$$

There is some $\mathcal{F}_{t_k}$-measurable nonnegative r.v. R_k such that $t_{k+1}\wedge\tau\wedge\sigma_\varepsilon^{(m)}(t)=(t_k+R_k)\wedge t_{k+1}$, $\{\tau\wedge\sigma_\varepsilon^{(m)}(t)\geq t_{k+1}\}=\{t_k+R_k\geq t_{k+1}\}$, and $(\tau\wedge\sigma_\varepsilon^{(m)}(t))1_{\{\tau\wedge\sigma_\varepsilon^{(m)}(t)<t_{k+1}\}}=(t_k+R_k)1_{\{\tau\wedge\sigma_\varepsilon^{(m)}(t)<t_{k+1}\}}$. Since $\Phi_{t_{k+1}}=\jmath(\Delta t_{k+1},\Phi_{t_k},\frac{\Delta X_{k+1}}{\sqrt{\Delta t_{k+1}}})$, last displayed expectation equals $e^{-\lambda t_k}$ times expectation of

$$\begin{aligned}(Jv_{k+1}^{(m)})&(\Delta t_{k+1},\Phi_{t_k},t-t_k,R_k\wedge\Delta t_{k+1})\\&+1_{[0,\Delta t_{k+1})}(R_k)(J_{R_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})-(J_{t-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k}),\end{aligned}\tag{A.2}$$

which, we shall show, $\mathbb{P}_\infty$-a.e. vanishes. Since $\sigma_\varepsilon^{(m)}(t)=\min\{s\geq t;\ \gamma^{(m)}(s)>-\varepsilon\}$, we have $0>-\varepsilon\geq\gamma^{(m)}(s)=e^{\lambda(s-t_k)}(J_{s-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ or $0>(J_{s-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ for every $t\leq s<t_{k+1}\wedge\sigma_\varepsilon^{(m)}(t)\wedge\tau\equiv t_{k+1}\wedge(t_k+R_k)$ or for every $t-t_k\leq s-t_k<R_k\wedge\Delta t_{k+1}$.

Basis case with $R_k<\Delta t_{k+1}$. On $\{R_k<\Delta t_{k+1}\}$, we have $0>(J_{s-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ for $t-t_k\leq s-t_k<R_k$, and by Lemma 4.1 (v) with $y_0=t-t_k$, $y_1=R_k-\delta$, $\Delta t=\Delta t_{k+1}$, $\phi=\Phi_{t_k}$, $y=y_0$, $z=y_1$, and $w=v_{k+1}^{(m)}$, we conclude $(J_{t-t_k}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})=\int_{t-t_k}^{R_k-\delta}e^{-\lambda u}(\varphi(u,\Phi_{t_k})-\frac{\lambda}{c})\mathrm{d}u+(J_{R_k-\delta}v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ for all $0<\delta<R_k-(t-t_k)$. By Lemma 4.1 (iv), $(J_y v_{k+1}^{(m)})(\Delta t_{k+1},\Phi_{t_k})$ is

continuous at $y = R_k \in [0, \Delta t_{k+1}]$, and $\delta \downarrow 0$ gives $(J_{t-t_k} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k}) = \int_{t-t_k}^{R_k} e^{-\lambda u}(\varphi(u, \Phi_{t_k}) - \frac{\lambda}{c})\mathrm{d}u + (J_{R_k} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k})$. Therefore, the r.v. in (A.2) equals zero $\mathbb{P}_\infty$-a.s. on $\{R_k < \Delta t_{k+1}\}$.

Basis case with $R_k \geq \Delta t_{k+1}$. On $\{R_k \geq \Delta t_{k+1}\}$, we have $0 > (J_{s-t_k} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k})$ for every $t - t_k \leq s - t_k \leq \Delta t_{k+1} \wedge R_k = \Delta t_{k+1}$. By Lemma 4.1 (v) with $y_0 = t - t_k$, $y_1 = \Delta t_{k+1} - \delta$, $\Delta t = \Delta t_{k+1}$, $\phi = \Phi_{t_k}$, $y = y_0$, $z = y_1$, and $w = v_{k+1}^{(m)}$, we conclude that $(J_{t-t_k} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k}) = \int_{t-t_k}^{\Delta t_{k+1}-\delta} e^{-\lambda u}(\varphi(u, \Phi_{t_k}) - \frac{\lambda}{c})\mathrm{d}u + (J_{\Delta t_{k+1}-\delta} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k})$ for every $0 < \delta < \Delta t_{k+1} - t + t_k \equiv t_{k+1} - t$. By Lemma 4.1 (iv), $y \mapsto (J_y v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k})$ is continuous on $[0, \Delta t_{k+1}] \ni y$, and letting $\delta \downarrow 0$ above implies that on $\{R_n \geq \Delta t_{n+1}\}$ $(J_{t-t_k} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k}) = \int_{t-t_k}^{\Delta t_{k+1}} e^{-\lambda u} \left(\varphi(u, \Phi_{t_k}) - \frac{\lambda}{c}\right) \mathrm{d}u + (J_{\Delta t_{k+1}} v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k}) = (J v_{k+1}^{(m)})(\Delta t_{k+1}, \Phi_{t_k}, t - t_k, \Delta t_{k+1})$ and shows that the random variable in (A.2) equals zero $\mathbb{P}_\infty$-a.s. on $\{R_k \geq t_{k+1}\}$. This completes the proof of the basis case $n = k+1$.

Inductive step. Suppose that $\mathbb{E}_\infty[M^{(m)}(t_n \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t))] = \mathbb{E}_\infty[M^{(m)}(t)]$ for some $k+1 \leq n \leq m-1$ and show that it also holds for $n+1$. Since $\mathbb{E}_\infty[M^{(m)}(t_{n+1} \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t))] - \mathbb{E}_\infty[M^{(m)}(t_n \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t))]$ equals $\mathbb{E}_\infty[1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}}(M^{(m)}(t_{n+1} \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t)) - M^{(m)}(t_n))]$, the result follows from the induction hypothesis, if the last expectation equals zero. Because $\gamma^{(m)}(t_{n+1}) = v_{n+1}^{(m)}(\Phi_{t_{n+1}})$ and $\gamma^{(m)}(\tau \wedge \sigma_\varepsilon^{(m)}(t)) = e^{\lambda(\tau \wedge \sigma_\varepsilon^{(m)}(t) - t_n)}(J_{\tau \wedge \sigma_\varepsilon^{(m)} - t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n})$ on $\{t_n \leq \tau \wedge \sigma_\varepsilon^{(m)}(t) < t_{n+1}\}$, we have $\mathbb{E}_\infty[1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}}(M^{(m)}(t_{n+1} \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t)) - M^{(m)}(t_n))] =$

$$\begin{aligned}\mathbb{E}_\infty\Big[1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}}\Big(&\int_{t_n}^{t_{n+1} \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t)} e^{-\lambda u}\Big(\varphi(u - t_n, \Phi_{t_n}) - \frac{\lambda}{c}\Big)\mathrm{d}u \\ &+ 1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_{n+1}\}} e^{-\lambda t_{n+1}} v_{n+1}^{(m)}(\Phi_{t_{n+1}}) \\ &+ 1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) < t_{n+1}\}} e^{-\lambda t_n}(J_{\tau \wedge \sigma_\varepsilon^{(m)} - t_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n}) - e^{-\lambda t_n} v_n^{(m)}(\Phi_{t_n})\Big)\Big],\end{aligned}$$

and since there is an $\mathcal{F}_{t_n}$-mble r.v. R_n such that $[t_{n+1} \wedge \tau \wedge \sigma_\varepsilon^{(m)}(t)]1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}} = [t_{n+1} \wedge (t_n + R_n)]1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}}$ and $\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_{n+1}\} = \{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n, t_n + R_n \geq t_{n+1}\}$, last displayed expectation equals $e^{-\lambda t_n}$ times the expectation of $1_{\{\tau \wedge \sigma_\varepsilon^{(m)}(t) \geq t_n\}}[(J v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n}, 0, R_n \wedge \Delta t_{n+1}) + 1_{[0, \Delta t_{n+1})}(R_n)(J_{R_n} v_{n+1}^{(m)})(\Delta t_{n+1}, \Phi_{t_n}) - v_n^{(m)}(\Phi_{t_n})]$ which vanishes $\mathbb{P}_\infty$-a.s. as in the basis case.

A.4. **Proof of Corollary 5.6.** Fix any $m \geq 1$, $0 \leq t \leq u \leq v \leq t_m$, and $\varepsilon \geq 0$. Let F be a $\mathcal{F}_u$-mble event, and define $\mathbb{F}$-stopping time $\tau := u 1_F + v 1_{\Omega \setminus F}$. Then by Proposition 5.5 $\mathbb{E}_\infty[M^{(m)}(v \wedge \sigma_\varepsilon^{(m)}(t))] = \mathbb{E}_\infty[M^{(m)}(t)] = \mathbb{E}_\infty[M^{(m)}(\tau \wedge \sigma_\varepsilon^{(m)}(t))] = \mathbb{E}_\infty[M^{(m)}(u \wedge \sigma_\varepsilon^{(m)}(t))1_F] + \mathbb{E}_\infty[M^{(m)}(v \wedge \sigma_\varepsilon^{(m)}(t))1_{\Omega \setminus F}]$, and $\mathbb{E}_\infty[M^{(m)}(u \wedge \sigma_\varepsilon^{(m)}(t))1_F] = \mathbb{E}_\infty[M^{(m)}(v \wedge \sigma_\varepsilon^{(m)}(t))1_F]$. Since $F \in \mathcal{F}_u$ is arbitrary, we have $M^{(m)}(u \wedge \sigma_\varepsilon^{(m)}(t)) = \mathbb{E}_\infty(M^{(m)}(v \wedge \sigma_\varepsilon^{(m)}(t)) \mid \mathcal{F}_u)$, $\mathbb{P}_\infty$-a.s.

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