1. Consider the functions

$$f(x) = \sqrt{x^2 + a} + bx \qquad \text{and} \qquad g(x) = x^3 + cx^2 + d$$

where a, b, c, d are <u>nonzero</u> real constants.

a. Give values to a, b, c, d by filling in the boxes below so that the resulting functions satisfy $f(1) = 0$ and $g(1) = 0$. No explanation is required in this part.

$a = \boxed{3}$ $\qquad b = \boxed{-2}$ $\qquad c = \boxed{1}$ $\qquad d = \boxed{-2}$

b. Compute the limit $\lim\limits_{x\to 1} \dfrac{f(x)}{g(x)}$ where a, b, c, d have the values given in **Part a.**
[Do not use L'Hôpital's Rule. This part will be graded only if Part a is completely correct.]

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{\sqrt{x^2+3} - 2x}{x^3+x^2-2} = \lim_{x\to 1} \frac{\sqrt{x^2+3} - 2x}{x^3+x^2-2} \cdot \frac{\sqrt{x^2+3}+2x}{\sqrt{x^2+3}+2x}$$

$$= \lim_{x\to 1} \frac{x^2+3-4x^2}{(x-1)(x^2+2x+2)\cdot(\sqrt{x^2+3}+2x)}$$

$$= \lim_{x\to 1} \frac{-3\cdot(x^2-1)}{(x-1)\cdot(x^2+2x+2)\cdot(\sqrt{x^2+3}+2x)}$$

$$= \lim_{x\to 1} \frac{-3\cdot(x-1)(x+1)}{(x-1)\cdot(x^2+2x+2)\cdot(\sqrt{x^2+3}+2x)}$$

$$= \lim_{x\to 1} \frac{-3\cdot(x+1)}{(x^2+2x+2)\cdot(\sqrt{x^2+3}+2x)}$$

$$= \frac{-3 : 2}{5\cdot(2+2)} = -\frac{3}{10}$$

2. In each of the following, if the given statement is true for all functions f that are defined on the entire real line, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample. No explanation is required.

a. If f is continuous on $(-\infty, \infty)$, then $f'(\pi)$ exists.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $|x-\pi|$

b. If $f(-1) = -1$ and $f(1) = 1$, then $f(c) = 0$ for some c in $(-1, 1)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

c. If $f''(x) = -f(x)$ for all x, then $f(x) = \sin x$ or $f(x) = \cos x$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ 0

d. If f is differentiable on $(-\infty, \infty)$, then $f'(\pi)$ exists.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$

e. If $\lim_{x \to 0} |f(x)| = 1$, then $\lim_{x \to 0} f(x) = 1$ or $\lim_{x \to 0} f(x) = -1$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

3. Suppose that f is a twice-differentiable function satisfying the following conditions:

- $y = x/2 + 1$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 4$.
- $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 8$.
- $y = -x/3 + 2$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 12$.

Consider the function $g(x) = (f(x^3))^2$.

a. Find an equation for the tangent line to the graph of $y = g(x)$ at the point with $x = 2$.

$f'(8) = \frac{1}{4}$, $f(8) = \frac{8}{4} - 3 = -1$

$g(2) = (f(8))^2 = (-1)^2 = 1$

$g'(x) = 2f(x^3)f'(x^3) \cdot 3x^2$

$g'(2) = 2f(8)f'(8) \cdot 3 \cdot 2^2 = 2 \cdot (-1) \cdot \frac{1}{4} \cdot 12 = -6$

An equation for the tangent line is:

$y - 1 = -6 \cdot (x-2) \Rightarrow y = -6x + 13$

b. Suppose that $g''(2) = 0$. Find $f''(8)$.

$g''(x) = 2f'(x^3)f'(x^3) \cdot (3x^2)^2 + 2f(x^3)f''(x^3) \cdot (3x^2)^2 + 2f(x^3)f'(x^3) \cdot 6x$

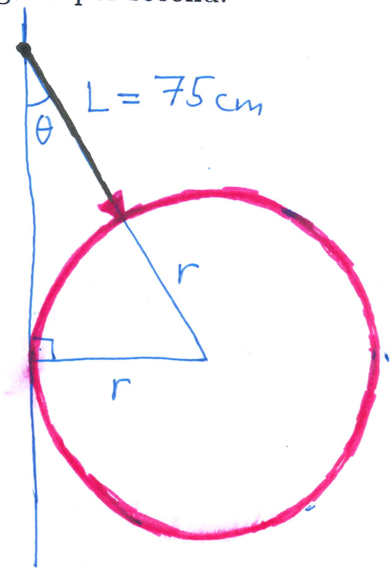
$\Rightarrow g''(2) = 2f'(8)f'(8) \cdot (3 \cdot 2^2)^2 + 2f(8)f''(8) \cdot (3 \cdot 2^2)^2 + 2f(8)f'(8) \cdot 6 \cdot 2$

$\Rightarrow 0 = 2 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot 12^2 + 2 \cdot (-1) \cdot f''(8) \cdot 12^2 + 2 \cdot (-1) \cdot \frac{1}{4} \cdot 12$

$\Rightarrow 288 f''(8) = 12 \Rightarrow f''(8) = \frac{1}{24}$

4. Air is escaping from a balloon that is hanging at the end of a 75 cm long string attached to a nail on a wall. Assume that at all times the balloon has the shape of a sphere which is tangent to the wall, and the string extends along a line which passes through the center of the sphere and lies in a vertical plane perpendicular to the wall.

Suppose that at a certain moment the volume of the balloon is decreasing at a rate of 1800π cm^3/s and its surface area is decreasing at a rate of 240π cm^2/s. Determine how fast the angle between the string and the wall is changing at this moment. Express your answer in units of degrees per second.



r = radius of the balloon

V = volume of the balloon

A = surface area of the balloon

θ = angle between the string and the wall

$V = \frac{4\pi}{3} r^3$ $\quad\Downarrow d/dt$

$A = 4\pi r^2$ $\quad\Downarrow d/dt$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ $\qquad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}$

$\Downarrow$ at our moment $\Downarrow$

$-1800\pi = 4\pi r^2 \frac{dr}{dt}$ $\qquad -240\pi = 8\pi r \frac{dr}{dt}$

$r = 15$ cm, $\frac{dr}{dt} = -2$ cm/s

$\sin\theta = \frac{r}{L+r} \overset{d/dt}{\Longrightarrow} \cos\theta \cdot \frac{d\theta}{dt} = \frac{L}{(L+r)^2} \cdot \frac{dr}{dt}$

at our moment

$\Longrightarrow \frac{\sqrt{90^2-15^2}}{90} \frac{d\theta}{dt} = \frac{75}{90^2} \cdot (-2 \text{ cm/s}) \Rightarrow \frac{d\theta}{dt} = -\frac{1}{9\sqrt{35}}$ radians/s

$\Rightarrow \frac{d\theta}{dt} = -\frac{1}{9\sqrt{35}} \cdot \frac{180}{\pi} \,°/s = -\frac{4}{\pi}\sqrt{\frac{5}{7}} \,°/s$

The angle is decreasing at a rate of $\frac{4}{\pi}\sqrt{\frac{5}{7}}$ °/s.