Date: 20 May 2018, Sunday

Time: **15:00-17:15**

NAME:..

STUDENT NO:..

YOUR DEPARTMENT:..

CIRCLE YOUR SECTION: **1 2 3 4 5 6**

Math 101 Calculus I - Final Exam – Solution Key

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Check that there are **4** questions on your booklet. Write your name on top of every page. Show your work in reasonable detail, unless asked otherwise. A correct answer without proper or too much reasoning may not get any credit.

The comprehensive list of *Math 101 Exam Rules* **in full detail is available on Moodle.** The following are only a few reminders:

- The exam consists of 4 questions.

- Read the questions carefully.

- **Solutions, not answers, get points, unless asked otherwise.** Show all your work in well-organized mathematical sentences and explain your reasoning fully, unless asked otherwise.

- **What can not be read will not be read.** Write clearly and cleanly.

- Simplify your answers as far as possible.

- Calculators and dictionaries are not allowed.

- **Turn off and leave your mobile phones with the exam proctor before the exam starts.**

- *This exam is being recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.*

Q-1) Science and engineering is all about precision! Do the following calculations and write the answers in the boxes provided. No partial credits are granted, especially in real life!

Let $f(x) = x^5 + x^4 - x^3 + 2x + 1$ and $g(x) = x^3 + \sin x - 4\sec x + 2x + 5$.

$$(f \circ g)''(0) = \boxed{202}$$

The equation $x^2 + x^3y + xy^4 + y\cos x = 1$ implicitly defines y as a differentiable function of x.

$$\left(y'' \Big|_{(x,y)=(0,1)} \right) = \boxed{7}$$

$$\lim_{x \to 0} \frac{(\tan x)^2 \ \ln(x^2 + 2\cos x)}{(e^x - 1)(x + x^2 + x^3)} = \boxed{\ln 2}$$

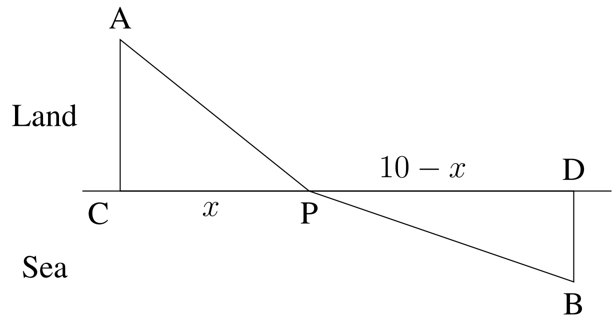
$$\left(\frac{d}{dx}\Big|_{x=1} \right) \left((x-1) \int_0^1 \frac{dt}{1 + (x-1)^2 t^2} \right) = \boxed{1}$$

Let V be the volume of the solid obtained by revolving around the y-axis the region bounded by the curve $y = x^3 - 2x^2 + x$ and the x-axis between $x = 0$ and $x = 1$.

$$V = \boxed{\frac{\pi}{15}}$$

Grading: 5 points each

Q-2



You want to transfer goods from A on land to B on sea. The cost of transportation is c_1 times the square of the distance travelled on land, and is c_2 times the square of the distance travelled on sea. You want to build a port at P so as to minimize the the total cost of transportation from A to B.

(i) Show that the position of P that minimizes the total cost depends only on c_1 and c_2. Find the corresponding x.

(ii) If P is exactly at the middle of CD, find c_1 and c_2.

Grading: 20+5 points

Solution: *Show your work in reasonable detail.*

(i) Let $f(x)$ be the cost function. Then

$$f(x) = c_1|AP|^2 + c_2|PB|^2 = c_1(x^2 + |AC|^2) + c_2((10-x)^2 + |DB|^2),$$

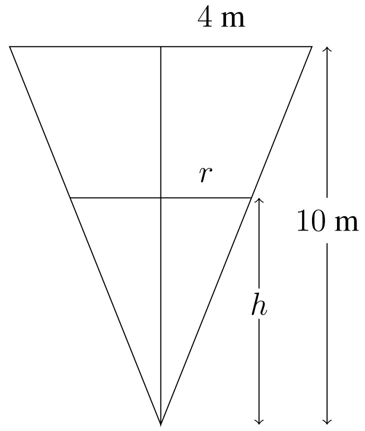
and taking derivatives we find

$$f'(x) = 0 \ \text{ when } \ x = \frac{10\,c_2}{c_1 + c_2},$$

and hence does not depend on the other values in the problem.

(ii) Note that when $x = \dfrac{10\,c_2}{c_1+c_2}$, then $10 - x = \dfrac{10\,c_1}{c_1+c_2}$. Hence if P is the midpoint, then $x = 10 - x$, and in this case $c_1 = c_2$.

Q-3



(i) A water tank has the shape of an inverted circular cone with base radius 4 m and height 10 m. If water is being pumped into the tank at a rate of 3 m^3/min, find the rate at which the water level is rising when the water is 7 m deep.
(ii) Assume now that the tank is full and we stop pumping in water. We notice immediately that there is now a leak and the water level is dropping at that moment at the rate of 0.5 m/min, and then we also notice that the water level drops at a rate proportional to the water depth. Show that it takes forever for the tank to become totally empty.
(iii) But in real life the tank becomes empty after a finite period of time. Explain why our model did not see this!
Hint: $1/e \approx 0.368$.

Grading: 10+10+5 points

Solution: *Show your work in reasonable detail.*

(i) First we have

$$V(t) = \frac{\pi}{3} r^2(t) h(t).$$

But we also have

$$\frac{4}{10} = \frac{r(t)}{h(t)}, \quad \text{so} \quad r(t) = \frac{2h(t)}{5},$$

and hence

$$V(t) = \frac{4\pi h^3(t)}{75}, \quad \text{and} \quad V'(t) = \frac{4\pi h^2(t) h'(t)}{25}.$$

We put $V'(t) = 3$ and $h(t) = 7$ to get

$$h'(t) = \frac{75}{196\pi} \text{m}^3/\text{min}.$$

(ii) We have $h'(t) = \alpha h(t)$ and $h'(0) = -0.5$ and $h(0) = 10$. This gives $\alpha = -5/100$. Then integrating we see that $h(t) = \beta e^{\alpha t}$ for some positive constant β. In fact $\beta = 10$. But setting $h(t) = 0$ and solving for t, we see that it takes forever to empty the tank.

(iii) When t is large, the amount of water in the tank, given by $h(t)$, will be eventually smaller than the size of a water molecule. So practically we don't want $h(t)$ to be zero, but equal to a very small number for us to declare the tank empty.

Q-4) Evaluate the following integrals:

(i) $\displaystyle\int \sin^{-1} x\, dx.$

(ii) $\displaystyle\int \frac{(\ln x)^2}{x^7}\, dx.$

(iii) $\displaystyle\int \frac{dx}{x^4\sqrt{9+x^2}}.$

(iv) $\displaystyle\int_0^\infty x^{2018} e^{-x}\, dx.$

Grading: 5+7+7+6 points

Solution: *Show your work in reasonable detail.*

(i) Integration by parts with $u = \arcsin x$.

Answer=$x \arcsin(x) + \sqrt{1-x^2} + C.$

(ii) Integration by parts twice, first with $u = (\ln x)^2$ and then with $u = \ln x$.

Answer=$-\dfrac{1}{6}\dfrac{(\ln(x))^2}{x^6} - \dfrac{1}{18}\dfrac{\ln(x)}{x^6} - \dfrac{1}{108}\dfrac{1}{x^6} + C.$

(iii) Put $x = 3\tan\theta$. simplify and get

$$\frac{1}{81}\int \frac{\cos^3\theta}{\sin^4\theta}\, d\theta.$$

Now put $u = \sin\theta$ to get

$$\frac{1}{81}\int \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du.$$

Integrate, put back $u = \sin\theta = \dfrac{x}{\sqrt{9+x^2}}$ to get

Answer=$\dfrac{1}{243}\dfrac{\sqrt{9+x^2}\,(-9+2\,x^2)}{x^3} + C.$

(iv) First application of by-parts gives

$$\int_0^\infty x^{2018} e^{-x}\, dx = 2018 \int_0^\infty x^{2017} e^{-x}\, dx.$$

It is now clear that

Answer=$2018! \approx 9 \times 10^{5794}.$