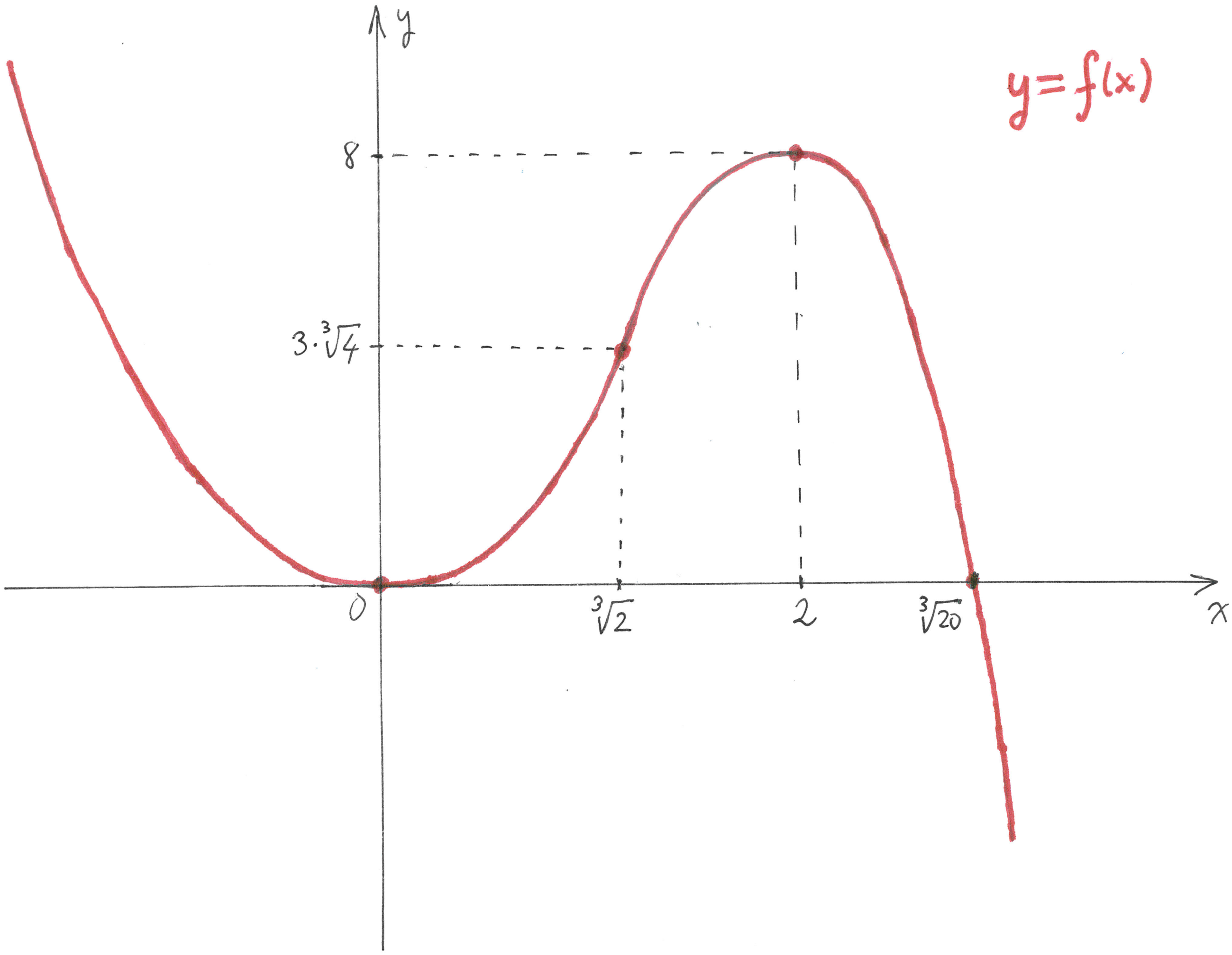
1. A function f that is twice-differentiable on the entire real line satisfies the following conditions:

❶ $f(0) = 0\,,\ f(\sqrt[3]{2}) = 3\sqrt[3]{4}\,,\ f(2) = 8\,,\ f(\sqrt[3]{20}) = 0$

❷ $f'(x) < 0$ for $x < 0$ and for $x > 2$; $f'(x) > 0$ for $0 < x < 2$

❸ $f''(x) > 0$ for $x < \sqrt[3]{2}$; $f''(x) < 0$ for $x > \sqrt[3]{2}$

a. Sketch the graph of $y = f(x)$ making sure that all important features are clearly shown.



b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = ax^b + cx^d$ satisfies the conditions ❶-❸ if a, b, c and d are chosen as

$a = \boxed{-\frac{1}{6}}$, $b = \boxed{5}$, $c = \boxed{\frac{10}{3}}$ and $d = \boxed{2}$.

We have $f(x) = ax^b + cx^d$, $f'(x) = abx^{b-1} + cdx^{d-1}$, and $f''(x) = ab(b-1)x^{b-2} + cd(d-1)x^{d-2}$.

If both of b and d are larger than 2, then $f''(0)$ would be zero. Since $f''(0) > 0$, we must have either $b = 2$ or $d = 2$. Let us take $d = 2$. (The case for $b = 2$ is similar.)

Now we have $f(x) = ax^2(x^{b-2} + \frac{c}{a})$. Since $f(20^{1/3}) = 0$, we have

$$(20^{1/3})^{b-2} = -\frac{c}{a}. \tag{1}$$

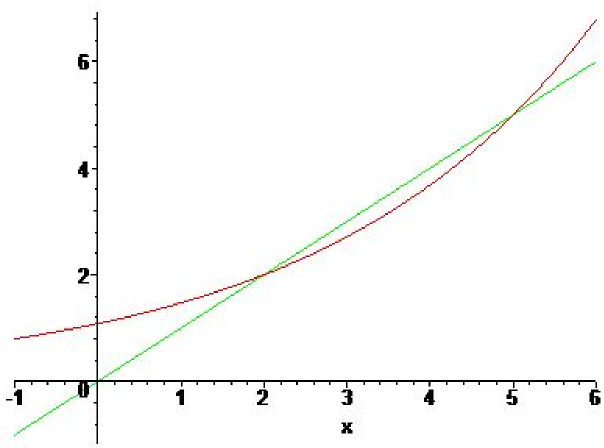
We also have $f'(x) = abx(x^{b-2} + \frac{2c}{ab})$. Since $f'(2) = 0$, we have

$$2^{b-2} = -\frac{2c}{ab}. \tag{2}$$

Dividing equation (1) by equation (2), and simplifying, we get

$$2\left(\frac{5}{2}\right)^{\frac{b-2}{3}} = b. \tag{3}$$

By inspection we see that $b = 2$ and $b = 5$ are two solutions. Since $y = 2(5/2)^{(x-2)/3}$ is concave up, and $y = x$ is a straight line, these two graphs can intersect at most at two points and we have just found all solutions. Here is the graph of these two functions:



Since b is assumed to be larger than 2, we must have $b = 5$.

Now equation (1) becomes

$$20 = -\frac{c}{a}, \qquad \text{or equivalently,} \qquad 20a + c = 0. \tag{3}$$

On the other hand $f(2) = 8$ gives

$$8a + c = 2. \tag{4}$$

Solving equations (3) and (4) together we get

$$a = -\frac{1}{6}, \text{ and } \quad c = \frac{10}{3}.$$

If at the beginning, instead of $d = 2$, we had chosen $b = 2$, then we would find $d = 5$, $a = 10/3$ and $c = -1/6$.

2. An isosceles triangle ABC with $|AB| = |BC|$ in the xy-plane satisfies the following conditions:

❶ The side $[BC]$ lies along the x-axis

❷ The side $[AB]$ passes through the point $P(0,6)$

❸ The side $[AC]$ passes through the point $Q(5,6)$

Determine the largest and smallest possible values of the area of the triangle ABC.

Let y be the y-coordinate of A and let S be the area of ABC.
The area of APQ is $\frac{1}{2}\cdot 5\cdot(y-6)$. Since ABC is similar to APQ with a ratio of $\frac{y}{y-6}$, the area of ABC is $\left(\frac{y}{y-6}\right)^2\cdot\frac{5}{2}(y-6)=\frac{5}{2}\cdot\frac{y^2}{y-6}$.

(Although there are two different triangles for any given y with $6<y<11$, they both have the same area.)

Since $|AP|=|PQ|=5$ and $|AP|\geq y-6$, y can be at most $5+6=11$.

Hence we want to:

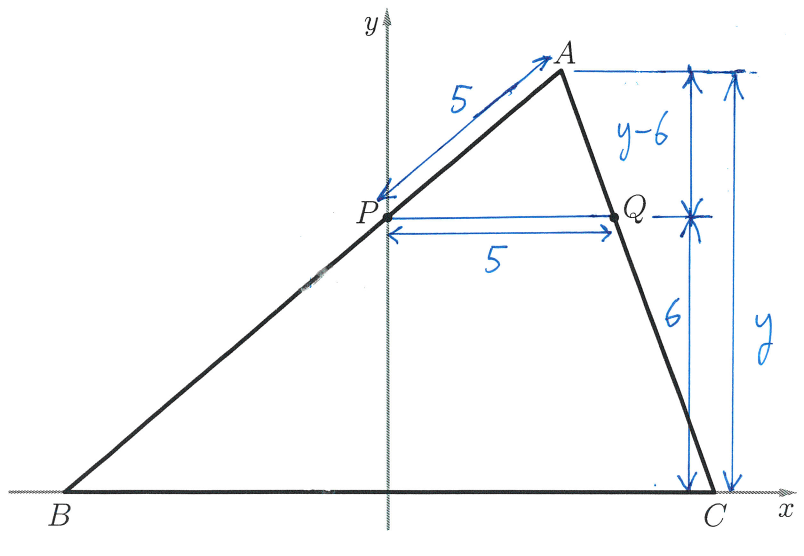
$$\text{Maximize/Minimize } S=\frac{5}{2}\cdot\frac{y^2}{y-6} \text{ for } 6<y\leq 11$$

Critical points: $\frac{dS}{dy}=\frac{5}{2}\cdot\left(\frac{2y}{y-6}-\frac{y^2}{(y-6)^2}\right)=\frac{5}{2}\cdot\frac{y(y-12)}{(y-6)^2}=0 \Rightarrow \underbrace{y=0 \text{ or } y=12}_{\text{Not in the interval}}$

Endpoints: $y=6$: $\lim_{y\to 6^+} S=\infty$

$y=11 \Rightarrow S=\frac{121}{2}$

The smallest possible area is $\frac{121}{2}$. There is no largest possible area as the area can be made arbitrarily large by choosing y sufficiently close to 6.



3a. Suppose that a continuous function f satisfies the equation

$$f'(x) = f(x) \int_0^x f(t)\,dt$$

for all x.

❶ In this part assume that $f(0) = -1$. Show that f has a critical point at $x = 0$, and determine whether it is a local maximum, a local minimum, or neither.

$f'(0) = f(0)\int_0^0 f(t)\,dt = -1 \cdot 0 = 0 \Rightarrow f$ has a critical point at $x=0$

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$f''(x) = f'(x)\int_0^x f(t)\,dt + f(x)\cdot\left(\int_0^x f(t)\,dt\right)' = f'(x)\int_0^x f(t)\,dt + f(x)^2$

$\Rightarrow f''(0) = f'(0)\cdot\int_0^0 f(t)\,dt + f(0)^2 = 0\cdot 0 + (-1)^2 = 1 > 0$

$\Rightarrow f$ has a local minimum at $x=0$

❷ In this part assume that $f''(2) = 0$. Express $f(2)$ in terms of $A = f'(2)$ only.

$f(x)f''(x) = f'(x)\cdot f(x)\int_0^x f(t)\,dt + f(x)^3 \Rightarrow f(x)f''(x) = f'(x)^2 + f(x)^3$

$\Rightarrow f(2)f''(2) = f'(2)^2 + f(2)^3 \Rightarrow 0 = A^2 + f(2)^3 \Rightarrow f(2) = -A^{2/3}$

3b. Suppose that a function g with continuous second derivative on $[0,1]$ satisfies:

$$g(0) = 2, \quad g'(0) = 3, \quad g''(0) = 4, \quad g(1) = -4, \quad g'(1) = -3, \quad g''(1) = -2$$

Evaluate $\int_0^1 g(x)\,g'(x)\left(1 + g'(x)^2 + g(x)g''(x)\right)dx$.

$\int_0^1 g(x)g'(x)\,dx = \int_{g(0)}^{g(1)} u\,du = \left.\frac{u^2}{2}\right]_2^{-4} = \frac{(-4)^2-2^2}{2} = 6$

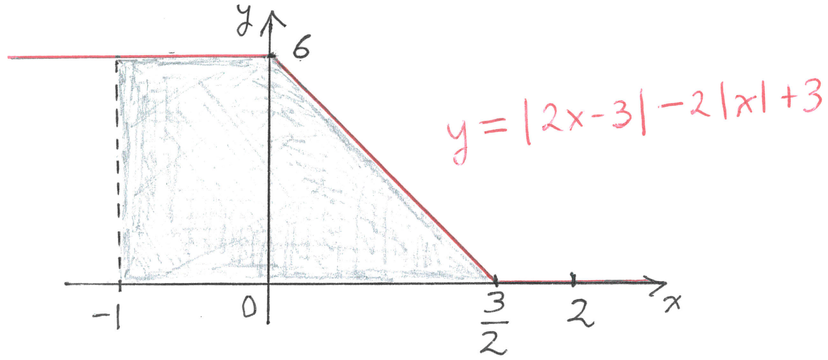
$\boxed{\begin{array}{l} u = g(x) \\ du = g'(x)\,dx \end{array}}$ $\boxed{\begin{array}{l} u = g(x)g'(x) \\ du = (g'(x)^2 + g(x)g''(x))\,dx \end{array}}$

$\int_0^1 g(x)g'(x)\left(g'(x)^2 + g(x)g''(x)\right)dx = \int_{g(0)g'(0)}^{g(1)g'(1)} u\,du = \left.\frac{u^2}{2}\right]_{2\cdot 3}^{(-4)\cdot(-3)} = \frac{12^2 - 6^2}{2} = 54$

$\int_0^1 g(x)g'(x)\left(1 + g'(x)^2 + g(x)g''(x)\right)dx = 6 + 54 = 60$

4. Evaluate the following integrals:

a. $\int_{-1}^{2} \left(|2x-3| - 2|x| + 3 \right) dx = 6 \cdot 1 + \frac{1}{2} \cdot 6 \cdot \frac{3}{2} = \frac{21}{2}$



b. $\int \frac{\tan\theta \sec^2\theta}{(1-\tan^2\theta)^2}\, d\theta = \frac{1}{2} \int du = \frac{1}{2} u + C = \frac{1}{2 \cdot (1-\tan^2\theta)} + C$

$u = \frac{1}{1-\tan^2\theta}$

$du = -\frac{1}{(1-\tan^2\theta)^2} \cdot (-2\tan\theta) \cdot \sec^2\theta \, d\theta$