**1.** Find $y'''|_{x=1}$ if $y$ is a differentiable function of $x$ satisfying the identity $y^3 + xy + x^2 = 1$.

$$\xrightarrow{x=1} \quad y^3 + y + 1 = 1 \Rightarrow y(y^2+1) = 0 \Rightarrow y = 0$$

$$\Big\downarrow \ d/dx$$

$$3y^2y' + y + xy' + 2x = 0$$

$$\xrightarrow{\boxed{x=1,\ y=0}} \quad 0 + 0 + y' + 2 = 0 \Rightarrow y' = -2$$

$$\Big\downarrow \ d/dx$$

$$6y(y')^2 + 3y^2y'' + y' + y' + xy'' + 2 = 0$$

$$\xrightarrow{\boxed{x=1,\ y=0,\ y'=-2}} \quad 0 + 0 + (-2) + (-2) + y'' + 2 = 0 \Rightarrow y'' = 2$$

$$\Big\downarrow \ d/dx$$

$$6(y')^3 + 12yy'y'' + 6yy'y'' + 3y^2y''' + 2y'' + y'' + xy''' = 0$$

$$\Big\downarrow \ \boxed{x=1,\ y=0,\ y'=-2,\ y''=2}$$

$$-48 + 0 + 0 + 0 + 4 + 2 + y''' = 0$$

$$\Downarrow$$

$$y''' = 42 \quad \text{when } x = 1$$

**2.** Find all pairs $(a, b)$ of constants for which the limit

$$\lim_{x\to 0} \frac{e^x - xe^{ax} - \cos bx}{x^4}$$

exists, and find the value of the limit for each of these pairs of constants.

$$\lim_{x\to 0} \frac{e^x - xe^{ax} - \cos bx}{x^4} \overset{L'H}{=} \lim_{x\to 0} \frac{e^x - e^{ax} - axe^{ax} + b\sin bx}{4x^3}$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{e^x - ae^{ax} - ae^{ax} - a^2xe^{ax} + b^2\cos bx}{12x^2}$$

L'H ← $\boxed{\text{provided that } 1-2a+b^2=0 \text{; otherwise, the limit does not exist}}$

$$= \lim_{x\to 0} \frac{e^x - 2a^2e^{ax} - a^2e^{ax} - a^3xe^{ax} - b^3\sin bx}{24x}$$

L'H ← $\boxed{\text{provided that } 1-3a^2=0 \text{; otherwise, the limit does not exist}}$

$$= \lim_{x\to 0} \frac{e^x - 3a^3e^{ax} - a^3e^{ax} - a^4xe^{ax} - b^4\cos bx}{24} = \frac{1-4a^3-b^4}{24}$$

$$1-3a^2=0 \implies a=\frac{1}{\sqrt{3}} \quad \text{or} \quad a=-\frac{1}{\sqrt{3}}$$

$$1-2a+b^2=0$$

$$b=\sqrt{\frac{2}{\sqrt{3}}-1} \quad \text{or} \quad b=-\sqrt{\frac{2}{\sqrt{3}}-1}$$

$b^2=-\frac{2}{\sqrt{3}}-1$, which is impossible

The limit exists exactly when $(a,b)=\left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{\sqrt{3}}-1}\right)$ or $\left(\frac{1}{\sqrt{3}}, -\sqrt{\frac{2}{\sqrt{3}}-1}\right)$, and in both cases the limit is

$$\frac{1-4a^3-b^4}{24} = \frac{1}{24}\left(1-\frac{4}{3\sqrt{3}}-\left(\frac{2}{\sqrt{3}}-1\right)^2\right) = \frac{1}{9\sqrt{3}}-\frac{1}{18}$$

**3.** In each of the following, if the given statement is true for all continuous functions $f$ that are defined on $(-\infty, \infty)$, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample.

**a.** $f$ has a derivative on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{|x|}$

**b.** $f$ has an antiderivative on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$ $\boxed{\phantom{x}}$

**c.** $\dfrac{d}{dx}\displaystyle\int_0^1 f(x)\,dx = f(x)$ for all $0 \le x \le 1$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{1}$

**d.** If $f$ is increasing on $(-\infty, \infty)$, then $f^2$ is increasing on $(-\infty, \infty)$.

☐ TRUE

☒ FALSE, because it does not hold for $f(x) =$ $\boxed{x}$

**e.** If $f$ is decreasing on $(-\infty, \infty)$, then $e^f$ is decreasing on $(-\infty, \infty)$.

☒ TRUE

☐ FALSE, because it does not hold for $f(x) =$ $\boxed{\phantom{x}}$

**4a.** Evaluate the integral $\displaystyle\int \frac{\tan x}{2\tan^2 x - 1}\,dx$

$$\int \frac{\tan x}{2\tan^2 x - 1}\,dx = \int \frac{\sin x \cos x}{2\sin^2 x - \cos^2 x}\,dx = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln|u| + C$$

$$\boxed{\begin{aligned} u &= 2\sin^2 x - \cos^2 x \\ du &= \big(2\cdot 2\sin x\cdot\cos x - 2\cos x\cdot(-\sin x)\big)\,dx \\ &= 6\sin x\cos x\,dx \end{aligned}}$$

$$= \frac{1}{6}\ln|2\sin^2 x - \cos^2 x| + C$$

**4b.** Suppose that a function $f$ with a continuous derivative satisfies the conditions:

❶ $\displaystyle\int_1^3 f'(x)\,dx = 1$ ❷ $\displaystyle\int_1^3 f(x)\,f'(x)\,dx = 1$ ❸ $\displaystyle\int_1^3 x^2 f'(x)\,dx = 1$ ❹ $\displaystyle\int_1^3 (f(x))^2\,dx = 2020$

Find $\displaystyle\int_1^3 x f(x)\,dx$.

$$① \Rightarrow 1 = \int_1^3 f'(x)\,dx = f(x)\Big]_1^3 = f(3) - f(1)$$

$$② \Rightarrow 1 = \int_1^3 f(x)f'(x)\,dx = \frac{1}{2}f(x)^2\Big]_1^3 = \frac{1}{2}f(3)^2 - \frac{1}{2}f(1)^2$$

$$f(3) = \frac{3}{2},\quad f(1) = \frac{1}{2}$$

$$③ \Rightarrow 1 = \int_1^3 x^2 f'(x)\,dx = \int_1^3 x^2\,d(f(x)) = x^2 f(x)\Big]_1^3 - \int_1^3 f(x)\,d(x^2)$$

$$= 9f(3) - f(1) - 2\int_1^3 x f(x)\,dx = \frac{27}{2} - \frac{1}{2} - 2\int_1^3 x f(x)\,dx$$

$$= 13 - 2\int_1^3 x f(x)\,dx$$

$$\Rightarrow \int_1^3 x f(x)\,dx = 6$$

5. Suppose that a function $f$ satisfies the following conditions:

❶ $f$ is positive and continuous on $[0, \infty)$.

❷ $f'(x) < 0$ for all $x > 0$.

❸ The improper integral $\int_0^\infty x f(x)\,dx$ converges.

Let $R$ be the region between the graph of $y = f(x)$ and the $x$-axis for $x \geq 0$, and let $V(a)$ be the volume of the solid generated by revolving the region $R$ about the line $x = a$ where $a$ is a nonnegative real number.

a. Show that the improper integral $\int_0^\infty f(x)\,dx$ converges.

$$\int_0^\infty x f(x)\,dx \text{ converges} \Rightarrow \left.\begin{array}{l}\int_1^\infty x f(x)\,dx \text{ converges} \\ 0 \le f(x) \le x f(x) \text{ for } x \geq 1\end{array}\right\} \Rightarrow \int_1^\infty f(x)\,dx \text{ converges by Comparison Test}$$

$$\Rightarrow \int_0^\infty f(x)\,dx \text{ converges as } f \text{ is continuous on } [0,1].$$

b. Express $V(a)$ using the cylindrical shells method.

$$V(a) = 2\pi \int_0^a (a-x) f(x)\,dx + 2\pi \int_{2a}^\infty (x-a) f(x)\,dx$$

c. Show that $V''(a) > 0$ for all $a > 0$.

$$V(a) = 2\pi \left( a \int_0^a f(x)\,dx - \int_0^a x f(x)\,dx + \int_{2a}^\infty x f(x)\,dx - a \int_{2a}^\infty f(x)\,dx \right)$$

FTC1

$$V'(a) \overset{\downarrow}{=} 2\pi \left( \int_0^a f(x)\,dx + a f(a) - a f(a) - 2a f(2a) \cdot 2 - \int_{2a}^\infty f(x)\,dx + a f(2a) \cdot 2 \right)$$

$$= 2\pi \left( \int_0^a f(x)\,dx - \int_{2a}^\infty f(x)\,dx - 2a f(2a) \right)$$

FTC1

$$V''(a) \overset{\downarrow}{=} 2\pi \left( f(a) + f(2a) \cdot 2 - 2 f(2a) - 2a f'(2a) \cdot 2 \right)$$

$$= 2\pi \left( f(a) - 4a f'(2a) \right) > 0 \text{ for } a > 0$$

as $f(a) > 0$ and $f'(2a) < 0$ for $a > 0$.