1. Suppose that $\lim_{x\to 0^+} f(x) = A$, $\lim_{x\to 0^-} f(x) = B$, $f(0) = C$, where A, B, C are distinct real numbers.

In each of the following, fill in the corresponding box by:

- Expressing the limit in terms of A, B, C if it is possible to do so using the given information;
- Writing `DNE` if it is possible to conclude that the limit does not exist using the given information; or
- Putting a ✗, otherwise.

No explanation is required. No partial points will be given. [The box should contain nothing except your answer!]

a. $\lim_{x\to 0^+} f(x-\sqrt{x}) = \boxed{B}$

b. $\lim_{x\to 0^+} f(x\sin^2(1/x)) = \boxed{\text{DNE}}$

c. $\lim_{x\to 0^+} f(x - x^2\sin(1/x)) = \boxed{A}$

d. $\lim_{x\to 0^+} f(x-\sin(x)) = \boxed{A}$

e. $\lim_{x\to 0^+} f(x-\tan(x)) = \boxed{B}$

Explanations for the answers of Question 1

Q-1-a) When $0 < x < 1$, we have $x - \sqrt{x} < 0$. If we let $t = x - \sqrt{x}$, then t goes to zero from the left while x is going to zero from the right. Hence

$$\lim_{x \to 0^+} f(x - \sqrt{x}) = \lim_{t \to 0^-} f(t) = B.$$

Q-1-b) When $0 < x < 1$, we have $x \sin^2(1/x) \geq 0$. In fact when $x_n = 1/(\pi + 2n\pi)$ for $n = 1, 2, 3, \ldots$, we have $f(x_n \sin^2(1/x_n) = f(0) = C$. This means that in any neighborhood $(0, \delta)$ with $\delta > 0$, we have infinitely many points where $f(x \sin^2(1/x)) = C$. When $x \to 0^+$ but $x \neq x_n$, then $x \sin^2(1/x) \to 0^+$. Hence

$$\lim_{\substack{x \to 0^+ \\ x \neq x_n}} f(x \sin^2(1/x)) = A, \text{ and } \lim_{\substack{x \to 0^+ \\ x = x_n}} f(x \sin^2(1/x)) = \lim_{n \to \infty} f(x_n \sin^2(1/x_n)) = C.$$

Since $A \neq C$, we have $\lim_{x \to 0^+} f(x \sin^2(1/x))$ does not exist. Hence

$$\lim_{x \to 0^+} f(x \sin^2(1/x)) = DNE$$

Q-1-c) When $0 < x < 1$, we have $t = x - x^2 \sin(1/x) = x(1 - x \sin(1/x)) > 0$. This shows that as x goes to zero from the right, t also goes to zero from the right. Hence

$$\lim_{x \to 0^+} f(x - x^2 \sin(1/x)) = \lim_{t \to 0^+} f(t) = A.$$

Q-1-d) Putting $t = x - \sin x$, we see that for $0 < x$ we have $t > 0$, and as x goes to zero from the right, t also goes to zero from the right. Hence

$$\lim_{x \to 0^+} f(x - \sin x) = \lim_{t \to 0^+} f(t) = A.$$

Q-1-e) Putting $t = x - \tan x$, we see that for $0 < x < \pi/2$, we have $t < 0$. [This can be seen as follows: Let $\phi(x) = x - \tan x$ for $0 < x < \pi/2$. Then $\phi(0) = 0$ but $\phi'(x) = 1 - \sec^2 x < 0$, so $\phi(x)$ is decreasing starting from $\phi(0) = 0$ and is negative on $0 < x < \pi/2$.] Also note as before that as x goes to zero from the right, then t also goes to zero but from the left. Hence

$$\lim_{x \to 0^+} f(x - \tan x) = \lim_{t \to 0^-} f(t) = B.$$

2. Consider the function $f(x) = \dfrac{\sin(\pi x)}{\sqrt{x^2+x^3} - x}$.

a. Find the domain of f.

$\circledast$ $\sqrt{x^2+x^3} - x \neq 0 \iff \sqrt{x^2+x^3} \neq x \iff x^2+x^3 \neq x^2$ and $x \geq 0 \iff x \neq 0$

$\circledast$ $x^2+x^3 \geq 0 \iff x^2(1+x) \geq 0 \iff 1+x \geq 0 \iff x \geq -1$

$\Rightarrow$ Domain of f is $[-1,0) \cup (0,\infty)$.

b. Compute $f(5/4)$.

$$f\left(\frac{5}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\sqrt{\left(\frac{5}{4}\right)^2+\left(\frac{5}{4}\right)^3} - \frac{5}{4}} = \frac{-\frac{1}{\sqrt{2}}}{\frac{5}{4}\cdot\left(\sqrt{\frac{9}{4}}-1\right)} = -\frac{4\sqrt{2}}{5}$$

c. Show that there is a positive real number x such that $f(x) = -1$.

$f\left(\frac{5}{4}\right) = -\sqrt{\frac{32}{25}} < -1 < 0 = f(1)$

Since f is continuous on $\left[1, \frac{5}{4}\right]$, by IVT there is a c in $\left(1, \frac{5}{4}\right)$ such that $f(c) = -1$.

d. Evaluate $\lim\limits_{x\to 0^-} f(x)$. [Do not use L'Hôpital's Rule!]

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin \pi x}{\sqrt{x^2+x^3} - x} = \lim_{x\to 0^-} \frac{\sin \pi x}{|x|\sqrt{1+x} - x} = \lim_{x\to 0^-} \frac{\sin \pi x}{-x\sqrt{1+x} - x}$$

$$= -\pi \cdot \lim_{x\to 0^-} \frac{\sin \pi x}{\pi x} \cdot \lim_{x\to 0^-} \frac{1}{\sqrt{1+x}+1} = -\pi \cdot 1 \cdot \frac{1}{2} = -\frac{\pi}{2}$$

e. Evaluate $\lim\limits_{x\to 0^+} x f(x)$. [Do not use L'Hôpital's Rule!]

$$\lim_{x\to 0^+} x f(x) = \lim_{x\to 0^+} \frac{x \sin \pi x}{\sqrt{x^2+x^3} - x} = \lim_{x\to 0^+} \frac{x \sin \pi x}{|x|\sqrt{1+x} - x} = \lim_{x\to 0^+} \frac{x \sin \pi x}{x\sqrt{1+x} - x}$$

$$= \lim_{x\to 0^+} \frac{\sin \pi x}{\sqrt{1+x} - 1} = \lim_{x\to 0^+} \frac{\sin \pi x \cdot (\sqrt{1+x}+1)}{(\sqrt{1+x}-1)\cdot(\sqrt{1+x}+1)} = \pi \cdot \lim_{x\to 0^+} \frac{\sin \pi x}{\pi x} \cdot \lim_{x\to 0^+} (\sqrt{1+x}+1)$$

$$= \pi \cdot 1 \cdot 2 = 2\pi$$

3. Suppose that f is a differentiable function and $g(x) = f(xf(x^2) + f(x))$.
Find $f(5)$ and $f'(5)$ if

❶ $f(2) = 3$, $f(3) = 5$, $f(4) = 1$, $f'(2) = -2$, $f'(3) = -4$, $f'(4) = -3$; and

❷ $y = 20x + 19$ is an equation for the tangent line to the graph of $y = g(x)$ at the point with $x = 2$.

$$(2) \Rightarrow g(2) = y\big|_{x=2} = (20x+19)\big|_{x=2} = 20\cdot 2 + 19 = 59$$

$$\|$$

$$(1) \Rightarrow f(2f(4)+f(2)) = f(2\cdot 1 + 3) = f(5)$$

$$\Longrightarrow f(5) = 59$$

$$g'(x) = f'(xf(x^2)+f(x))\cdot(1\cdot f(x^2) + x\cdot f'(x^2)\cdot 2x + f'(x))$$

$$\Downarrow$$

$$(1) \Rightarrow g'(2) = f'(2f(4)+f(2))\cdot(f(4)+8f'(4)+f'(2))$$

$$= f'(2\cdot 1+3)\cdot(1+8\cdot(-3)+(-2))$$

$$= -25 f'(5)$$

$$(2) \Rightarrow g'(2) = 20$$

$$\Longrightarrow f'(5) = -\frac{4}{5}$$

4. The points P and Q are moving along the parabola $y = x^2$ in the xy-plane in such a way that their coordinates are differentiable functions of time and the distance between them is constant. (Assume that the coordinates are measured in meters and the time is measured in seconds.)

Determine the rate of change of the distance between the point Q and the origin at the moment when P is at $(2, 4)$, Q is at $(-1, 1)$, and the distance between the point P and the origin is increasing at a rate of 3 m/s.

Let (a, a^2) and (b, b^2) be the coordinates of P and Q, respectively.

$$|OP|^2 = a^2 + a^4 \Rightarrow 2|OP| \cdot \frac{d}{dt}|OP| = (2a + 4a^3) \cdot \frac{da}{dt}$$

$$\Rightarrow 2 \cdot \sqrt{2^2 + 2^4} \cdot 3 = (2 \cdot 2 + 4 \cdot 2^3) \cdot \frac{da}{dt} \Rightarrow \frac{da}{dt} = \frac{\sqrt{5}}{3} \text{ m/s}$$

$$\boxed{a = 2\text{m}, \ \frac{d}{dt}|OP| = 3\text{ m/s}}$$

$$|PQ|^2 = (a-b)^2 + (a^2 - b^2)^2 \Rightarrow 0 = 2 \cdot (a-b) \cdot \left(\frac{da}{dt} - \frac{db}{dt}\right) + 2(a^2 - b^2) \cdot \left(2a\frac{da}{dt} - 2b\frac{db}{dt}\right)$$

$$\Rightarrow \frac{\sqrt{5}}{3} - \frac{db}{dt} + (2 + (-1)) \cdot \left(2 \cdot 2 \cdot \frac{\sqrt{5}}{3} - 2 \cdot (-1) \cdot \frac{db}{dt}\right) = 0 \Rightarrow \frac{db}{dt} = -\frac{5\sqrt{5}}{3} \text{ m/s}$$

$$\boxed{a = 2\text{m}, b = -1\text{m}, \frac{da}{dt} = \frac{\sqrt{5}}{3}\text{ m/s}, |PQ| = \text{const.}}$$

$$|OQ|^2 = b^2 + b^4 \Rightarrow 2|OQ| \frac{d}{dt}|OQ| = (2b + 4b^3) \cdot \frac{db}{dt}$$

$$\Rightarrow 2 \cdot \sqrt{(-1)^2 + (-1)^4} \cdot \frac{d}{dt}|OQ| = (2 \cdot (-1) + 4 \cdot (-1)^3) \cdot \left(-\frac{5\sqrt{5}}{3}\right) \Rightarrow \frac{d}{dt}|OQ| = 5\sqrt{\frac{5}{2}} \text{ m/s}$$

$$\boxed{b = -1\text{m}, \frac{db}{dt} = -\frac{5\sqrt{5}}{3}\text{ m/s}}$$

The distance between the point Q and the origin is increasing at a rate of $5\sqrt{\frac{5}{2}}$ m/s at that moment.