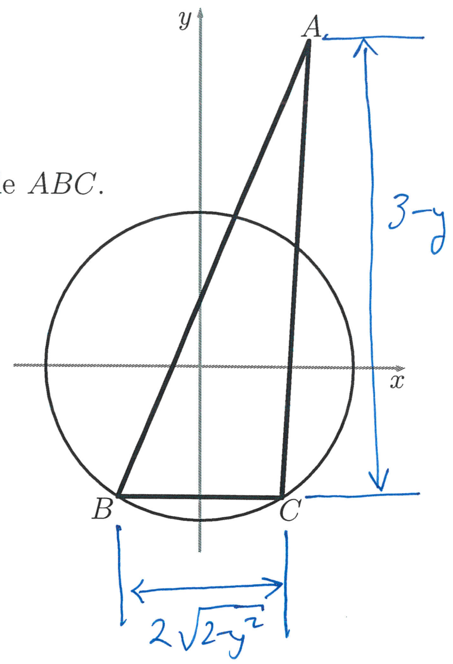
1. A triangle ABC in the xy-plane satisfies the following conditions:

❶ The coordinates of the vertex A are $(1,3)$.

❷ The vertices B and C lie on the circle $x^2+y^2=2$.

❸ The side $[BC]$ is parallel to the x-axis.

Find the largest and smallest possible values of the area of the triangle ABC.



Let y be the y-coordinate of B.

Then:

$$S = (\text{Area of } ABC) = \frac{1}{2}\cdot 2\sqrt{2-y^2}\cdot(3-y)$$

We want to:

Maximize/Minimize $S=(3-y)\cdot\sqrt{2-y^2}$ for $-\sqrt{2}\le y\le\sqrt{2}$

Critical points:

$$\frac{dS}{dy} = -\sqrt{2-y^2} + (3-y)\cdot\frac{-2y}{2\sqrt{2-y^2}} = 0 \Rightarrow 2y^2-3y-2=0$$

$$\Rightarrow y = \frac{3\pm5}{4} = \begin{cases} 2 & \otimes \leftarrow \text{not in the interval} \\ -1/2 \end{cases}$$

$$y=-\frac{1}{2} \Rightarrow S = \frac{7\sqrt{7}}{4}$$

End points:

$$y=\sqrt{2} \Rightarrow S=0$$

$$y=-\sqrt{2} \Rightarrow S=0$$

The largest possible area is $\frac{7\sqrt{7}}{4}$.

The smallest possible area is 0.

2. Evaluate the following integrals.

a. $\displaystyle\int \frac{\sin^3 x}{1+\cos x}\,dx = \int \frac{\sin^2 x}{1+\cos x}\sin x\,dx = \int \frac{1-\cos^2 x}{1+\cos x}\sin x\,dx$

$$= \int (1-\cos x)\cdot \sin x\,dx = \int (1-u)\cdot(-du) = -u + \frac{1}{2}u^2 + C$$

$u = \cos x$

$du = -\sin x\,dx$

$$= -\cos x + \frac{1}{2}\cos^2 x + C$$

b. $\displaystyle\int_0^{\pi/2} \frac{\sin^3 x}{1+\cos^2 x}\,dx = \int_0^{\pi/2} \frac{\sin^2 x}{1+\cos^2 x}\cdot \sin x\,dx = \int_0^{\pi/2} \frac{1-\cos^2 x}{1+\cos^2 x}\cdot \sin x\,dx$

$$= \int_1^0 \frac{1-u^2}{1+u^2}\cdot(-du) = \int_0^1 \left(\frac{2}{1+u^2} - 1\right) du = \Big[2\arctan u - u\Big]_0^1$$

$u = \cos x$

$du = -\sin x\,dx$

$$= 2\arctan(1) - 1 - 2\arctan(0) + 0$$

$$= 2\cdot\frac{\pi}{4} - 1 = \frac{\pi}{2} - 1$$

3. Consider the function

$$f(x) = \exp(-(\ln x)^k) = e^{-((\ln x)^k)}$$

where k is a constant.

a. In this part, let $k = 3$ and compute $f'(e^2)$.

$$f(x) = e^{-(\ln x)^3} \Rightarrow f'(x) = e^{-(\ln x)^3} \cdot \left(-3(\ln x)^2 \cdot \frac{1}{x}\right)$$

$$\Rightarrow f'(e^2) = e^{-(\ln(e^2))^3} \cdot \left(-3 \cdot (\ln(e^2))^2 \cdot \frac{1}{e^2}\right) = e^{-2^3} \cdot \left(-3 \cdot 2^2 \cdot \frac{1}{e^2}\right) = -\frac{12}{e^{10}}$$

b. In this part, let $k = 2$ and compute $\int_1^\infty f(x) \frac{\ln x}{x}\, dx$. [Here you may use the fact that $\int_a^\infty e^{-x}\, dx = e^{-a}$.]

$$\int_1^\infty f(x) \cdot \frac{\ln x}{x}\, dx = \int_1^\infty e^{-(\ln x)^2} \cdot \frac{\ln x}{x}\, dx = \int_0^\infty e^{-u} \cdot \frac{1}{2}\, du = \frac{1}{2} e^0 = \frac{1}{2}$$

$$\boxed{\begin{array}{l} u = (\ln x)^2 \\ du = 2 \ln x \cdot \frac{1}{x}\, dx \end{array}}$$

c. In this part, assume that $f(x^2) = (f(x))^3$ for all $x > 1$ and find k.

$$f(x^2) = f(x)^3 \text{ for all } x > 1 \iff e^{-(\ln(x^2))^k} = \left(e^{-(\ln x)^k}\right)^3 \text{ for all } x > 1$$

$$\iff e^{-(2\ln x)^k} = e^{-3(\ln x)^k} \text{ for all } x > 1 \iff 2^k (\ln x)^k = 3(\ln x)^k \text{ for all } x > 1$$

$$\iff 2^k = 3 \iff k = \log_2 3$$

4a. Find $f(8)$ if f is a contiuous function satisfying

$$\int_0^{x^3} f(t)\,dt + \int_0^x f(t^3)\,dt = x$$

for all x.

$$\Big\Downarrow \ d/dx$$

$$\frac{d}{dx}\int_0^{x^3} f(t)\,dt + \frac{d}{dx}\int_0^{x} f(t^3)\,dt = \frac{d}{dx}x$$

$$\Big\Downarrow \leftarrow \boxed{\text{FTC1}}$$

$$f(x^3)\cdot 3x^2 + f(x^3) = 1$$

$$\Big\Downarrow \leftarrow \boxed{x=2}$$

$$f(8)\cdot 12 + f(8) = 1$$

$$\Big\Downarrow$$

$$f(8) = \frac{1}{13}$$

4b. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{e^{x-\frac{1}{2}x^2} - x - 1}{x^3}$.

$$\lim_{x\to 0} \frac{e^{x-\frac{1}{2}x^2} - x - 1}{x^3} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{e^{x-\frac{1}{2}x^2}\cdot(1-x) - 1}{3x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{e^{x-\frac{1}{2}x^2}\cdot(1-x)^2 - e^{x-\frac{1}{2}x^2}}{6x}$$

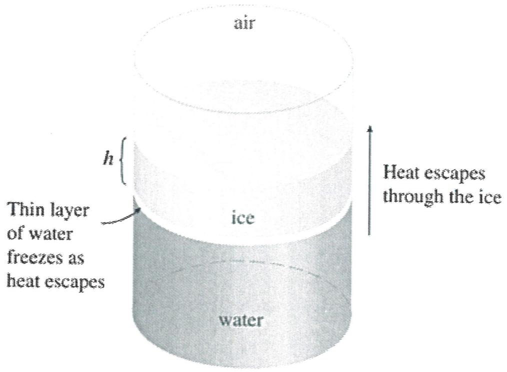
$$= \lim_{x\to 0}\left(\frac{1}{6}\cdot e^{x-\frac{1}{2}x^2}\cdot \frac{1-2x+x^2-1}{x}\right) = \frac{1}{6}\cdot e^0\cdot \lim_{x\to 0}(-2+x) = \frac{1}{6}\cdot 1\cdot(-2) = -\frac{1}{3}$$

5. The thickness h of a sheet of sea ice as a function of time t satisfies the equation

$$\frac{dh}{dt} = \frac{k}{LDh}(T_w - T_a)$$

for $h > 0$ where

- T_a is the air temperature,
- T_w is the water temperature,
- k is the thermal conductivity of the ice,
- L is the latent heat of the water, and
- D is the mass density of the water.



In this question we will assume that T_a, T_w, k, L, and D are constants.

We measure the thickness of a sheet of sea ice daily, and observe that it is 3 cm on day 0, and 5 cm on day 10.

Determine when the thickness of the sheet will be 13 cm.

Let $A = \frac{k}{LD} \cdot (T_w - T_a)$.

$$\frac{dh}{dt} = \frac{A}{h} \Rightarrow h\,dh = A\,dt \Rightarrow \int h\,dh = \int A\,dt$$

$$\Rightarrow \frac{1}{2}h^2 = At + C$$

$\boxed{t=0, \ h=3}$ $\boxed{t=10, \ h=5}$

$$\frac{9}{2} = C \quad \text{and} \quad \frac{25}{2} = 10A + C \Rightarrow A = \frac{4}{5}$$

$$\frac{1}{2}h^2 = \frac{4}{5}t + \frac{9}{2}$$

$\boxed{h=13}$

$$\frac{169}{2} = \frac{4}{5}t + \frac{9}{2} \Rightarrow t = 100 \text{ days}$$

The thickness of the sheet will be 13 cm on day 100.