Bilkent University

Quiz # 07
Math 101-Section 12 Calculus I
25 November 2021 Thursday
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Solution Key

Q-1) Evaluate the limit

$$\lim_{n\to\infty} \frac{1}{n^8} \sum_{i=1}^{n} (i^7 + i^5 n^2 + i^3 n^4),$$

by interpreting the sum as a Riemann sum.

Show your work. Simplify as much as possible.

Solutions:

The general term of the summation can be written as

$$\frac{1}{n^8}\left(i^7 + i^5 n^2 + i^3 n^4\right) = \frac{1}{n}\left[\left(\frac{i}{n}\right)^7 + \left(\frac{i}{n}\right)^5 + \left(\frac{i}{n}\right)^3\right].$$

Now consider the function

$$f(x) = x^7 + x^5 + x^3, \text{ where } x \in [0,1].$$

Divide the interval $[0,1]$ into n equal subintervals, and on each subinterval choose the right end point. Then we have for this partition and sampling

$$\Delta x = \frac{1}{n},\; x_i^* = \frac{i}{n},\; i = 1,\ldots,n,$$

and the above summation becomes the Riemann sum of $f(x)$ for this particular partition and sampling. Since f is continuous, the Riemann sum converges to the integral of the function f on $[0,1]$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^8} \sum_{i=1}^{n} (i^7 + i^5 n^2 + i^3 n^4) &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^7 + \left(\frac{i}{n}\right)^5 + \left(\frac{i}{n}\right)^3\right] \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \\
&= \int_0^1 f(x)\,dx \\
&= \int_0^1 (x^7 + x^5 + x^3)\,dx \\
&= \left(\frac{x^8}{8} + \frac{x^6}{6} + \frac{x^4}{4}\Big|_0^1\right) \\
&= \frac{1}{8} + \frac{1}{6} + \frac{1}{4} \\
&= \frac{13}{24}.
\end{aligned}$$