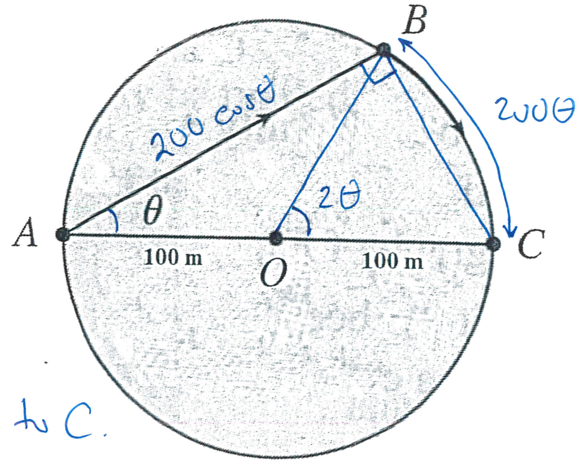
1. You are standing at a point A on the shore of a circular lake with radius 100 m, and you are planning to go to the point C, where $[AC]$ is a diameter of the circle. You will swim from A to a point B on the shore along a straight line and then either walk or run from B to C along the shore.

- You can swim with a speed of 1 m/s.
- You can walk with a speed of $\sqrt{2}$ m/s.
- You can run with a speed of 2 m/s.

Determine the angle $\theta = \widehat{CAB}$ that will take you from A to C in the shortest possible time

a. if you walk from B to C, and

b. if you run from B to C.



Let T be the time it takes to go from A to C.
Let v be the velocity on land. Then:

$$T = 200\cos\theta + \frac{200\,\theta}{v} = 200\cdot\left(\cos\theta + \frac{\theta}{v}\right) \quad \text{for } 0 \le \theta \le \frac{\pi}{2}$$

Hence:

$$\frac{dT}{d\theta} = 200\cdot\left(-\sin\theta + \frac{1}{v}\right) = 0 \Rightarrow \sin\theta = \frac{1}{v}$$

(a) If you walk, then $v = \sqrt{2}$ m/s.

Critical point: $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \Rightarrow T = 200\cdot\left(\frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right)$

Endpoints: $\theta = 0 \Rightarrow T = 200$

$\theta = \frac{\pi}{2} \Rightarrow T = 200\cdot\frac{\pi}{2\sqrt{2}}$

Shortest time occurs for $\theta = 0$ as $\pi > 3 > 2\sqrt{2}$ and $4 + \pi > 7 > 4\sqrt{2}$.

(b) If you run, then $v = 2$ m/s.

Critical point: $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow T = 200\cdot\left(\frac{\sqrt{3}}{2} + \frac{\pi}{12}\right)$

Endpoints: $\theta = 0 \Rightarrow T = 200$

$\theta = \frac{\pi}{2} \Rightarrow T = 200\cdot\frac{\pi}{4}$

Shortest time occurs for $\theta = \frac{\pi}{2}$ as $4 > \pi$ and $6\sqrt{3} > 8 > 2\pi$.

2a. Suppose that f is a continuous function satisfying

$$f(x) = x - x^2 - x\int_0^x f(t)\,dt \qquad \circledast$$

for all x, and c is a real number such that $f'(c) = 0$. Express $f(c)$ in terms of c only.

$$\circledast \xrightarrow{d/dx} f'(x) = 1 - 2x - \int_0^x f(t)\,dt - x f(x) \quad \text{by FTC 1}$$

$$\Downarrow x=c$$

$$0 = f'(c) = 1 - 2c - \int_0^c f(t)\,dt - c f(c) \quad (1)$$

$$\circledast \xrightarrow{x=c} f(c) = c - c^2 - c\int_0^c f(t)\,dt \quad (2)$$

$$-c \times (1) + (2): \quad f(c) = c^2 + c^2 f(c)$$

$$\Downarrow$$

$$f(c) = \frac{c^2}{1-c^2}$$

2b. Evaluate the limit $\displaystyle\lim_{n\to\infty} \frac{1}{n^3}\left(\sum_{i=1}^{3n}\sqrt{i}\right)^2$.

$$\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i} = \sum_{i=1}^{3n}\sqrt{\frac{i}{n}}\cdot\frac{1}{n}$$ is a right Rieman sum for $f(x) = \sqrt{x}$ over the interval $[0,3]$.

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i} = \int_0^3 \sqrt{x}\,dx = \left.\frac{x^{3/2}}{3/2}\right]_0^3 = \frac{2}{3}\cdot 3^{3/2} = 2\cdot 3^{1/2}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n^3}\left(\sum_{i=1}^{3n}\sqrt{i}\right)^2 = \left(\lim_{n\to\infty}\frac{1}{n^{3/2}}\sum_{i=1}^{3n}\sqrt{i}\right)^2 = \left(2\cdot 3^{1/2}\right)^2 = 12$$

3. Evaluate the following integrals.

a. $\displaystyle\int (3x-1)(x+2)(x^2-2)^{2022}(2x+5)^{2022}\,dx = \int u^{2022}\cdot \frac{1}{2}\,du$

$$\boxed{\begin{aligned} u &= (x^2-2)(2x+5) = 2x^3+5x^2-4x-10 \\ du &= (6x^2+10x-4)\,dx = 2(3x-1)(x+2)\,dx \end{aligned}}$$

$$= \frac{1}{2}\cdot\frac{u^{2023}}{2023} + C' = \frac{(x^2-2)^{2023}(2x+5)^{2023}}{4046} + C'$$

b. $\displaystyle\int_{\pi/4}^{5\pi/4} \frac{\cos^2 x}{(x+\sin x\cos x)^2}\,dx = \int_{\frac{\pi}{4}+\frac{1}{2}}^{\frac{5\pi}{4}+\frac{1}{2}} \frac{1}{u^2}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_{\frac{\pi}{4}+\frac{1}{2}}^{\frac{5\pi}{4}+\frac{1}{2}} u^{-2}\,du$

$$\boxed{\begin{aligned} u &= x+\sin x\cos x \\ du &= (1+\cos^2 x-\sin^2 x)\,dx = 2\cos^2 x\,dx \end{aligned}}$$

$$= -\frac{1}{2}u^{-1}\Big]_{\frac{\pi}{4}+\frac{1}{2}}^{\frac{5\pi}{4}+\frac{1}{2}} = -\frac{1}{2}\cdot\left(\frac{1}{\frac{5\pi}{4}+\frac{1}{2}} - \frac{1}{\frac{\pi}{4}+\frac{1}{2}}\right) = \frac{8\pi}{(5\pi+2)(\pi+2)}$$

4. Let $R(a)$ be the region bounded by the graph of $f(x) = ax - x^2$ and the x-axis for $0 \leq x \leq a$, where a is a positive constant.

a. Compute the volume $V(a)$ of the solid generated by revolving $R(a)$ about the x-axis.

$$V(a) = \pi \int_0^a (ax-x^2)^2 \, dx = \pi \int_0^a (a^2x^2 - 2ax^3 + x^4) \, dx$$

$$= \pi \left[\frac{a^2}{3} x^3 - \frac{a}{2} x^4 + \frac{1}{5} x^5 \right]_0^a = \pi \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) a^5 = \frac{\pi}{30} a^5$$

b. Compute the volume $W(a)$ of the solid generated by revolving $R(a)$ about the y-axis.

$$W(a) = 2\pi \int_0^a x \cdot (ax - x^2) \, dx = 2\pi \int_0^a (ax^2 - x^3) \, dx$$

$$= 2\pi \left[\frac{a}{3} x^3 - \frac{1}{4} x^4 \right]_0^a = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) a^4 = \frac{\pi}{6} a^4$$

c. Find all values of a for which $V(a) = W(a)$.

$$V(a) = W(a) \Rightarrow \frac{\pi}{30} a^5 = \frac{\pi}{6} a^4 \Rightarrow a = 5$$