Bilkent University

Quiz # 07
Math 101-Section 04 Calculus I
9 November 2023 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Let $f(x) = x^5$ on the interval $[0,1]$. Subdivide this interval into n equal subintervals as $0 = x_0 < x_1 < \cdots < x_n = 1$. For this function and for this partition we define $L_n = \sum_{i=0}^{n-1} \frac{1}{n} f(x_i)$, $U_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i)$, $R_n = \sum_{i=1}^{n} \frac{1}{n} f(x_i^*)$, where each $x_i^* \in [x_{i-1}, x_i]$ is an arbitrarily chosen points.

(a) Calculate $\lim_{n\to\infty} L_n$. **(b)** Calculate $\lim_{n\to\infty} U_n$. **(c)** Calculate $\lim_{n\to\infty} R_n$.

Hint: $1^5 + \cdots + n^5 = \frac{1}{12}[2n^6 + 6n^5 + 5n^4 - n^2]$ Grading: 3+3+4=10 points

Solution: (Grader: `rbulakguler71@gmal.com`)

(a) Note that $x_i = i/n$. Then we have

$$L_n = \sum_{i=0}^{n-1} \frac{1}{n}\frac{i^5}{n^5} = \frac{1}{n^6}\sum_{i=0}^{n-1} i^5 = \frac{1}{n^6}[\frac{1}{12}[2(n-1)^6 + 6(n-1)^5 + 5(n-1)^4 - (n-1)^2]]$$

Now we clearly have $\lim_{n\to\infty} L_n = \frac{1}{6}$.

(b) As above we have $x_i = i/n$ and we have

$$U_n = \sum_{i=1}^{n} \frac{1}{n}\frac{i^5}{n^5} = \frac{1}{n^6}\sum_{i=1}^{n} i^5 = \frac{1}{n^6}[\frac{1}{12}[2n^6 + 6n^5 + 5n^4 - n^2]]$$

Now we have $\lim_{n\to\infty} U_n = \frac{1}{6}$.

(c) Since f is increasing we have $L_n < R_n < U_n$.
Using the squeeze theorem we have $\lim_{n\to\infty} R_n = \frac{1}{6}$.