Date: 21 June 2003, Saturday Instructor: Ali Sinan Sertöz

**Math 102 Calculus – Midterm Exam I**
**Solutions**

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**Q-1)** Evaluate the integral $\displaystyle\int \frac{x^2+5}{(x-1)^2(x^2+1)}\,dx$

**Solution:** Here you need to simplify the integrand using the technique of partial fractions:

$$\frac{x^2+5}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Bringing the RHS to common denominator and equating the numerators of LHS with that of the RHS gives

$$x^2+5 = (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D).$$

From here it follows that $A=-2$, $B=3$, $C=2$ and $D=0$, so

$$\frac{x^2+5}{(x-1)^2(x^2+1)} = -2\frac{1}{x-1} + 3\frac{1}{(x-1)^2} + \frac{2x}{x^2+1},$$

which can now be integrated easily to give

$$\int \frac{x^2+5}{(x-1)^2(x^2+1)}\,dx = -2\ln|x-1| - \frac{3}{x-1} + \ln(x^2+1) + C.$$

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**Q-2-A)** Evaluate the integral $\displaystyle\int x\,(\ln x)^2\;dx$

**Solution:** Let $u=(\ln x)^2$ and $dv = xdx$. Then $du = (2/x)\ln x dx$, $v=(1/2)x^2$. This gives

$$\int x(\ln x)^2 dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x dx.$$

For the second integral let $u=\ln x$, $dv = xdx$. Then $du=(1/x)dx$, $v=(1/2)x^2$ and

$$\begin{aligned}\int x\ln x dx &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int xdx \\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.\end{aligned}$$

Combining these we get

$$\int x(\ln x)^2 = \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2\ln x + \frac{1}{4}x^2 + C.$$

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**Q-2-B)** Evaluate the integral $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$.

**Solution:** Put $x = \sec\theta$, $dx = \sec\theta\tan\theta d\theta$. Then $\sqrt{x^2-1} = \tan\theta$ and

$$\begin{aligned} \int \frac{\sqrt{x^2-1}}{x^2}\,dx &= \int \frac{\tan^2\theta}{\sec\theta}\,d\theta \\ &= \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta \\ &= \int \sec\theta\,d\theta - \int \cos\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| - \sin\theta + C \\ &= \ln|x + \sqrt{x^2-1}| - \frac{\sqrt{x^2-1}}{x} + C. \end{aligned}$$

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**Q-3-A)** Does the improper integral $\int_0^\infty \frac{dx}{\sqrt{64x^7+2003}}$ exist? Show your reasoning in detail.

**Solution:** First observe that

$$\int_0^\infty \frac{dx}{\sqrt{64x^7+2003}} = \int_0^1 \frac{dx}{\sqrt{64x^7+2003}} + \int_1^\infty \frac{dx}{\sqrt{64x^7+2003}}$$

and the integral from $0$ to $1$ is finite. So we have to examine only the integral from $1$ to $\infty$. For this we recall that $\int_1^\infty \frac{dx}{x^{7/2}}$ converges since $7/2 > 1$. On the other hand

$$\lim_{x\to\infty} \frac{\left(1/x^{(7/2)}\right)}{\left(1/\sqrt{64x^7+2003}\right)} = 8$$

and by the Limit Comparison Test the original integral **converges**.

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**Q-3-B)** Does the improper integral $\int_0^1 \frac{x}{\sin^3 x}\,dx$ exist? Show your reasoning in detail.

**Solution:** First recall that $\int_0^1 \frac{dx}{x^2}$ diverges. Then observe that

$$\lim_{x\to 0^+} \frac{(1/x^2)}{(x/\sin^3 x)} = \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^3 = 1$$

and by the Limit Comparison test the original integral **diverges**.

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**Q-4-A)** Write the equation of the plane passing through the points
$P = (1,2,3)$, $Q = (2,3,2)$ and $R = (3,5,4)$.

**Solution:** First find two vectors parallel to the plane:
$\vec{PQ} = Q - P = (1,1,-1)$, $\vec{PR} = R - P = (2,3,1)$.
Then find a direction $\vec{n}$ orthogonal to both of these vectors: $\vec{n} = \vec{PQ} \times \vec{PR} = (4,-3,1)$.
Now observe that $\vec{n}\cdot P = \vec{n}\cdot Q = \vec{n}\cdot R = 1$. So the equation of this plane is

$$4x - 3y + z = 1.$$

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**Q-4-B)** Find the point of intersection of the line
$x = 1 + 2t$, $y = 3 + 4t$, $z = 5 + 6t$, $t \in \mathbb{R}$, with the plane $7x + 8y + 9z = 10$.

**Solution:** Substitute the parametric equations of the line into the equation of the plane to obtain

$$7(1 + 2t) + 8(3 + 4t) + 9(5 + 6t) \quad = \quad 10$$

which gives $t = -\dfrac{33}{50}$. Putting this value of $t$ into the parametric equation of the line gives the point of intersection as

$$x = -\frac{8}{25}, \quad y = \frac{9}{25}, \quad z = \frac{26}{25}.$$

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I hope you had all the answers right.

If you have any comments or questions please write to me at : `sertoz@fen.bilkent.edu.tr`

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