MATH 102 MIDTERM I-Solutions

Date: June 18, 2004 Goncharov & Sertöz

1. a) Evaluate $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(x+1)}$.

Putting $u = \sqrt{x}$ gives $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(x+1)} = 2\int_1^\infty \frac{du}{1+u^2} = 2\,(\arctan u|_1^\infty) = \frac{\pi}{2}$.

b) Test the following integral for convergence $\displaystyle\int_0^\infty \frac{1+sin^2x}{e^x+x^2+1}\,dx$.

Let $f(x) = \dfrac{1+sin^2x}{e^x+x^2+1}$ and $g(x) = e^{-x}$.

Then $\dfrac{f(x)}{g(x)} = \dfrac{1+sin^2x}{1+x^2e^{-x}+e^{-x}} < 2$, so $0 \le f(x) < 2g(x)$.

Since $\displaystyle\int_0^\infty g(x)dx$ converges, the given integral also converges by direct comparison.

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2. Find the sum $\displaystyle\sum_4^\infty \frac{1}{n^2-n-2}$.

Let $S_n = \displaystyle\sum_{k=4}^n \frac{1}{n^2-n-2} = \sum_{k=4}^n \left(\frac{1/3}{n-2} - \frac{1/3}{n+1}\right)$. Then $S_n = \dfrac{1}{3}\left(\dfrac{13}{12} - \dfrac{1}{n-1} - \dfrac{1}{n} - \dfrac{1}{n+1}\right)$.
The sum is then found as $\displaystyle\lim_{n\to\infty} S_n = \frac{13}{36}$.

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3. Determine if each of the following series is convergent or divergent.

a) $\displaystyle\sum_3^\infty \frac{ln(ln\,n)}{ln^2n}$

Since $\displaystyle\lim_{n\to\infty} \ln\ln n = \infty$, we have $\ln\ln n > 1$ for large n.

Similarly since $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0$, we have $(\ln n)^2 < n$ for large n.

Then $\dfrac{ln(ln\,n)}{ln^2n} > \dfrac{1}{n}$, and the series diverges by comparing with the harmonic series.

b) $\sum_1^\infty (\sqrt{n^3+1} - \sqrt{n^3})$.

$$(\sqrt{n^3+1} - \sqrt{n^3}) = (\sqrt{n^3+1} - \sqrt{n^3})\frac{\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}} = \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}} < \frac{1}{2\sqrt{n^3}} = \frac{1}{2n^{3/2}}.$$

Then the series converges by direct comparison with the converging p-series.

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4. Find the radius of convergence of the series $\sum_1^\infty \frac{2\cdot 5\cdot 8\ \cdot \cdots \cdot (3n+2)}{2\cdot 4\cdot 6\ \cdot \cdots \cdot (2n)}\ (2x)^n$.

$$a_n = 2\cdot \frac{5\cdot 8\cdots(3n+2)}{2\cdot 4\cdots(2n)}\ (2x)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3n+5)2x}{2n+2}\right| = 3|x|.$$

For convergence we need $3|x| < 1$, so the radius of convergence is $1/3$.

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5. Evaluate the limit $lim_{x\to 0}\frac{sin(x^2) - x^2\, cosx}{(e^{x^2}-1)\, x^2}$.

Using the Taylor expansions of $\sin t$, $\cos t$ and e^t we get

$$\begin{aligned}\lim_{x\to 0}\frac{sin(x^2) - x^2\, cosx}{(e^{x^2}-1)\, x^2} &= \lim_{x\to 0}\frac{(x^2 - x^6/6 + \cdots) - (x^2 - x^4/2 + \cdots)}{x^4 + x^8/2 + \cdots} \\ &= \lim_{x\to 0}\frac{1/2 + x(junk)}{1 + x(another\ junk)} \\ &= \frac{1}{2}.\end{aligned}$$
