Date: July 26, 2008, Saturday NAME:...
Time: 15:00-17:00
Ali Sinan Sertöz STUDENT NO:...

Math 102 Calculus II – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

PLEASE READ:
Check that there are 5 questions on your exam booklet. Write your name on the top of every page. Show your work in reasonable detail. A correct answer without proper reasoning may not get any credit.

Q-1) For any ϵ with $0 < \epsilon < \pi/6$, define the region R_ϵ as the region in $\mathbb{R}^2$ bounded by the curves $y = 1/x$, $y = 2/x$, $x = \epsilon$ and $x = \pi/6$. Calculate

$$\lim_{\epsilon \to 0} \int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dxdy.$$

Solution:

$$\begin{aligned}
\int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dxdy &= \int_\epsilon^{\pi/6} \int_{1/x}^{2/x} x^2 \sec^2(x^2 y)\, dydx \\
&= \int_\epsilon^{\pi/6} \left[\tan(x^2 y)\Big|_{1/x}^{2/x} \right] dx \\
&= \int_\epsilon^{\pi/6} (\tan(2x) - \tan(x))\ dx \\
&= \left[-\frac{1}{2} \ln \cos 2x + \ln \cos x \Big|_\epsilon^{\pi/6} \right] \\
&= \frac{1}{2} \ln \frac{3}{2} - [-\frac{1}{2} \ln \cos 2\epsilon + \ln \cos \epsilon].
\end{aligned}$$

And since ln and cos are continuous functions,

$$\lim_{\epsilon \to 0} \int\int_{R_\epsilon} x^2 \sec^2(x^2 y)\, dxdy = \frac{1}{2} \ln \frac{3}{2}.$$

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Q-2) Find the volume of the region bounded by the paraboloid $x^2 + y^2 + z = 4$ and the cylinder $x^2 - 2y + y^2 = 0$ above the xy-plane.

Hint: $\int \sin^4 t \, dt = \frac{3t}{8} - \sin(2t)\left(\frac{3}{16} + \frac{1}{8}\sin^2 t\right) + C.$

Solution:

$$\begin{aligned}
\text{Volume} &= \int_0^2 \int_{-\sqrt{2y-y^2}}^{\sqrt{2y-y^2}} \int_0^{4-x^2-y^2} dz\, dx\, dy \\
&= \int_0^{\pi} \int_0^{2\sin\theta} (4 - r^2) r dr\, d\theta \\
&= \int_0^{\pi} \left[2r^2 - \frac{1}{4} r^4 \Big|_0^{2\sin\theta} \right] d\theta \\
&= \int_0^{\pi} \left(8\sin^2\theta - 4\sin^4\theta\right) d\theta \\
&= \left[\sin^3\theta\cos\theta - \frac{5}{2}\sin\theta\cos\theta + \frac{5t}{2} \Big|_0^{\pi} \right] \\
&= \frac{5\pi}{2}.
\end{aligned}$$

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Q-3) Let C be the curve parameterized as $r(\theta) = \left(\frac{\theta}{8},\ \sin^2\theta,\ 1-\cos^4\theta\right)$, with $0 \leq \theta \leq 2\pi$. Calculate

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds,$$

where $\mathbf{F} = (-\tan(x+y^2+z^3),\ -2y\tan(x+y^2+z^3),\ -3z^2\tan(x+y^2+z^3))$.

Solution: In the last homework we showed that this is a conservative field with potential function $f = \ln\cos(x+y^2+z^3)+C$. This gives

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = f(r(2\pi)) - f(r(0)) = f(\pi/4,0,0) - f(0,0,0) = -\frac{1}{2}\ln 2.$$

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Q-4) Find the surface area of the piece of the paraboloid $x^2 + y^2 + z = 4$ with $z \geq 0$.

Solution: If $f = x^2 + y^2 + z - 4$ and D is the projection of the paraboloid S to xy-plane, then the surface area is given by

$$\begin{aligned}
\int_S d\sigma &= \int_D \frac{|\nabla f|}{|\nabla f \cdot \mathbf{k}|}\, dxdy \\
&= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\; r\; drd\theta \\
&= (2\pi)\left[\frac{1}{12}(1 + 4r^2)^{3/2} \Big|_0^2 \right] \\
&= \frac{\pi}{6}(17\sqrt{17} - 1).
\end{aligned}$$

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Q-5) Evaluate the integral

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma$$

where S is the level surface given by $x^2 + z^2 - 4 + y^4 = 0$, $y \geq 0$,

$$\mathbf{F} = \left(x^2 z + \ln(y^2+1),\ \cosh(x^2+y^2) - \ln(z^2+1),\ \frac{y^3}{y^2+1} - xz^2 \right),$$

and $\mathbf{n}$ is the unit normal of S pointing out.

Solution: Let $D = \{(x,z) \in \mathbb{R}^2 \mid x^2 + z^2 \leq 4\}$ and let $C = \partial D$. Then using Stokes' theorem twice, we find that

$$\begin{aligned}\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma &= \int_C \mathbf{F} \cdot \mathbf{T} \, ds \\ &= \int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1} \, d\sigma\end{aligned}$$

where $\mathbf{n_1}$ is the unit normal of D pointing towards y-direction to be compatible with the orientation on C which in turn is induced by $\mathbf{n}$. Thus $\mathbf{n_1} = \mathbf{j}$ and $\nabla \times \mathbf{F} \cdot \mathbf{n_1} = x^2 + z^2$. This gives

$$\begin{aligned}\int\int_D \nabla \times \mathbf{F} \cdot \mathbf{n_1} \, d\sigma &= \int\int_D (x^2 + z^2) \, dx dz \\ &= \int_0^{2\pi} \int_0^2 r^3 \, dr d\theta \\ &= (2\pi)(\frac{16}{4}) = 8\pi.\end{aligned}$$